THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

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VOLUME XLIV

1937

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EDITORIAL FOREWORD

In this first number of the MONTHLY issued under the direction of a new editorial board, it appears desirable to say a few words concerning the nature of the MONTHLY and concerning our editorial policy.

As announced on its cover, the MONTHLY is the official journal of the Mathematical Association of America which is devoted to the interests of collegiate mathematics. Accordingly the MONTHLY publishes official reports of the Association and of its various Sections, and such other material as may be deemed best fitted for the promotion of collegiate mathematics. The word "collegiate" calls attention to the fact that the MONTHLY is devoted neither to technical research nor to elementary or pedagogical questions but it presents papers on such topics as they appear of significance for the promotion of collegiate mathematics.

The material presented in the MONTHLY should, we believe, be of interest and service to one or more of the following groups—undergraduates, graduates, and teachers in colleges whose interests are mathematical. Furthermore, each number of the MONTHLY should contain something of interest for each of these groups.

The MONTHLY includes three departments which are primarily intended to serve the needs of undergraduates—mathematics clubs, elementary problems and solutions, and notes and discussions. It is hoped that these departments will stimulate the enthusiasm of undergraduates for mathematics, broaden their knowledge of the subject, and serve to perpetuate their interest after graduation. In the handling of these departments the present editors expect to follow the course laid out by their predecessors with minor innovations.

Graduate students and college teachers may find things of interest in the three departments mentioned above, but other departments may be of greater service to them. Those who enjoy the stimulation of a difficult problem may turn to the department of advanced problems and solutions. Teachers may profit by reference to the department of recent publications which contains notices and reviews of recent books in the fields of collegiate mathematics, and will be interested in the department of news and notices which includes items of personal interest concerning mathematicians, their degrees, honors, appointments, promotions, or deaths.

In addition to the material mentioned above, the MONTHLY includes a considerable number of longer articles, reports and minor research papers. The editors believe that the MONTHLY should not become a competitor of the research journals in handling these papers. Clarity of exposition based upon a more moderate mathematical background should characterize papers presented in the MONTHLY. The editors hope that contributors will submit a variety of general expository articles, not too technical in nature, which will either present historical sketches of the development of significant mathematical topics or will give clear elementary expositions of such topics not elsewhere readily accessible, both in pure and in applied mathematics.

THE TWENTIETH ANNUAL MEETING OF THE KENTUCKY SECTION

The twentieth annual meeting of the Kentucky Section of the Mathematical Association of America was held in joint session with the Kentucky Academy of Science at the Western State Teachers College, Bowling Green, on Saturday, May 9, 1936. Professor H. M. Yarbrough, acting for Professor R. S. Park, presided at both sessions. The attendance was thirty, including the following eleven members of the Association: J. G. Black, P. P. Boyd, L. A. Fair, A. R. Fehn, R. D. Perry, F. Elizabeth LeStourgeon, W. L. Moore, Sister Charles Mary Morrison, Wallace Smith, Guy Stevenson, H. M. Yarbrough. Dinner was served at the Helm Hotel. Professor W. L. Moore, University of Louisville, was elected chairman, and Professor A. R. Fehn, Centre College, secretary, for the year 1936–37.

The following papers were given:

1. "Fundamental mathematical concepts for mathematics majors" by Professor Guy Stevenson, University of Louisville.

2. "The defense of high school mathematics" by Wallace Smith, New River State College, Montgomery, West Virginia.

3. "Teaching the binomial theorem" by Tryphena Howard, Western State Teachers College.

4. "Some statistical aspects of methods used in scoring test papers" by Professor R. D. Perry, Western State Teachers College.

5. "Advice for calculating mechanically the square root deviation from the mean" by Professor J. G. Black, Morehead State Teachers College.

6. "Curves and surfaces of flotation" by Dr. Fritz John, University of Kentucky, introduced by Professor Boyd.

7. "On linear measures of point sets composed of any number of rectifiable arcs" by Susan J. Howard, Western State Teachers College, introduced by Professor Yarbrough.

8. "Differentials" by L. P. Hutchison, University of Kentucky, introduced by Professor Yarbrough.

9. "Euclidean algorithm in algebraic fields" by Dr. E. D. Jenkins, University of Kentucky, introduced by Professor Yarbrough.

Abstracts of some of the papers follow, numbered in accordance with their place on the program:

2. Mr. Smith stated that the present generation of leaders in the high school field have evolved a philosophy of easy education which has developed a set of teachers inadequately prepared to teach high school mathematics. The results are a decline in the study of mathematics and a gradual deletion of this subject from the high school curriculum. As a corrective he recommends that administrative leaders be required to submit to a five-year training program with emphasis on at least one academic subject and that mathematics teachers be required to complete a five-year program of training, emphasizing mathematics in the last two years.

4. Professor Perry derived formulas to be used in scoring various types of tests. For instance, for a test of the multiple-choice type, he gave the formula $S=R-W/(n-1)$ where S stands for the score the student should receive, R stands for the number of right responses, W the number of wrong responses, and n the number of choices the student may make in a response.

5. By means of a simple arrangement of metric scales to form a right triangle with a sliding hypotenuse, Professor Black stated it was feasible, if the median is known, and without knowing the value of any individual deviation save the first to calculate with great rapidity the standard deviation of a series containing as many as four hundred terms. The device should be extremely valuable for all types of statistical work. By the use of protractors and variation of the triangle it is possible also to set up quickly on a large and accurate scale the graphical solution of the general triangle, making a quick check for work in engineering, surveying and triangulation in warfare. It is possible to solve other problems in algebra and trigonometry with the device.

6. In the theory of floating bodies two kinds of surfaces are associated with a solid body, surfaces of flotation and surfaces of buoyancy. Whereas the surfaces of buoyancy are always convex, the surfaces of flotation are in general not convex. As a consequence of a theorem of Blaschke a necessary condition for the convexity of all surfaces of flotation of a convex body is that the body has a center, i.e. is symmetrical with respect to some point. Dr. John proved in his paper that in the corresponding two-dimensional case this condition is also sufficient, i.e., the centers of the secants cutting off from a convex region with a center a constant area form a convex curve.

8. Mr. Hutchison stressed the importance of the role of the differential in mathematics. He gave a general definition of a differential and showed how it may be used to obtain implicit function theorems and Lagrange multiplier theorems in terms of general linear metric spaces as well as spaces of infinitely many dimensions.

9. Quadratic fields for which a Euclid algorithm is known to exist and those for which an algorithm is known not to exist, were discussed by Dr. Jenkins. A substitute method for finding a greatest common divisor of integers of an algebraic field was then given.

A. R. FEHN, *Secretary*

This year 1937 marks the two hundred fiftieth anniversary of the publication of Newton's "Principia," in which he sets forth his theory of gravitation, characterized by Whewell, in his "History of the Inductive Sciences," (Bk. VII, ii, §5) as "indisputably and incomparably the greatest scientific discovery ever made." Lagrange is said to have remarked that Newton was the greatest genius that ever lived, and the most fortunate, since we can find only once a system of the universe to be established. Whether Einstein's theory of relativity, or some other discovery of recent times, may eventually invalidate Lagrange's statement, made over a century ago, remains to be seen.

and let $f_n(x)$ be defined by (9.3) so that $f(x)$ is 0, then the right member of (9.4) is 0 while the left member is log 2.

10. *Conclusion.* I have attempted to set forth, in a way intelligible to those who do not know the theory of convergence in mean and Lebesgue integration, some of the reasons why analysts should study and use these concepts. I shall be content if each hearer of this sermon can write the following three planks in his credo.

Introduction of convergence in mean into Fourier analysis has brought not a complication but instead a tremendous simplification and a gratifying completeness to the theory.

Just as an analysis based on the class of real algebraic numbers alone is incomplete and inelegant as compared with that based on the class of all real numbers, so also is an analysis based on convergence and Riemann integrals incomplete and inelegant as compared with that based on convergence in mean and Lebesgue integration.

The fact that one's interest may lie in a small class of functions such as continuous functions can furnish no excuse for ignorance of and failure to use the complete and elegant analysis based on the concepts of convergence in mean and Lebesgue integration.

THE ELEMENTARY FOUNDATION OF MATHEMATICAL PHYSICS

By RONOLD KING, Lafayette College

For many centuries man has been curious about the structure of nature. In attempting to understand the mysteries of its changing form and the rhythm of its stupendous continuity, he has progressed slowly, yet amazingly. Thus, he has flown blindly on the powerful wings of imagination; he has trudged boldly along the winding road of consistent logic; he has used his senses to look with penetration and to listen with patience. But it was not until he learned to send a logical mind soaring high into the abstract spaces of mathematics, while a skillful hand guided its flight with the compass of observed reality, that he began to find in nature not the whims of capricious gods, but unity and consistency.

In the early nineteenth century the great mathematician Gauss* called mathematics the "Queen of the sciences." In the year 1935 a distinguished American scholar wrote: "Had Gauss known—that in respect of aim, method, and content, the enterprise of mathematics and the enterprise of natural science are separated by a chasm as deep and unbridgeable as that which sunders logical deduction from experimental observation, it is certain—that the great 'prince of mathematicians' would not deliberately have so spoken as to imply that one of two essentially disparate enterprises could be the 'Queen' of the other."

* Will someone supply a reference for this quotation? E.J.M.

If there is a bottomless gulf between mathematics and the natural sciences, how is it possible that it is precisely in the application of mathematics to these sciences that the human mind has been so successful? If it is true that mathematics rises above the clouds into the boundless sunshine of an infinite space, while physics is tied by its very nature to earth itself with its tides and its tremors, with winter and the magic of spring, then by what agency are they connected?

This is a modern example of an age-old problem, that of explaining mind and matter. Mathematics is mind-stuff; natural science is earth-stuff. But it is not the purpose of this analysis to become entangled in the confusing "isms" of philosophy, however interesting these may be. For, whatever may be the relationship between mind and matter, it is certainly true that a correspondence has been discovered between the logic of mathematics and the laws of nature as revealed by experimental science. This correspondence is the cornerstone of applied mathematics, and in particular of mathematical physics. It is to a clearer insight into those mental and physical processes which are the foundation of that phenomenal structure, theoretical physics, that this study is dedicated.

1. *Mathematics.* The position of mathematics in the scheme of knowledge has occupied the keenest minds. The Pythagorean Greeks characterized number as "great and perfect and omnipotent and the principle and guide of divine and human life." More recently it has been described as the principle by which the world "instead of being a chaos becomes a cosmos." These are eloquent words, but are they not questions rather than explanations?

The modern concept of mathematics is that of a pure, formal science completely identified with logic. It emphasizes that pure mathematics does not deal with the material world, and, hence, is not concerned with natural science. This view is in striking contrast with historically earlier ones which, by failing to distinguish clearly between mathematics and its applications, encountered insurmountable logical obstacles. The true field of mathematics is that of relations. Specifically, it is the field of relations characterized by precision, completeness, and sharpness. Ordinary ideas, most often interrelated in vague and indefinite ways, are outside its realm. Since mathematics thus limits itself to those precise relations which are independent of the uncertainties of a physical universe, its laws are fundamentally and invariably true. But, clearly, the correctness of mathematical relationship is due not to any especially charmed mode of thought, but rather to the nature of the ideas and concepts with which it operates.

The method of mathematics is that of logical deduction. It depends upon one of the great achievements of the human mind, namely, the discovery by Pythagoras and Thales of the art of proving theorems. This art is based on the fundamental principle that no proposition is considered established until it has been proved, i.e., logically deduced from other propositions previously estab-

lished. Evidently the application of this principle presupposes the specification of certain initial assumptions or premises. These may be selected arbitrarily to suit the convenience or the interest of the investigator. This method is illustrated in the development of Euclidean geometry which, although originally a practical art of land measurement, has become a model of abstract logical coherency. In this system all geometrical quantities can be defined in terms of a few indefinables such as "point" and "straight line," and all propositions can be deduced from a relatively small number of axioms about these indefinables. The axioms (for example, the well-known statement that only one straight line can be drawn through two points) used to be regarded as self-evident truths. They are now recognized to be merely assumptions, since it has been found possible to construct other no less consistent systems of geometry on the basis of quite different premises. And it is entirely outside the sphere of the mathematician to attempt to decide whether Euclidean or Riemannian geometry is true in any particular universe. The mathematician merely states that if anything whatsoever has the properties specified in the axioms, then of necessity it will also have the properties described in the theorems.

Mathematics is an excursion into the uncharted spaces of the logically thinkable. Give a mathematician a group of entities satisfying certain conditions specified in a set of hypotheses, and he will proceed to investigate what logical deductions can be made. To accomplish this end he will invent convenient symbols and a code of rules to govern their use. This he will do on the basis of the principle of logical continuity and to suit his own convenience. His only guide is his mathematical curiosity; his compass is his experience in dealing with highly abstract relationships. Thus, urged on by an adventurous mind, he blazes new trails in ever more intricate regions of abstract space. His range is unlimited, it is bound by no dimensions, tied to no facts. It is the hunting ground of pure mind stalking the logically thinkable. It is the domain of correspondence, of definite precision, of positive relationships. In it value answers to value, state corresponds to state, condition relates to condition, change responds to change.

Can anything new be created in a process of such complete logical sequence? Is it possible for more to be derived from conclusions than is first included in the premises? It is beside the point to state that no human process can ever create in the sense of divine creation. But if age-old stone is cut from a quarry and shaped into man-made building blocks; and if these are piled upon each other to build a cathedral, then man has created something new. What is new is not the structure of the stone, which may persist into eternity, but the form in which it appears. In much the same way mathematics creates new forms in the shape of deductions. But it does more than this. It reveals in such deductions many an implication never supposed to be contained in the hypotheses. In other words, the quest for mathematical truth is in a very large degree an attempt to discover what the assumed hypotheses really mean. And in the measure that the human mind reveals to itself what it never suspected to exist in its postulates, it has created something new—new, not in an absolute or uni-

versal sense, but in the restricted meaning of enlightened human understanding. Thus a complex number, a vector, or a determinant is a new form created by mathematics.

But let it not be supposed that mathematics is unlimited and all-powerful. Such is not the case. Mathematics is a very human endeavor, and as such it is restricted by the limitations of the human mind to think what is logical and to comprehend what is complex.

2. *Experimental Physics.* The aim of experimental physics in particular, and of all empirical science in general, is unlike that of mathematics. Experimental science is not at all interested in what is logically thinkable; its entire concern is with a material universe the structure of which it seeks to reveal in the form of observed fact.

As Thales' and Pythagoras' art of proving theorems is the basis of mathematical logic, so Galileo's example of experimental demonstration is the foundation of scientific exactitude. This demands that no statement about the structure of nature be accepted as true, until it has been exhaustively verified by direct observation. All observation begins with one of the five senses, and it depends (as was enunciated by Kant) upon an elementary intuition or awareness of natural phenomena. But even a well-trained eye can provide only a kind of qualitative knowledge which is piecemeal and inexact. The scientist has, therefore, been compelled to reinforce the simple senses with delicate instruments which he has been able to invent as his knowledge about natural phenomena increased. With their aid he has succeeded in recording quantitative knowledge about selected events which he isolates more or less completely in his laboratory. In the last analysis the science of measurement is an ingenious technique for comparing particular phenomena with previously established standards. It depends, therefore, upon the establishment, the preservation and the reproduction of a few fundamental standards, and upon experimental observations which may be duplicated at will. If it is found convenient, new quantities may be derived from the established standards and defined in terms of these and of the actual operations performed in measuring them.

All experimental measurements are essentially in the form of pointer readings on an arbitrary scale. They are definitely and inevitably limited in accuracy by the size of the smallest scale division, and by the fact that they are necessarily made in terms of small areas and not in terms of abstract points and lines. Thus, for example, if velocity is to be defined in terms of the ratio of a measured distance traversed in a measured time interval, it must always be an average and an approximate velocity. The mathematical concept of instantaneous velocity, or of velocity at a point, is excluded as an unmeasurable abstraction.

Every measurement involves a calibrated instrument, a phenomenon to be measured, and a contact in the form of an energy transfer between them. The correctness of every experimental observation must, therefore, depend upon the effect of this contact on the measurement. In ordinary large scale determinations this may be made entirely insignificant. In the atomic field, on the other hand,

even the smallest possible energy transfer, namely that due to a single quantum, may represent so large a contact as to make the measurement totally meaningless. This is found to be the case when an attempt is made to measure the position and the velocity of an electron in a microscopic sense. Furthermore, since the nature of the measurement depends upon an interaction between an observer and a given phenomenon, the results observed may vary with interactions due to different types of measuring contrivances. For example, clouds of electrons interacting with one measuring device are observed to give an effect interpreted as due to waves, whereas an interaction with a different kind of instrument produces a so-called particle effect. It is evident, then, that the experimental method is limited not only by technical problems of precision and sensitivity, but by the structure of nature itself. The observer, regardless of his mechanical or electrical disguise, is always a part of this structure and unavoidably plays a role in every measurement. Thus a so-called experimental fact must always be merely a more or less exact estimate of nature itself.

The method of mathematics was described as logical deduction from self-imposed hypotheses; the method of experimental science is careful comparison of specific observations with self-selected standards. But whereas the blade of abstract logic is infinitely sharp and thin, that of experimental comparison is only finitely and relatively so; at times it proves to be a hammer.

3. *Mathematical Physics*. Knowledge which is purely factual is sterile. Nothing new can be learned by repeatedly measuring the same fact no matter how accurately it is ascertained. But, if when a large number of similar facts has been carefully observed, the investigator is led to suspect the existence of an invariant relation between the phenomena and the circumstances under which they were measured, the seed for something new has been planted. Such is the case, for example, when a smooth curve is drawn through the plotted points corresponding to the empirically observed times and distances of a falling body. This is a simple illustration of the process of induction as practiced in the physical sciences, a process of generalization from a few discrete facts to infinitely many. The essence of physical induction is that in the presence of experimental observations showing that all examined x's have certain properties, one concludes that all existing x's have the same properties. The process might be described as the discovery of a frame into which the observed facts will fit. The intuitive awareness of a correspondence between a series of empirical observations and a definite, though perhaps not readily visualized relation, is the transition from matter to mind, from experimental to theoretical physics. But the field of positive relations is the field of mathematics, and the expression of such a relationship is the expression of a mathematical function. Thus, the fundamental basis of mathematical physics is the observed fact that the logically necessary relations which hold between mathematical expressions hold for natural phenomena themselves.

The representation of natural phenomena in any form which is to persist

in time depends on the existence of something in nature which is permanent. The old metaphysics conceived this ultimate invariant substance to be matter. Modern physics finds the element of permanence in the functional relationships of pure mathematics. Thus, the reason why logic and mathematics apply to nature is because they describe the invariant relations which are actually found in it. This discovery at once makes available the entire mathematical superstructure for the study not of experimental facts of observation, but of the relations which govern them, of those abstract properties commonly called the laws of nature. "When we consider natural objects purely as the embodiment of the invariant relations found in nature, we are said to idealize those objects, or to consider them as ideal limits. But such idealization gives us the essential conditions for what truly exists." In this light one may represent rigid bodies by points and consider their motion along mathematical lines. One may take advantage of the concepts of the infinitesimal calculus such as instantaneous velocity and acceleration at a point. In other words, relations between idealized physical objects may be expressed in the form of differential or integral equations. For example, the motion of the center of gravity of a rigid body is described by the differential equation due to Newton, $F(s, v, t) = d(mv)/dt$.

But let it be clearly noted that this entire mathematical representation is an abstraction which in itself has no real measureable physical significance. One must not forget that one is dealing with relations, not with measured facts. To demand a physical significance for all mathematical symbols occurring in relations which are derived from observed facts is to presume the experimentally verifiable existence of a framework in nature corresponding to every conceivable logically performed operation in formal mathematics. Such experimental verification must be forever lacking in any general sense, since experimental measurement can be performed and expressed in terms of only a limited number of physically real quantities. Thus, mathematical physics does not bridge the chasm between pure mathematics and experimental science. It operates entirely on the mathematical side. In form and in method it resembles mathematics, but its aim is different. The purpose of theoretical physics is not to think the logically thinkable, but by logical thinking to calculate relations which can be verified by direct experiment. Birkhoff writes that "the chief function of mathematical symbolism is to enable the mind to carry through certain processes of logical thought." In mathematical physics the process is immaterial, the logical character is presumed, and the interest centers on specific results which it is desired to obtain in as simple a form as possible. Thus the only condition imposed by the mathematical physicist upon his symbolism is that it ultimately must be useful to calculate definite relations corresponding to a set of empirical facts. In pure mathematics one begins with arbitrary hypotheses and concerns one's self with possible logical deductions. In mathematical physics the procedure is essentially the reverse. Given is a mathematical relation obtained inductively from a set of experimental facts; to find is a general hypothesis from which it may be deduced.

The principal and most successful method of mathematical physics is to guess at a solution, i.e., an hypothesis, and then to verify its correctness deductively. This may, perhaps, be done most elegantly and most significantly by assuming a variety of possible hypotheses of a very general nature, and by then proceeding mathematically to obtain a number of possible deductions. These may then be compared with the measured data of a series of related experiments to determine whether the observed facts fit into any of them exactly or approximately. But since the functional relations found in nature are usually highly complex, it is frequently impossible to establish mathematical forms which are at the same time sufficiently general to be exact and sufficiently simple to be soluble with available mathematics in a form useful for purposes of calculation. It is then convenient to derive relations involving only a few variables and parameters which have been selected with due regard for the nature of the problem and reasonable mathematical simplicity. From such relations it is often possible to compute approximate but nevertheless useful results which possess what the German calls *Uebersichtlichkeit*. Indeed such idealized solutions are the ones from which one may obtain the clearest insight into the individual processes of nature.

With increasingly powerful mathematical tools and a broader understanding for relations in the physical world, the mathematical physicist has become bolder and more successful. Instead of merely seeking relations to fit known facts, he proposes broad generalizations from which he predicts what experimental facts must be found if the superstructure he has invented is to be a correct one. There are innumerable examples of successful prediction from theory, indicating that the mathematical physicist is gaining a real understanding of the structure of nature. It is by this method of bold mathematical generalization that simple relations suggested by specialized experiments are transformed according to the logical rules of mathematics into symbolically more imposing but theoretically simpler and more universal forms. It is in such forms that unity in nature becomes apparent. For, from extremely general relations a great variety of special ones may be logically deduced and made to serve as frameworks for the observations of new experiments.

4. *Consistency in Nature.* The method of mathematical physics of calculating from differential equations the observable facts of the physical world depends upon four fundamental conditions. First, it presupposes the experimental availability of accurate quantitative facts about natural phenomena. Second, it depends upon the discovery in sets of such facts of relations between conveniently selected variables. Third, it assumes the invariance in space and time of the discovered relations. Fourth, it depends upon the invention of mathematical functions and equations which represent variables and relations discoverable in nature.

The statement that physical facts are connected according to invariant laws condenses these four basic assumptions into a single principle of consistency in

nature or of causality. This principle is nothing more than the assertion that the mathematical-logical sequence from hypothesis to deduction is valid in mathematical physics as, indeed, it must be, since it is the very foundation of the mathematical method. So long as physical laws are expressed in terms of mathematical relations, they must be presumed to obey this sequence. The law of cause and effect in physics is, therefore, in the nature of a comprehensive postulate or maxim. Obviously it cannot be proved by a method which presupposes its validity and which alone explains its meaning. It merely asserts and thus assumes that mathematical functional relationships can, in principle, be invented to fit permanently the observed facts in nature. In mathematics one states that every deduction is the logical consequence of definite, preëstablished hypotheses because deduction was so defined. In mathematical physics one asserts that every hypothesis (cause) has for its logical deductions (effects) the observed facts of nature, because the hypothesis was constructed so that this logical sequence would be found true. Its validity, moreover, must be invariant to transformations in space and time.

It is assumed, for example, that the differential equation which is used to describe the motion of a vibrating string at a given instant and place will be useful in the same way at a different time and a different place. If nature were found to be inconsistent in terms of a certain differential equation, then one would be compelled to assume that the mathematical representation, since it does not fit this physical world, has no general significance in it. A different mathematical formulation using other variables to represent the physical state in question would then have to be sought. In quantum mechanics, for example, it has been found impossible to calculate certain observed facts from relations using position and momentum coordinates to represent the physical state. On the other hand, the experimental data are correctly calculated from relations in which the variable is a complicated probability function. The aim of mathematical physics, to calculate observed facts, is thus achieved in the latter but not in the former case. To be sure, it may be of metaphysical interest to speculate on the kind of universe in which the former representation would lead to correct results, but this is outside the realm of physics. For the problem in physics is not to consider the properties of conceivable universes, but to find mathematical relations from which the experimentally observed facts of this universe may be calculated.

While mathematical relations are successfully used to represent nature, the rules of logic as expressed in the law of cause and effect must be found verified. Let it be emphasized that the aim of physics is to fit mathematics to relations found experimentally in nature, and not to explain the physical world in terms of a mathematical tradition.

The elementary foundation of mathematical physics has been analysed in three major parts. These are, first, the experimental technique which makes it possible to intercompare natural phenomena to a high degree of precision; second, the framework of permanent relations which is discovered in experi-

mental facts and which is expressible in terms of mathematical functions; third, the vast and highly articulate system of mathematical logic. Upon this foundation the human mind has constructed its most powerful and most dependable tool—the mathematical-scientific method. It is used wherever mathematics serves a practical purpose. It is the leaven of modern knowledge.

A GEOMETRIC REPRESENTATION OF A LINE INTEGRAL*

By W. H. ROEVER, Washington University

The linear differential expression

(1) $$dz = P(x, y)dx + Q(x, y)dy$$

defines at each point of space (x, y, z) a *surface element* (σ) (Fig. 1). For all points which lie on the same perpendicular to the x,y plane, such surface elements are parallel. A curve c of the x,y plane has at each of its points (x, y) a tangent line t. A vertical plane (i.e., one parallel to the axis Oz) which passes through the tangent t cuts the surface element σ in a *lineal element* l. It thus follows that on a vertical cylinder (i.e., one whose generators are parallel to Oz) which passes through the curve c, there are on each generator g an infinitude of the first order of lineal elements, all of which are parallel (Fig. 2). Consequently there are on

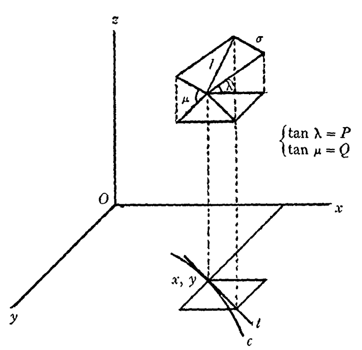

FIG. 1

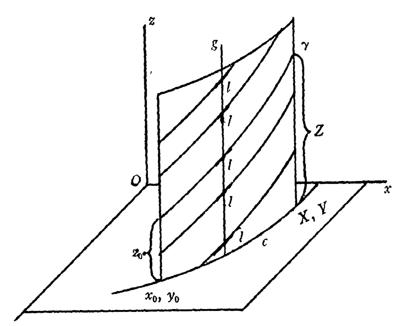

FIG. 2

the cylinder an infinitude of the second order of lineal elements, and these are the tangents to a one-parameter family (infinitude of the first order) of curves γ which lie upon the cylinder. Furthermore, these curves are so related that one may be obtained from another by a vertical displacement.

Let us consider now that portion of one of the curves γ which lies between the generators (of the cylinder) determined by the points (x_0, y_0) and (X, Y) of the curve c, and let us denote by z_0 and Z the ordinates of the points in which this particular curve γ cuts these generators respectively (Fig. 2).

It is then evident that *the line integral*

(2) $$\int_{(c)} Pdx + Qdy,$$

* Read before the American Mathematical Society, December 1, 1928.

taken along the curve c of the x,y plane from the point (x_0, y_0) *to the point* (X, Y), *has for value the difference of ordinates* $Z - z_0$.

The differential expression (1) defines in space an infinitude of the second order of surface elements σ. When the functions $P(x, y)$ and $Q(x, y)$ are general, there do not exist surfaces which have the surface elements σ for tangent planes. A condition, necessary and sufficient, that there should exist such surfaces is expressed by the relation

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

Under this condition the expression (1) defines a one-parameter family of surfaces S, any one of which may be obtained from another by a vertical displacement. In this case, one easily sees that the curves γ on the vertical cylinder, determined by the curve c, are the intersections of this cylinder with the surfaces S (Fig. 3).

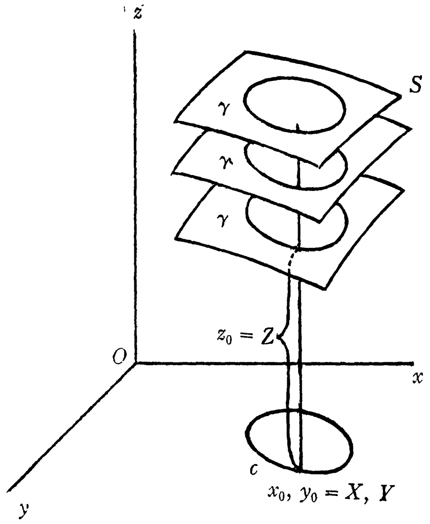

FIG. 3

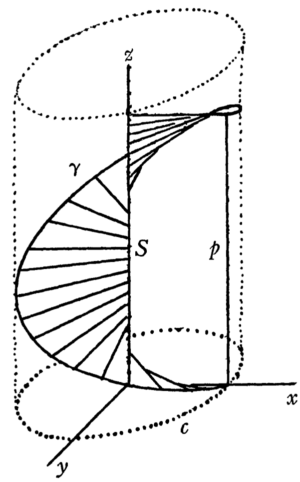

FIG. 4

If the surfaces S are continuous throughout the region of the x,y plane which is determined by a closed curve c, the curves γ are also closed, and hence in this case, for which the points (x_0, y_0, z_0) and (X, Y, Z) coincide, we have

$$\int_{(c)} Pdx + Qdy = 0.$$

It may, of course, happen that the surfaces S are not continuous throughout the region determined by a closed curve c, and then the curves γ, while still the intersections of the surfaces S with the cylindrical surface on c, may not be closed, and consequently the line integral around this closed curve c may not be zero. Such, for example, is the case when

$$P = \frac{-y}{x^2 + y^2}, \qquad Q = \frac{x}{x^2 + y^2}.$$

The surfaces S are then the right helicoids:*

$$z = \arc \tan \frac{y}{x} + \text{constant},$$

which are discontinuous at the point $x = y = 0$. For a closed curve c having this point in its interior, it is evident that

$$\int_{(c)} Pdx + Qdy = p,$$

where p is the pitch of the helicoids (Fig. 4).

THE METHOD OF CASCADES

By JULIUS SHAIN, Bronx, N.Y.

Michel Rolle is generally recognized as the inventor of what mathematicians call Rolle's Theorem. His outstanding contribution to mathematics is his Method of Cascades, a process for the general solution of numerical equations of the form $x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots a_n = 0$. Rolle gave its steps in his *Traité d'Algèbre* in 1690. In 1691 at the suggestion of mathematicians he published a proof of the method in his *Demonstration.*

The method of cascades has an important historical significance. Some basic principles of the calculus and the theory of equations can definitely be traced to their origin as incidental propositions of the method. It amplified the concepts of limits of roots of equations, provided the fundamentals from which Maclaurin derived his formula, began modern methods of series for determining roots, and discussed the relationship of imaginary roots in equations and their derivatives. Rolle's theorem, an important proposition of the calculus, also owes its origin to the method. The history of Rolle's theorem was investigated by several scholars. A complete account can be found in an article by Florian Cajori.† Cajori indicates some consequences of the *Demonstration* but concerns himself mainly with Rolle's theorem.

In order to appreciate the significance with which the Method of Cascades was endowed in the mind of Rolle, it is necessary to understand that he was working on the problem of finding the roots (which at the time meant, of course, only the real roots) of polynomials of any degree. To this end, in the early pages

* Here,

$$Pdx + Qdy = \frac{xdy - ydx}{x^2 + y^2} = d \arc \tan \frac{y}{x}.$$

† Cajori, Florian; Bibliotheca Mathematica, Leipzig, 1911.

of his Algebre he had developed a routine which, as cut-and-try methods go, was not a bad one for converging upon a root once two numbers were known between which it must lie. We will explain the details of this method below. If then a method could be developed for finding these two numbers, which Rolle called "hypotheses,"* his task would be complete. The method of cascades was this last step.

An Application of the Method

The simplest way to explain the method of cascades is to apply it to a particular equation. This we shall do exactly as Rolle would have done it.

Problem: To find the roots of the equation

$$x^4 + 2x^3 - 13x^2 - 14x + 24 = 0 \tag{1}$$

by the method of cascades.

Step 1, *Preparation of the equation.* Rolle first requires that the equation be transformed so that its terms alternate in sign. The purpose of this is to render all its roots positive. He is thus able to assert that 0, which he calls the "petite hypothese," is less than every root. He makes this transformation by means of the formula

$$x = \left(\frac{a}{c} + 1\right) - y \tag{1.1}$$

where a is the absolute value of the largest negative coefficient in equation (1) and c is the coefficient of the term of highest degree.

In our case we substitute $15 - y$ for x which gives us

$$y^4 - 62y^3 + 1427y^2 - 14446y + 54264 = 0. \tag{2}$$

The assertion that (1.1) leads to an equation having only positive roots is equivalent to the statement that no root of the original equation exceeded $a/c+1$. Rolle states this as true of polynomials in general, and calls the upper bound to which it leads the "grande hypothèse." Applying the theorem to (2), we learn that no root can exceed 14447.

Step 2, *Forming the Cascades.* After the equation has been prepared the following method is applied to it according to Rolle.

a. Multiply every term of the equation by the exponent of the unknown variable and divide by the unknown.

b. Do the same with the second equation formed from the first. Continue until an equation of the first degree is obtained.

Each of these equations is called by Rolle in his *Traité d'Algèbre* a cascade.

The above process is equivalent to taking successive derivatives of the equation. Thus a cascade of an equation is its derivative.

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* Rolle used this word in the sense of "any assumed value of the variable." It is only when qualified by an adjective as explained below, that it takes on the special significance of a "bound." For that reason, we leave it untranslated.

In our case the cascades are:

$$y^4 - 62y^3 + 1427y^2 - 14446y + 54264 = 0 \tag{3a}$$

$$4y^3 - 186y^2 + 2854y - 14446 = 0 \tag{3b}$$

$$12y^2 - 372y + 2854 = 0 \tag{3c}$$

$$24y - 372 = 0. \tag{3d}$$

Step 3. *Solving the equation.* Rolle states in his Algébre that the roots of an equation are separated by the roots of its cascade. He proves the statement in his Demonstration. Thus the roots of the cascade are "hypothèses moyennes" (intermediate bounds) which together with the "petite hypothèse" and the "grande hypothèse" referred to above define a set of intervals, each of which contains at most a single root.

If then the roots of (3b) can be found, the roots of (3a) can also be found by following Rolle's routine. But the roots of (3b) are likewise separated by those of its cascade (3c), and the latter by the roots of (3d) which is easily solved. Hence the roots of (3a) can certainly be found.

It remains to explain the routine which Rolle devised for carrying out the successive stages of this solution. To this end it will suffice to indicate the process of solving (3a) once the roots of (3b) are known to be approximately 13, 15 and 18; for the solutions of (3c) and (3b) are merely small editions of this process.

y	$3a$
0	54264
13	−24
6	6864
9	1200
11	144
12	0
15	24
14	0
18	−24
16	24
17	0

As data for our solution of (3a) we have the "petite hypothèse" 0, the "hypothèses moyennes" 13, 15 and 18, and the "grande hypothèse" 14447. As shown in the accompanying table, we begin by computing the values of (3a) for 0 and 13. As the signs of the results are opposite, we next try the mean of 0 and 13, *rounded to a single digit.** The results for 6 and 13 are opposite in sign; hence we next try $(6+13)/2=9$. Eventually we find that 12 is a root.

Then we use the pairs of hypotheses 13 and 15, 15 and 18, 18 and 14447

* Rolle's rules regarding the retention of digits are quite explicit.

in the same way. Two of these are shown in the table. The last would require the use of the trial hypotheses 14447, 7000, 3000, 1000, 500, 200, 100, 50, 30, 20, 19, of which the last is the desired root.

The roots of (3a) being 12, 14, 17 and 19, those of (1) are readily seen to be 3, 1, -2, -4.

Historical Significance

The "Grande Hypothèse." As we have said above, Rolle stated that $h=a/c+1$ is an upper bound of the roots of an equation.

That "the greatest negative coefficient of any equation increased by unit always exceeds the greatest root of the equation" was proven by Maclaurin in his *Algebra.**

Lagrange in *Traité de la Resolution des Equations*, pp. 11, 12, and 13, credits Maclaurin with a method for proving that $(k+1)$ is an upper limit if k is the greatest negative coefficient.

Cajori† writes, "He (Rolle) then puts $y=h-x$ and obtains an equation whose signs alternate. That his process always actually yields the desired result is not proved by Rolle either in his *Algebra* or in his *Demonstration.*

However there is in *Traité d'Algèbre* a paragraph which describes a method for proving that the transformation will yield an equation with alternate signs. The wording of this paragraph may be transformed into mathematical symbols in the following way.

Rolle on p. 123 of *Traité d'Algèbre* says, "Pour faire la quatriéme preparation on peut suppose l'inconnue égale à la difference de deux indeterminées, & se servir facilement de l'indetermination pour donner aux termes du résultat la disposition requise, parce que prenant l'une des deux indéterminées pour l'origine du résultat, chaque terme qui doit devenir negatif aura la plus haute puissance de l'autre indéterminée avec le signe $-$, & chaque terme qui doit devenir positif aura la plus haute puissance de l'inconnue avec le signe $+$."

Following Rolle's description, if we start with the equation

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots a_{n-1}x + a_n = 0 \tag{4a}$$

suppose $x=e-y$, we get

$$\left.\begin{array}{rrr} +(-1)^n y^n \cdots + \dfrac{n(n-1)}{2} e^{n-2}y^2 & - ne^{n-1}y & + e^n \\ +(-1)^{n-1}a_1y^{n-1} \cdots & -(n-1)a_1e^{n-2}y & + a_1e^{n-1} \\ + a_2(-1)^{n-2}y^{n-2} \cdots & -(n-2)a_2e^{n-3}y & + a_2e^{n-2} \\ \vdots & \vdots & \vdots \\ & - a_{n-1}y & + a_{n-1}e \\ & & + a_n \end{array}\right\} = 0. \tag{4b}$$

* Maclaurin; *Algebra*, p. 172, 1748.

† Cajori; Bibliotheca Mathematica, Leipzig, p. 302, 1911.

If $e=k+1=h$, where $-k$ is the greatest negative coefficient of equation (4a), the signs of the terms of (4b) will be the same as the signs of the terms with the highest power of e. For example in the last term of (4b) which is

$$e^n + a_1e^{n-1} + a_2e^{n-2} \cdots + a_{n-1}e + a_n \tag{4c}$$

e^n is greater than the sum of all negative terms of (4c).* And since the sign of e^n is + the term (4c) must be positive.

In the same way in the preceding term of (4b) which is

$$[-ne^{n-1} - (n-1)a_1e^{n-2} - (n-2)a_2e^{n-3} \cdots - a_{n-1}]y \tag{4d}$$

the absolute value of $-ne^{n-1}$ is greater than the sum of all positive terms of (4d), where $e=k+1$ and in this case a_r changes signs and k becomes the greatest positive coefficient of equation (4d). Therefore (4d) is negative since $-ne^{n-1}$ is negative. And so on for all other terms of (4b).

Therefore (4b) may be written

$$y^n - A_1y^{n-1} + A_2y^{n-2} \cdots + (-1)^nA_n = 0 \tag{4'b}$$

where the A's are positive.

We have, by means of Rolle's description of the proof, shown that the transformation always yields the desired equation with alternating signs.

The Method of Series and Maclaurin Formula. The foundation of the method of cascades is a theorem which later writers referred to as Rolle's theorem. In 1729 Colin Maclaurin† published a new proof of Rolle's theorem. He also stated that, "In general, the Roots of the Equation

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} \text{ \&c.} = 0, \tag{5a}$$

are the Limits of the Roots of the Equation

$$nx^{n-1} - \overline{n-1} \times Ax^{n-2} + \overline{n-2} \times Bx^{n-3} \text{ \&c.} = 0, \tag{5b}$$

or of any Equation that is deduced from it by multiplying its Terms by any Arithmetical Progression $l \mp d$, $l \mp 2d$, $l \mp 3d$, &c. and conversely the Roots of this new Equation will be the Limits of the Roots of the proposed Equation" (5a).

From Rolle's theorem, the theorem of the mean and also Maclaurin's and Taylor's series follow immediately.‡

Rolle believed that subsequent methods for evolving roots would owe their origin in some way to cascades. In the first paragraph of the *Demonstration*§ he

* The proof that e^n is greater than the absolute value of the sum of all other negative terms of (4b) was no doubt familiar to Rolle as it can easily be derived by a simple substitution of $k+1$ for e in (4b).

† Maclaurin; Philosophical Transactions of the Royal Society of London, p. 88, 1729.

‡ E. B. Wilson; *Advanced Calculus*, p. 55.

§ Rolle; *Demonstration*, p. 1, 1691.

wrote, "Pour répondre aux principales Objections que l'on a faites contre la Methode des Cascades Algebriques, il faut démontrer que cette Méthode est infaillible dans tous ces cas; & même vous faire sentir qu'on ne peut pas en former une autre pour le même subjet à moins que d'y employer directement, ou indirectement ces Cascades."

Maclaurin's article in Philosophical Transactions is a definite link between cascades and modern methods of series. Here Maclaurin proved the fundamental proposition of the method by the use of transformation (4b). The coefficients of y in this transformation that Maclaurin calls the XIth lemma are a series of cascades. He wrote,* "From this XIth Lemma some important Theorems in the Method of Series · · · are demonstrated · · · "

Imaginary and Double Roots. In the Demonstration, p. 38, Rolle states and proves that, "Il y aura pour le moins autant de racines defaillantes de la seconde espèce dans chaque égalité, qu'il y en a dans sa Cascade immédiate." Maclaurin† shows that the equation (5a) will have as many imaginary roots as the equation (5b).

Rolle states,‡ "Sil arrive que les hypotheses donnent θ [that is 0] au lieu de donner + ou −, alors chacune de ces hypotheses sera une des racines de la cascade où se fait la substitution, & il faudra compter dans cette cascade une racine défaillante de la premiere espece, pour chacune de ces hypotheses qui produiront cet effet." Maclaurin§ shows that when two values of x are equal in the equation (5a), one of them must be a root of the equation (5b).

This year marks the three-hundredth anniversary of the publication of Descartes' "La Géométrie" which appeared as an appendix to a great treatise on method in science. "It was in this appendix, a small handbook of only about a hundred pages, that analytic geometry first appeared in print. The fundamental idea in Descartes' mind was not the revolutionizing of geometry so much as it was the elucidating of algebra by means of geometric intuition and concepts; in a word, the graphic treatment of the equation. His imagination extended far beyond this, however, to the establishing of a universal mathematics in which algebra, geometry, and arithmetic should be closely related members. He began by extending the ancient idea of latitude and longitude, showing that any point in a plane is uniquely determined by two coordinates, x and y, the equation $F(x, y)=0$ expressing a property which is true for every point of the curve. By studying the equation, therefore, he could, through the principle of a one-to-one correspondence, transfer his results at any time to the curve itself. It constituted what John Stuart Mill says was 'the greatest single step ever made in the progress of the exact sciences.'" (D. E. Smith, "History of Mathematics," vol. I, p. 375.)

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* Maclaurin; Philosophical Transactions of the Royal Society, p. 94, 1729.

† Maclaurin; Philosophical Transactions of the Royal Society, p. 88, 1729.

‡ Rolle; *Traité d'Algèbre*, p. 129, 1690.

§ Maclaurin; Philosophical Transactions of the Royal Society of London, p. 92, 1729.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions, and Notes in the Monthly is open to all forms of activity in collegiate mathematics including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

SIMPLIFICATION OF EQUATIONS OF CONICS

By L. S. JOHNSTON, University of Detroit

Mr. H. B. Thornton has pointed out in a recent note* that the process given in most texts on analytic geometry for simplifying the equation of a conic by means of substitutions from trigonometric formulas is usually long and laborious, and he exhibits a simpler method of accomplishing such simplifications. I offer a method which uses those same trigonometric formulas, but uses them in such manner as materially to shorten the actual work of simplification, accomplishing, I believe, the desired results with much less difficulty than in even Mr. Thornton's method. This method is not to my knowledge mentioned in any text, but is mentioned rather sketchily in my note "Additional Marginal Notes," this MONTHLY, October 1932.

Consider the general equation of the conic

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \tag{1}$$

Under the usual transformation rotating axes this becomes

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0 \tag{2}$$

the coefficients being well known functions of the original coefficients and of the angle θ defined by the equation

$$\tan 2\theta = \frac{B}{A - C}, \quad \text{with } 2\theta \text{ less than } \pi. \tag{3}$$

Most texts in analytics either prove or set as exercises the relations

$$A' + C' = A + C \tag{4}$$

$$A' - C' = \pm\sqrt{B^2 + (A - C)^2}, \tag{5}$$

but make little or no mention of their consequences. Now it is evident that if the proper sign to use before the radical in (5) can be definitely determined, then from (4) and (5) we can at once determine A' and C' without ambiguity, and without the explicit use of θ at all.

It is easily shown that if $\tan 2\alpha = m/n$, then

$$\tan\alpha = \frac{m}{n \pm \sqrt{m^2 + n^2}}.$$

(Incidentally, this formula, or the equivalent identity

* Simplification of the equations of conics, this MONTHLY, January 1934, p. 36.

$$\tan\alpha = \frac{\tan 2\alpha}{1+\sec 2\alpha} = \frac{\tan 2\alpha}{1 \pm \sqrt{1+\tan^2 2\alpha}},$$

is not to my knowledge mentioned in any text in trigonometry, though it is a much simpler method than the one usually used for finding $\tan\alpha$ when $\tan 2\alpha$ is given). Using this formula, we have

$$\tan\theta = \frac{B}{A - C \pm \sqrt{B^2 + (A-C)^2}}.$$

Now since this fraction must always be positive, we must choose the sign before the radical to be the same as that of B. Furthermore, the calculations of $\sin\theta$ and $\cos\theta$ which enter into the constants D', E', and F' of (2), are much simpler by this method than by the usual formulas $\sin\theta = \sqrt{(1-\cos 2\theta)/2}$, and $\cos\theta = \sqrt{(1+\cos 2\theta)/2}$.

For the central conics it is usually the practice to translate axes first, removing the linear terms from (1). Since this does not affect in any way the coefficients of the second degree terms, it is immaterial whether the angle of rotation be determined before or after the translation, for either the central conics or the parabola. Hence all the geometric properties of any conic except the position of the center (for the central conics) or the vertex (for the parabola) are very rapidly and conveniently calculated from A, B, and C without recourse to the angle θ.

The increasing tendency to omit simplification of conics by rotation from elementary courses is no doubt traceable in large part to the difficulties inherent in the computations involved. The method shown here seems to eliminate most of the difficulties, and I have found the method so satisfactory in my own classes that several years ago I abandoned the usual methods entirely, changing what had formerly been to most students a very distasteful section of the course to one which has been one of the most interesting.

A Note on Taylor's Theorem

By R. E. Moritz, University of Washington

Taylor's Theorem without the remainder was first published in 1715, but it was not until almost a century later that Lagrange and Cauchy succeeded in deriving the approximations of the remainder which bear their names. Both Lagrange and Cauchy used methods based on successive integrations. The remainder appeared first in the form of a definite integral, from which the derivative forms of the remainder were obtained by an application of the first law of the mean for integrals. The proofs of Taylor's Theorem which have found their way into textbooks on the calculus are variations or modifications of a proof first given by Homersham Cox in 1851[1]. This proof is more elementary than the proofs by Lagrange and Cauchy and is now generally considered the

1. Cambridge and Dublin Mathematical Journal, vol. 6, 1851, p. 80.

simplest of all existing proofs of the theorem. Yet notwithstanding Pringsheim's dictum[2] who refers to Cox's proof as "der an Einfachheit und Strenge wohl kaum etwas zu wünschen lassende Beweis," every teacher knows that the proof has the objection of high artificiality. Todhunter remarks that the demonstration will probably disappoint the reader, who, though unable to point out a flaw in the reasoning, will complain of the artificial and tentative character of the whole. This objection, however, may be raised against all the other proofs. Todhunter[3] goes on to say "Without denying the justice of these objections, we may reply that the highly general character of the theorem may be some excuse for the complexity and indirect nature of the investigation." The objection, which Todhunter refers to, is of course the arbitrary assumption of the auxiliary function employed in the proof.

The following proof avoids the assumption of this arbitrary function and, I believe, introduces no compensating difficulties.

Let $f(x)$ represent any function which, together with its first n derivatives, is continuous in the closed interval (a, b). Let us integrate $f^{(n)}(x)$ successively with respect to x, and determine the integration constant in each case so that the integral vanishes when $x=a$. Denoting the successive integrals by $F^{n-1}(x)$, $F^{n-2}(x)$, $\cdots$ we obtain

$$(1)\quad \begin{cases} F^{(n-1)}(x) = f^{(n-1)}(x) - f^{(n-1)}(a), \\ F^{(n-2)}(x) = f^{(n-2)}(x) - f^{(n-2)}(a) - (x-a)f^{(n-1)}(a), \\ F^{(n-3)}(x) = f^{(n-3)}(x) - f^{(n-3)}(a) - (x-a)f^{(n-2)}(a) - \dfrac{(x-a)^2}{2!}f^{(n-1)}(a), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F'(x) = f'(x) - f'(a) - (x-a)f''(a) - \cdots - \dfrac{(x-a)^{n-2}}{(n-2)!}f^{(n-1)}(a), \\ F(x) = f(x) - f(a) - (x-a)f'(a) - \dfrac{(x-a)^2}{2!}f''(a) - \cdots \\ \qquad\qquad - \dfrac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a). \end{cases}$$

Now consider the function

$$(2)\qquad \phi(x) = F(x) - \frac{(x-a)^n}{n!}R,$$

where R is defined by

2. Bibliotheca Mathematica, vol. 1, 1900, p. 455.
3. Differential Calculus, 1890, p. 75.

$$\phi(b) = F(b) - \frac{(b-a)^n}{n!} R = 0. \tag{3}$$

Since $F^{(i)}(a)$ vanishes for $i=0, 1, 2, \cdots, n-1$, so does $\phi^{(i)}(a)$, and we have (Rolle's theorem),

$$\begin{array}{llll}
\phi(a) = 0, & \phi(b) = 0, & \text{therefore} & \phi'(x_0) = 0, \ a < x_0 < b; \\
\phi'(a) = 0, & \phi'(x_0) = 0, & \text{therefore} & \phi''(x_1) = 0, \ a < x_1 < x_0 < b; \\
\phi''(a) = 0, & \phi''(x_1) = 0, & \text{therefore} & \phi'''(x_2) = 0, \ a < x_2 < x_1 < b; \\
\cdots & \cdots & \cdots & \cdots \\
\phi^{(n-1)}(a) = 0, & \phi^{(n-1)}(x_{n-1}) = 0, & \text{therefore} & \phi^{(n)}(x_n) = 0, \ a < x_n < x_{n-1} < b.
\end{array}$$

But from (2) and (1),

$$\phi^{(n)}(x) = F^{(n)}(x) - R = f^{(n)}(x) - R,$$

hence

$$R = f^{(n)}(x_n), \qquad a < x_n < b.$$

This value of R substituted in (3) gives Taylor's theorem in the form

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!} f''(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(b-a)^n}{n!} f^{(n)}(x_n).$$

REPEATING DESIGNS IN SURFACES OF NEGATIVE CURVATURE

By C. A. RICHMOND, Brooklyn, N.Y.

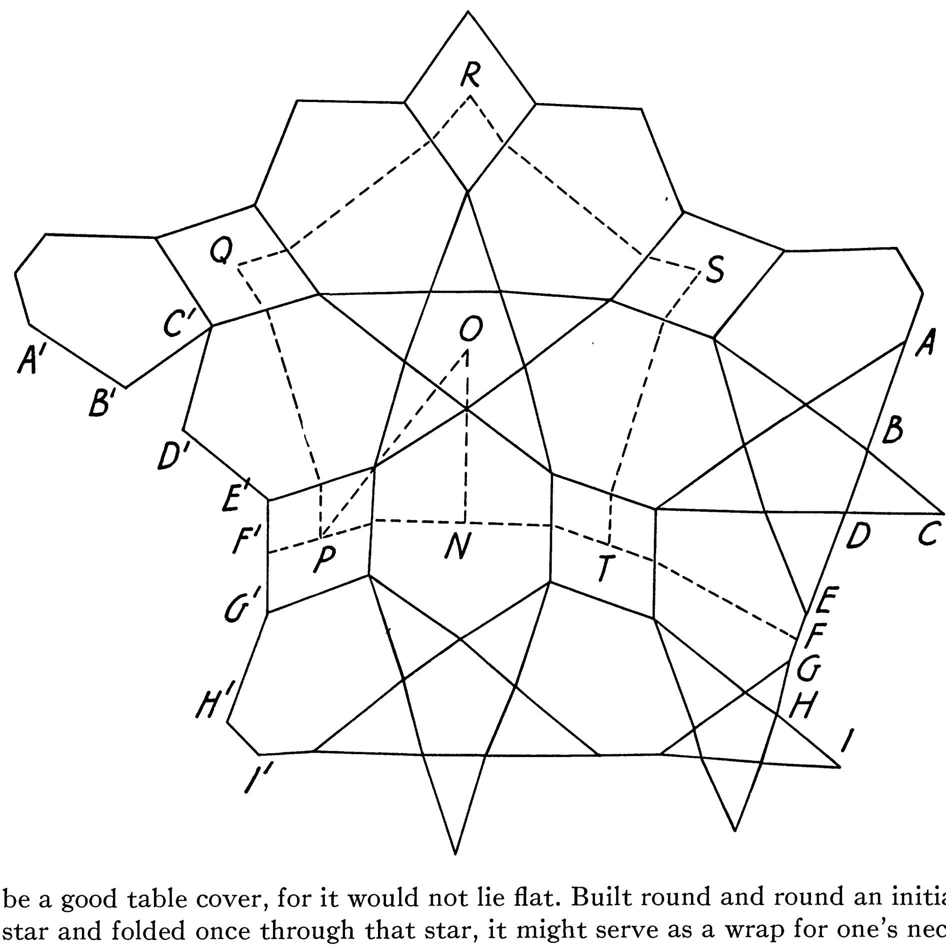
If equilateral triangles of cardboard are hinged edge to edge, seven around each vertex, one has a model of a flexible surface which will not lie flat. At each vertex the angle excess over 360° is 60°. If such a model comprises many triangles assembled round and round an initial triangle, it is excessively crumpled. To obviate such crumpling, an angle sum at each vertex only slightly more than 360° would seem desirable. Regular heptagons can be assembled, three at a vertex, only by flexing many or all of them; in this case the angle excess at each vertex is between 25° and 26°.

Consider the model constructed as follows.* Let a regular pentagon be hinged

* NOTE BY EDITOR. This paper presents some very interesting problems in the construction of certain types of surfaces. The interest is perhaps increased rather than lessened by the fact that the writer does not specify in conventional mathematical terms the distortion of geometrical figures involved in some of the constructions. If one actually attempts to make a cardboard model he will discover that some of the triangles, squares, pentagons, and hexagons, which are plane figures, must be distorted into non-planar figures in making the proposed model. The surfaces deserve further study. E.J.M.

at its edges to the respectively equal bases of five isosceles triangles, in which each leg is 3/2 times the base; the assembly is a kind of five-rayed star, with flexibly attached rays. Fit regular hexagons alternately between the rays, and fit squares at the star tips between the hexagons. The design repeats. The excess at each vertex is about 9° and averages exactly that. A fragment of this design has been distorted to reduce it to the accompanying plane diagram, in which a repeating unit is indicated at *PQRST*.

With suitably chosen colors for its stars, squares and hexagons, perhaps this design, done in cloth, would make a pleasing piece of patchwork. This would not



be a good table cover, for it would not lie flat. Built round and round an initial star and folded once through that star, it might serve as a wrap for one's neck and shoulders. It would approximate to a surface of negative curvature, somewhat as the shell of a regular icosahedron approximates to a surface of positive curvature, a sphere surface. The possessor of such a wrap might properly call it his Lobachefskian shawl. Whether or not it would serve as a garment for its owner, there is no doubt that it would wrap neatly around a pseudosphere of

the right size.* If one wanted a poncho or skirt, he might match the edges A-I and A'-I' together and get a surface model somewhat like an hyperboloid of one sheet. This would be folded double along the circuit of the line $F'PTF$, which would go around the wearer's neck or waist. Even with an excess of only 9°, the shawl or poncho would be too much crumpled, if the stars, squares and hexagons were made as small as a superior patchwork artist would wish. A design with a vertex excess of, say, about 3° would seem desirable.

The foregoing discussion introduces the following statement of a problem. Instead of attempting to state it generally, I make it rather specific, and follow with a few informal suggestions for attaining greater generality.

Show how to assemble regular polygons of 2 (or 3, or n) different kinds, all sides equal, to form a simply connected surface, so that the design shall repeat from place to place over the surface, and the sum of the angles at each vertex shall exceed 360°, but the average excess shall be the least possible.

The requirement that the polygons shall be regular may be relaxed; for example rhombs may be admitted as well as squares. Going further, as in our shawl design, the requirement for equality of the sides or edges may be waived; if the triangles of that design had been equilateral, there would have been a defect from 360° at some vertices and an excess at others; but with the isosceles triangles as they were chosen, the excess is nearly the same at all vertices. Another variation would be to require a small *defect* from 360°, instead of an excess. The subject matter of A. D. Bradley's *Geometry of Repeating Design* (Teachers College, Columbia Univ., New York, 1933) is restricted to surfaces of zero curvature. Our problem leads to the geometry of repeating design in surfaces of appreciable finite curvature, which, though less useful, should be no less rich in varied possibilities.

This year, 1937, marks the two hundred fiftieth anniversary of the death of Nicolaus Mercator, a writer on cosmography, trigonometry, astronomy, and methods of computing logarithms. In his "Logarithmotechnia," published in 1668, he gives the series which bears his name,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

* To get an approximate value for the curvature of this pseudosphere, we may look upon the lines of the diagram as geodesics of its surface. In the triangle PON, right angled at N,

$$\cos P = \sin O \cdot \cos(ON\sqrt{K})$$

where K is the Gaussian curvature. This is a familiar formula of spherical trigonometry, when K is positive, in which case $K=1/R^2$, R being the sphere radius. But in our case, K is negative; therefore the formula becomes

$$\cos P = \sin O \cdot \cosh(ON\sqrt{-K}).$$

It is easy to see that angles P and O are respectively 45° and 36°: hence $\cosh(ON\sqrt{-K}) = \cos 45°/\sin 36° = 1.203$, whence $K = 0.393/ON^2$. K is the negative squared reciprocal of the tangent intercept which is so conspicuous in the generation of the tractrix, whose rotation about its axis generates the pseudosphere.

A NEW METHOD FOR THE SOLUTION OF CUBIC EQUATIONS*

By H. A. NOGRADY, Detroit, Mich.

A usual method for solving a cubic, $x^3+bx^2+cx+d=0$, with real or complex coefficients b, c, d reduces the cubic first to the form $y^3+py+q=0$, making the solution depend on the two coefficients p and q. The method to be explained in this article further reduces the equation to the form

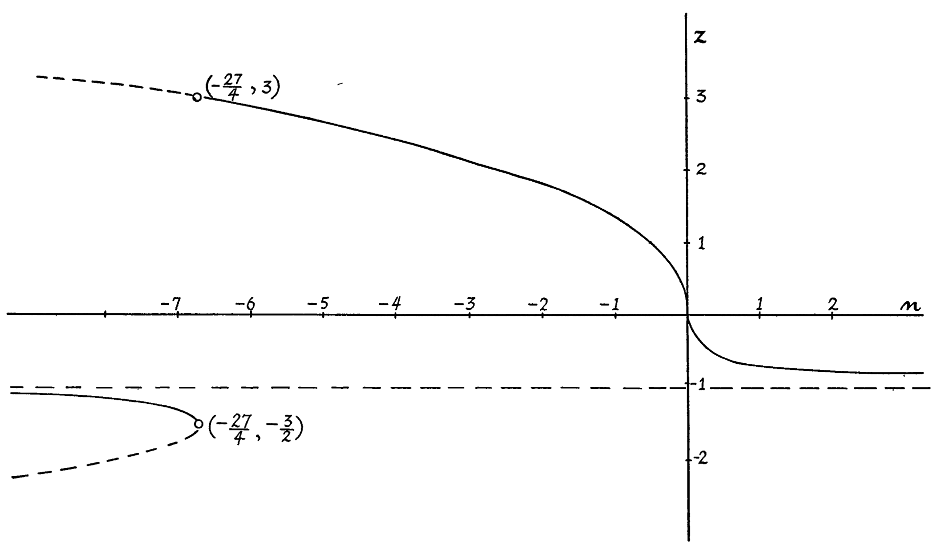
$$z^3 + nz + n = 0 \tag{1}$$

and makes the solution depend on the value of only one coefficient, n. After making this reduction it will be shown how to get all three values of z for each n and obtain in turn the three values of y, and finally the three values of x.

Starting with the equation in the form $y^3+py+q=0$, $pq\neq 0$, put $y=qz/p$ obtaining

$$z^3 + \frac{p^3}{q^2}z + \frac{p^3}{q^2} = 0,$$

which is in form (1) with $n=p^3/q^2$.



From equation (1) we have $n=-z^3/(z+1)$, and n is therefore a single-valued function of z. The relation between n and z is shown graphically in the accompanying figure. There is, for any real value of n, one and only one real value of z

* The author acknowledges his indebtedness to Dr. H. E. Spencer for his kind assistance in putting this theory in form for publication.

lying in the interval between $-3/2$ and 3, as shown by the solid part of the graph. We may designate this value of z by z_1.

The other two roots z_2 and z_3 are obtained by solving the quadratic equation $(z^3+nz+n)/(z-z_1)=0$, or $z^2+z_1z+z_1^2/(z_1+1)=0$. This gives

$$(2)\qquad z_2 = \frac{z_1}{2}\left(-1+\sqrt{\frac{z_1-3}{z_1+1}}\right) \quad\text{and}\quad z_3 = \frac{z_1}{2}\left(-1-\sqrt{\frac{z_1-3}{z_1+1}}\right).$$

It is obvious that it suffices to record in a table a single value of $z=z_1$ for each n, and then use formulas (2) for finding the values of z_2 and z_3.

The author has constructed the table* by assuming values of z_1, $z_1\neq -1$, $-3/2\leqq z_1<3$, at intervals of .001 and computing the corresponding n correct to six decimal places.

The method just described for solving a cubic with a real n consists then of the following steps:

(a) reduce the equation to the form (1);

(b) from the table obtain z_1 corresponding to the value of n at hand;

(c) obtain z_2 and z_3 from the formulas (2);

(d) obtain the three values of y from z_1, z_2, z_3.

An illustration of the method will be furnished by the solution of the equation

$$x^3+x^2-2x+7=0.$$

(a) Put $x=y-1/3$ and obtain $y^3+py+q=0$, in which $p=-7/3$, $q=209/27$. Then put $y=qz/p$, obtaining an equation of the form (1) with $n=p^3/q^2=-.212014$.

(b) From the table corresponding to $n=-.212014$, get $z_1=.714$ correct to three decimal places.

(c) Using formulas (2)

$$z_2 = \frac{.714}{2}\left(-1+\sqrt{\frac{-2\cdot 286}{1\cdot 714}}\right) = .357(-1+\sqrt{1\cdot 3337222}i)$$
$$= -.357+.412i$$

and $z_3=-.357-.412i$.

(d) From part (a),

$$y_1 = \frac{q}{p}z_1 = -\frac{209}{63}z_1 = -2\cdot 369$$

and

$$y_2 = 1\cdot 184 - 1\cdot 368i \quad\text{and}\quad y_3 = 1\cdot 184 + 1\cdot 368i.$$

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* A copy of the table with instructions for its use and with the complete theory may be obtained from the author, 445 E. Euclid Ave., Detroit, Mich.

Finally, since $x=y-1/3$,

$$x_1 = -2\cdot 702, \quad x_2 = .851 - 1\cdot 368i, \quad x_3 = .851 + 1\cdot 368i.$$

For the illustration just given, z_1, as obtained from the table, was correct to three decimal places. It is possible to get a more accurate value for z_1, without more accurate tables, as follows. Let z_1 be the value obtained from the table for a particular n, and let z_1+d be the correct value. Then

$$(z_1 + d)^3 + (z_1 + d)n + n = 0,$$

or

$$z_1^3 + z_1 n + n + d(3z_1^2 + n) + d^2(3z_1) + d^3 = 0.$$

By neglecting the terms in d^2 and d^3, we obtain as an approximation to d,

$$d = -\frac{z_1^3 + z_1 n + n}{3z_1^2 + n}, \qquad \text{or} \qquad z_1 + d = \frac{2z_1^3 - n}{3z_1^2 + n}.$$

In the above illustration, with $n=-.212014$ and $z_1=.714$, we get

$$z_1 + d = \frac{2(.714)^3 + .212014}{3(.714)^2 - .212014} = .713543.$$

If still further accuracy is desired, it can be obtained by repeating the above process.

If n is *not* real, then Cardan's solution of the equation $y^3+py+q=0$ leads us to $y=\sqrt[3]{r+si}+\sqrt[3]{t+ui}$.

Now take the following steps:

From the table obtain Z_1 and z_1 of the equations

$$Z^3 - \frac{27(r^2+s^2)}{4r^2}Z - \frac{27(r^2+s^2)}{4r^2} = 0$$

and

$$z^3 - \frac{27(t^2+u^2)}{4t^2}z - \frac{27(t^2+u^2)}{4t^2} = 0,$$

which are in the form (1) with $N=-27(r^2+s^2)/(4r^2)$ and $n=-27(t^2+u^2)/4t^2$. Obtain Z_2, Z_3 and z_2, z_3 from the formulas (2). Substitute the values of Z and z in $V^3=r(Z+1)/4$ and $v^3=t(z+1)/4$; then the values of V and v in $W^2=(V^3-r)/(3V)$ and $w^2=(v^3-t)/(3v)$. After properly choosing the signs of the imaginary terms, we find the following results:

$$y_1 = (V_1 + v_1) + (W_1 + w_1)i;$$
$$y_2 = (V_2 + v_2) + (W_2 + w_2)i;$$
$$y_3 = (V_3 + v_3) + (W_3 + w_3)i.$$

RECENT PUBLICATIONS

EDITED BY W. R. LONGLEY, Yale University

All books for review should be sent directly to the editor of this department, at the American Mathematical Monthly, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Descriptive Geometry. By Frank W. Bubb. New York, The Macmillan Co., 1935, xiv+234 pages.

Descriptive Geometry Problem Book. By Frank W. Bubb. New York, The Macmillan Co., 1936, 12 pages+91 problem plates.

The text which Professor Bubb offers to students and teachers of descriptive geometry is in the true sense of the word an original work and is further distinguished by the fact that it can be studied by the intelligent student working on his own. And this quality of the book should, it seems, make it all the more serviceable to the teacher.

Departing from the traditional method of treatment, which made of descriptive geometry a hybrid of mechanical drawing with a vague theoretic overlay, the author has given his material a logical structure built on the inner unity of an autonomous subject. To achieve this Professor Bubb has formulated a set of seven *fundamental space operations* which suffice for the solution of any problem considered. They are:

1. Assuming a point upon a line in space;
2. Passing a straight line through two given points;
3. Passing a plane through two given intersecting or parallel lines;
4. Determining the point of intersection of a line and a plane;
5. Passing a line through a given point, perpendicular to a given plane;
6. Passing a plane through a given point, perpendicular to a given line;
7. Performing a given plane construction upon a given plane in space (that is, upon a plane oblique to the plane of the drawing board).

These operations, used in the solution of any problem, are so fundamental that they may be looked upon as, in a sense, the set of postulates for descriptive geometry. Then the subject matter flows naturally and smoothly from a consistent minimum set of accepted principles, rather than as a patch work of drawing and geometry.

The author's presentation is very clear and simple. A thumbing of the pages will show an adequacy of illustrative drawings, but an absence of those elaborately confusing plates which on a first view give a text a forbidding appearance. The notation is set down at the outset and is adhered to throughout. The fundamental space operations are developed in complete detail, and so illustrated by both orthographic and perspective views that their validity can be inferred either from word or picture alone. Auxiliary views are introduced prior to fundamental space operation seven which requires one for its rendition. Certain fundamental drawing operations are presented as construction theorems, called

drawing board constructions, and are presented as needed. Fundamental space operations are alternatively applied with the use of auxiliary views where convenient. This basic material is taken up in the first half of the book and applied to the solution of special problems and to the classification of surfaces. A short chapter on mining problems is included, and the book closes with a complete but concise summary.

The importance of the author's method of presentation is that it prepares the student for the analysis of a problem in the application of the working tools developed in the first three chapters of the text, in particular, the application of the fundamental space operations. Thus if the student is so well schooled in them that he can describe them in words alone, he should be able to express in words alone the solution of any descriptive geometry problem. The drawing out of the problem becomes in reality then, mechanical drawing.

There are so many well chosen exercises that most teachers will not need to seek further for assignment material. As a valuable teaching aid the author has prepared a book of three hundred problems, presented in an order following the development of his text. The problems are printed on 8 by 11 sheets, perforated so that they can be easily removed. The given of each exercise is drawn out, making for a uniformity that minimizes the effort in grading. In order that the problem book may be used where the author's text is not in use an adequate summary of the fundamental material is included.

The reviewer can earnestly recommend Professor Bubb's original text and problem book to the serious attention of teachers and students of descriptive geometry.

S. B. LITTAUER

Descriptive Mathematics. By John Maclean. Bombay, Macmillan and Co., 1935. xvi+143 pages.

The author published in 1926 a slightly larger volume entitled *Graphs and Statistics*, in an attempt to broaden the usual mathematical curriculum of first year college students at the University of Bombay. This first book contained a wealth of problems from recent scientific sources, chiefly in medicine, physiology, and engineering, which illustrated various elementary applications of mathematical methods. The use of functional scales, nomograms, and the slide rule were largely stressed. The applications so overshadowed the mathematics, however, that a new text was demanded, which the author has supplied in the volume now under review.

Mr. Maclean sent along with his books several reprints of articles which he had written for the "Times of India," and the "Calcutta Review," which further elaborated his thesis that the old curriculum was numbing to the student who does not desire to become a mathematician, and that such a student's interests are furthered by preparing him to face fairly complicated problems in other sciences with a confidence that his mathematical tools can safely be applied, and with some training in making such applications.

"Descriptive Mathematics" certainly affords an earnest student many glimpses of fascinating developments of modern science. Among the topics listed are functional scales, inverse functions, empirical curves, triangular graphs, nomograms, frequency tables, histograms, probability, and finite differences. The student will find himself urged by the text to dig fairly deeply into numerous applications.

The book is hard to read. Cross references to the future paragraphs are made as freely as to the past. The diagrams are often confusing at first sight, and are not generally very close to the explanatory text. There are a few misprints, and the reviewer feels that many statements might have been better phrased.

The experiment in teaching which Mr. Maclean has made is an interesting one, and one that might be duplicated in this country at some of those institutions which depart from the usual ideas of college students.

College teachers will find the book stimulating. It is obviously not suitable for the majority of our students under the present regime.

C. A. RUPP

Business Mathematics. By I. L. Miller, D. Van Nostrand Company, 1935, 376 pages. $3.50.

This text is similar to one discussed by this reviewer in the American Mathematical Monthly, vol. 39, 1932, pp. 540–541, but it differs from the book discussed there in that it has a fuller treatment of insurance.

There are the usual chapters on elementary algebra and mathematics of finance. Illustrative examples and exercises are to be found in reasonable quantity.

Six of the twelve chapters are devoted to the mathematics of insurance for single lives. The treatment is longer than that given in most elementary books on the subject, and is well done.

J. H. BUSHEY

Differential Equations. By Norman Miller. Oxford University Press, 1935, 147 pages. $2.50.

This is another elementary textbook which, in common with most elementary mathematical texts, suffers from lack of precise fundamental concepts. It suffers further from lack of proofs that are ordinarily considered to be within the scope of an elementary text on the subject. In fact, there are no theorems in this book but only outlines of methods which are applicable in the most simple cases. There are the usual chapters on applications such as trajectories, law of growth, motion of particles, chains, etc. The selection of problems is good, although the author's attempts at precision of physical conditions are neither consistent nor successful (cf. Examples 1, 2, 4 pp. 87–89). The treatment of simultaneous differential equations and of the total equation in more than two variables is most superficial; partial differential equations are barely mentioned.

M. S. KNEBELMAN

the illustrative material are splendid, definitely outranking the field in this regard. The book is an intensely practical one.* Topics are presented in an interesting and convincing way, illustrations of the need for mathematical developments preceding and vivifying the more theoretical portions, which, in turn, are followed by clear and concise summary directions. Finally, valuable emphasis is placed throughout upon the succession of steps which are involved in the scientific approach to practical problems.

WARREN WEAVER

MATHEMATICS CLUBS

EDITED BY F. W. OWENS AND HELEN B. OWENS, State College, Pa.

All reports of club activities should be sent to F. W. Owens, 462 East Foster Ave., State College, Pennsylvania.

ATTENTION CLUBS!

Response to the letter forwarded in December by this department to all clubs listed in the MONTHLY directory has been generous, but many letters have not as yet received attention. If this department is to be of service to the clubs, cooperation is necessary. If your club has answered, accept our thanks; if it has not answered, please get it done ahead of term examinations.

If your club is not listed write at once for your communication.

BOOKS USEFUL TO MATHEMATICS CLUBS

Bibliography of books useful in a mathematics club; United States Bureau of Education Bulletin, 1911, No. 16.

List of books for a College library; this MONTHLY, 1917, vol. 24, pp. 368–376.

What other books do you suggest?

CONTEST PAPERS

A fairly large number of papers reached this department in answer to the suggestion made in the February 1936 issue of this MONTHLY. These have been classified into three types: (1) historical, (2) expository, (3) original.

Abstracts of two papers of the historical type are presented here.

I. THE LIFE AND WORK OF EULER

By ALICE BRYKCZYNSKI, University of Wisconsin, Extension Division

In this carefully written paper, the outstanding incidents of Leonhard Euler's life are sketched from his birth in Switzerland, 1707, until his death at St. Petersburg in 1783 after seventeen years of blindness.

His principal works in mathematics, astronomy, philosophy and mechanics are listed. The discussion of several of these works is included. That concerning his Part I of Introductio Analysis Infinitorum is quoted verbatim in the following paragraphs from the student's paper.

Euler's Introductio in 1748 caused a revolution in analytical mathematics, because of its general and systematized manner of presentation. It was intended to serve as an introduction to

* The reviewer has, he hopes, made it clear that this is a virtue which carries with it certain faults and limitations.

pure analytical mathematics. This was divided into two parts. The first part contains the bulk of the matter which is to be found in modern texts of algebra, theory of equations, and trigonometry. In the algebra he paid particular attention to the expansion of various functions in series, and to the summation of given series; and pointed out that an infinite series cannot be safely manipulated unless it is convergent. In the trigonometry, most of which was based on F. C. Mayer's Arithmetic of Sines which had been published in 1727, Euler developed the idea of John Bernoulli, that the subject was a branch of analysis and not a mere section of astronomy or geometry. He also introduced (contemporaneously with Simpson) the present abbreviations for the trigonometrical functions, and showed that the trigonometrical and exponential functions were connected by the relation

$$\cos \theta + i \sin \theta = e^{i\theta}.$$

Among these new abbreviated formulas he employed the notations A, B, and C to designate the angles of a triangle, and the opposite sides by a, b, and c, respectively. In 1750, Euler used S to denote the half-sum of the sides of a triangle; in 1755 he introduced Σ to signify "summation"; and in 1777 he used i for $+\sqrt{-1}$. He also was responsible (1734) for the notation $f(x)$ to indicate the "function of x." Euler defined a function in two ways:

1. Every analytical expression in x, i.e., every expression made up of powers, logarithms, trigonometric functions, etc., is called a function of x.
2. A function is the relation between y and x expressed in the XY plane by any curve drawn freehand.

It was also in Euler's Introductio that the symbol e was used to denote the base of the Napierian logarithms, namely, the incommensurable number $2.71828 \cdots$, and the symbol π used to denote the incommensurable number $3.14159 \cdots$. The use of a single symbol to denote the number $2.71828 \cdots$ seems to be due to Cotes who denoted it by M. Newton was probably the first to employ the literal exponential notation, and Euler, using the form a^z, had taken a as the base of any system of logarithms; it is probable that the choice of e for a particular base was determined by its being the vowel consecutive to a. The use of a single symbol to denote the number $3.14159 \cdots$ appears to have been introduced by John Bernoulli who represented it by c; Euler in 1734 denoted it by p, and in a letter of 1736 (in which he stated the theorem that the sum of the squares of the reciprocals of the natural numbers is $\pi^2/6$) he used the letter c; Goldbach in 1742 used π; after the publication of Euler's *Analysis* the symbol π was generally employed.

Works referred to include: Theory of equations, by Burnside and Panton; Elementary treatise on pure mathematics, by N. R. Dockeray; History of mathematical notations, by F. Cajori; Elements of mathematical theory of limits, by J. G. Leathems; Higher mathematics for engineers and physicists, by I. S. and E. S. Sokolnikoff.

II. WOMEN EMINENT IN MATHEMATICS

By ADELE SIDEK, Mount Mary College

Beginning with Hypatia, a neo-platonic philosopher of the fourth century, A.D., one of whose works on Diophantus is preserved in the Vatican library, Miss Sidek discusses six women who have won permanent notice in mathematical history.

Emilie de Brenteuil of the seventeenth century, more physicist than mathematician, is given first rank among French women students of the sciences in the first half of that century. Her Institute of Physics created much interest and, as Madame du Châtelet, her salons attracted Voltaire and prominent scientists to her circle.

Sophie Germain, the mathematical physicist, born in the year 1776, carried to greater heights the reputation of women as scientific students. Called the "most intellectual woman France has produced" she was the first woman allowed to attend the sessions of the Paris Academy of Sciences.

Between the times of these two French workers, lived Maria Gaetana Agnesi, better known to college students than the others because of the "witch of Agnesi" named for her. But contemporary with Sophie Germain was the Scotch woman, Mary Somerville, whose translation and criticism of Laplace's Mécanique Céleste is still a text at Cambridge, England. Her long life was spent largely in study and composition, her last works being a treatise on Theory of Differences and one on Molecular and Microscopic Science.

When Mary Somerville reached the traditional three score and ten years of age, Sonya Krukovsky, that vivacious and spirited Russian woman was born in Moscow. Her marriage to young Kovalévsky in order to escape bans placed on unmarried women in university classes, her years of study in Germany, and her brief period as professor at the University of Stockholm where she was a co-laborer with Mittag-Lefler were preliminaries to her election to the Academy of Sciences at St. Petersburg, only a year before her death.

Among the works referred to are: Famous Women, by J. Adelman; Sonya Kovalévsky, by I. F.; Les Femmes dans la Science, by A. Rebière; Women in Science, by J. A. Zahm.

Club Reports
1935–1936

The State Teachers Mathematics Club, New York State College for Teachers

President, Rosa Peters; Vice-President, D. Rogers; Secretary, N. Gunderson; Treasurer, Laura Hendricks; Faculty Adviser, Elizabeth Stokes. "A practice teacher's nightmare" was the stunt which welcomed new members. A Christmas Party and a picnic completed the social program. The club discussed at length the matter of high school mathematics clubs and mathematics in art and physics. Talks on prime numbers, the ellipse and constructions with a fixed circle completed the work of the regular monthly meetings. A bulletin board for mathematical news clippings is maintained.

Kappa Mu Epsilon, Florence State Teachers College

President, Orpha A. Culmer; Vice-President, W. A. Graham; Secretaries, Carrie M. Moore, Mary A. Denton, Mary A. Richeson; Treasurers, C. Thomas and Thisbe Riley. This new chapter has shown commendable originality in its social affairs, especially a banquet at which each member represented a mathematician and "talked in character."

Junior Mathematics Club, University of Wisconsin Extension Division

President, R. Christophe; Secretary-Treasurer, Alice Brykczynski; Faculty Adviser, Florence Jeffries. Two outstanding activities mark the work of the year, first the preparation and presentation of papers in competition for the Euler prize offered annually by the mathematics faculty; second, the annual spring exhibit of mathematical charts, machines and models. The Euler papers included "A philosophical interpretation of Zeno's four paradoxes"; "Hyperbolic functions"; "History of number symbols"; and "Life and work of Euler." This last paper, by Alice Brykczynski,

won the prize, was submitted in the MONTHLY contest and an abstract appears in this issue. Other papers presented to the club concerned divisions of space and theory of numbers. The exhibit included string models of surface intersections; solids of revolution, revolved by electricity; wooden models of quadric surfaces, home-made telescopes, etc. Most of the models have been made by students and presented to the department. The exhibit was open one evening. During the evening, in different rooms, brief talks were given on various mathematical subjects by club members. Over eight hundred high school students visited the exhibits and most of them listened to one or more of the talks. This is an annual exhibit, growing better and larger each year.

The club does its full share in the Milwaukee Intercollegiate Mathematical Association.

Pi Mu Epsilon, Brooklyn College

Director, Professor E. Fleisher; Vice-Director, J. Lorell; Secretary, Margaret Schwenzteger; Treasurer, M. Reade; Librarian, Jean Anderson. Installations of new members gave motive for two parties while the more serious meetings were given over to papers by members on such subjects as "Construction of seventeen sided polygons"; "Polygenic functions"; "Solution of a cubic by hyperbolas"; and "Transcendental property of gamma functions."

Pi Mu Epsilon, Lehigh University

Director, J. E. Raynor; Vice-Director, F. R. Malhlin; Secretary, D. L. Vaidelich. A joint meeting with the Physics Society of Lehigh University was held with talks and demonstrations of probability. An open lecture by Professor J. L. Coolidge of Harvard University on "Where did geometry come from?" was sponsored by the chapter. Other subjects discussed at regular meetings were "The relativity of Newton and Einstein"; "Cross classification"; "Calculus of prime numbers"; "N-dimensional geometry"; "Principle of finite differences"; and "Geometric constructions with compass alone". The year closed with the initiation banquet.

Mathematics Club, Ohio State University

Organized in January, 1936, this club sponsored by the local chapter of Pi Mu Epsilon conducts a program of two or three short papers at each meeting, prepared by students on the simpler subjects of more general interest. The following list is typical and covers the subjects most popular, most commonly found in club programs: "Waring's problem"; "Mathematical magic"; "The planimeter"; "Trisection of an angle"; "Magic squares"; "Fermat's last theorem"; "Summary of mathematical history"; "Squaring the circle"; "Concept of infinity in mathematics"; "Sorting machines for mathematical data"; "Mathematical prodigies"; "Origin of logarithms". For the current year officers are: President, C. Semmelman; Vice-President, L. Cohen; Secretary-Treasurer R. Wells.

Pi Mu Epsilon, New York University

Director, L. Baron; Vice-Director, S. Feith; Secretary, H. Moss; Treasurer, D. Gurinsky; Librarian, Louis Hajek. The Chapter held quarterly meetings at which lectures were heard on "Rise of European universities"; "Some ideas in quantum mechanics"; "Theory of surface areas" and "Chinese mathematics." The annual initiation banquet was well attended. The most important activity of the chapter was that of conducting the Third Annual Pi Mu Epsilon Interscholastic Mathematics Contest on April 25, 1936. There were 115 high schools which entered, of which 92 sent full teams of four students each. Highest score was made by the team from Boys High School of Brooklyn, N. Y., which was awarded a large cup. Stuyvesant High School of New York placed second and Abraham Lincoln High School of Brooklyn was third. Four smaller cups were awarded to teams making highest scores in their sections. The Chapter assisted in the preparation and demonstration of a mathematical exhibit shown after the contest. This exhibit was planned to make the principles of mathematics clear and attractive by means of charts and apparatus designed for that purpose.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 252. *Proposed by N. A. Court, University of Oklahoma.*

If the planes drawn through a given point M parallel to the faces BCD, CDA, DAB and ABC of the tetrahedron $ABCD$, whose centroid is at G, meet the respective medians of $ABCD$ in the points P, Q, R and S, prove that,

$$\frac{GP}{GA}+\frac{GQ}{GB}+\frac{GR}{GC}+\frac{GS}{GD}=0.$$

E 253. *Proposed by V. Thébault, Le Mans, France.*

Find two positive integers differing by five, with the sum of their squares a perfect cube, and show that the solution is unique.

E 254. *Proposed by D. L. MacKay, Evander Childs High School, New York.*

Given the vertices B and C, and the altitude from A, construct the triangle ABC so that $a^4=b^4+c^4$, where a, b and c are the sides of the triangle.

E 255. *Proposed by Cezar Coşniţă, Roumanian Mathematical Institute.*

Determine the locus of the centers of the spheres which pass through the two fixed points, A and B, and are tangent to the given plane P. What is the locus of the points of tangency on the plane?

E 256. *Proposed by L. J. Adams, Santa Monica Junior College, Calif.*

Prove that

$$\frac{\sin\theta+\sin 2\theta+\cdots+\sin n\theta}{\cos\theta+\cos 2\theta+\cdots+\cos n\theta}=\tan\frac{n+1}{2}\theta.$$

E 257. *Proposed by Mannis Charosh, New Utrecht High School, Brooklyn, N. Y.*

Construct the triangle ABC, given the altitude and median from A, and the difference, $b-c$, of the adjacent sides.

E 258. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

The sum and difference of $S\ L\ A\ P$ and $D\ E\ B$ are respectively $D\ U\ D\ E$ and $P\ I\ P\ S$, where each digit is replaced by a code letter. Determine the digit represented by each letter, and show that the solution is unique.

SOLUTIONS

E 185 [1935, 622]. *Proposed by C. A. Richmond, Tyngsboro, Mass.*

A thin, straight wire is marked off into m equal lengths by $m-1$ points. It is then bent at a right angle at each of one or more of these points, making each segment parallel to one of two rectangular axes. The bent wire may be self-intersecting, but not self-coincident over a finite length. How many different shapes may it have? Extend the problem to three dimensions by permitting the segments of the wire to be parallel to any of three rectangular axes.

Discussion by R. M. Foster, Bell Telephone Laboratories.

Since no solution of this problem has yet been published, it may be of some interest to present the results of straightforward enumeration when the number of bends is small, and to compare these results with those obtained for a modification of the problem for which it is possible to find a solution. Only the two-dimensional case is considered.

Let M denote the number of different shapes satisfying the conditions of the problem, including the original straight wire. By a systematic method of enumeration, Miss Marion C. Gray has found the values of M up to and including the case $m=9$. These are shown in the fourth column of the accompanying table.

Now let N be the total number of distinct shapes the wire may have, including the original straight wire and those shapes where the wire is self-coincident over a finite path. The shape of the wire is completely characterized by assigning a symbol ($+$, 0, or $-$) at each of the $n=m-1$ nodes; the symbol $+$ indicates a turn to the right, 0 indicates no turn, $-$ indicates a turn to the left. The ordered set of n symbols

$$e_1e_2e_3 \cdots e_n \qquad (e_i = +, 0, -)$$

then characterizes the wire shape. Two wire shapes are equivalent if, and only if, they have the same set of symbols or symbols which become the same upon reversing the order of the set, or upon interchanging $+$ and $-$, or both.

Let N_1 represent the total number of sets of n symbols, N_2 the number of these sets which are symmetrical with respect to a reversal of order, N_3 the number which are symmetrical with respect to interchange of $+$ and $-$, N_4 the number symmetrical with respect to both operations. Then

$$N = \tfrac{1}{4}(N_1 + N_2 + N_3 + N_4),$$

since every distinct shape is represented four times in the sum enclosed in parentheses. These individual terms can be computed:

$N_1=3^n$, since every one of the n symbols can be chosen independently in 3 different ways.

$N_2=3^{n/2}$ if n is even, since the first half of the set can be chosen independently, the second half being thereby determined. $N_2=3^{(n+1)/2}$ if n is odd, since

the first $(n-1)/2$ symbols can be chosen independently, the last $(n-1)/2$ symbols being thereby determined; the middle symbol can be chosen independently in 3 different ways.

$N_3=1$, since every symbol must be 0.

$N_4=3^{n/2}$ if n is even, as in the case of N_2. $N_4=3^{(n-1)/2}$ if n is odd, as in the case of N_2 but with the exception that the middle symbol must be 0.

Thus, if n is even,

$$N = \tfrac{1}{4}(3^n + 2\cdot 3^{n/2} + 1) = \tfrac{1}{4}(3^{n/2} + 1)^2,$$

and, if n is odd,

$$N = \tfrac{1}{4}(3^n + 3^{(n+1)/2} + 3^{(n-1)/2} + 1) = \tfrac{1}{4}(3^{(n-1)/2} + 1)(3^{(n+1)/2} + 1).$$

The values of N are readily found by first computing the series of values $(3^k+1)/2$ for $k=0, 1, 2, \cdots$. These values are 1, 2, 5, 14, 41, $\cdots$, each term being one less than three times the preceding term.

Segments	Nodes	Total distinct shapes	Non-coincident distinct shapes	Ratio
m	n	N	M	M/N
1	0	$1\cdot 1 = 1$	1	1.000
2	1	$1\cdot 2 = 2$	2	1.000
3	2	$2\cdot 2 = 4$	4	1.000
4	3	$2\cdot 5 = 10$	10	1.000
5	4	$5\cdot 5 = 25$	24	.960
6	5	$5\cdot 14 = 70$	66	.943
7	6	$14\cdot 14 = 196$	176	.898
8	7	$14\cdot 41 = 574$	493	.859
9	8	$41\cdot 41 = 1681$	1361	.810

The number N is thus readily computed. The ratio of M to N is shown in the last column of the table. This ratio decreases with increasing m so far as M has been found by enumeration. Whether the ratio approaches zero or not is an open question. In other words, the original problem, of finding a general expression for M, still remains to be solved.

E 211 [1936, 304]. *Proposed by V. Thébault, Le Mans, France.*

One liter of wine is drawn from a full cask, and replaced by water. Then one liter of the mixture is drawn and replaced by water. This is repeated until thirty-five liters have been drawn off and replaced, when an analysis determines that the cask contains equal parts of water and wine. What is the capacity of the cask?

Solution by R. K. Allen, Dartmouth College.

Let x be the initial number of liters of wine in the cask. After the first drawing and replacement, the number of liters of wine in the cask is $x[(x-1)/x]$,

and after the second drawing and replacement it will be $x[(x-1)/x]^2$. After thirty-five such drawings and replacements, there will be $x[(x-1)/x]^{35}$ liters of wine in the cask, which we are told is equal to half x. Then $[x/(x-1)]^{35}=2$, or $x/(x-1)=2^{1/35}$. From this we find $x=(2^{1/35})/(2^{1/35}-1)$, which is readily found by logarithms to be about four cubic centimeters less than fifty-one liters.

Also solved by W. E. Buker, Wm. Douglas, Sister M. Emeran, Daniel Finkel, E. L. Harp, Jr., Elmer Latshaw, C. A. Murray, J. Rosenbaum, C. E. Springer, E. P. Starke, J. E. Trevor, Simon Vatriquant, B. C. Zimmerman and the proposer.

E 212 [1936, 304]. *Proposed by C. W. Trigg, Cumnock College, Los Angeles.*

Any point P on the minor circle of an ellipse is connected to the foci of the ellipse. Show that the sum of the areas of the circles constructed on these connectors as diameters, is equal to one-half the area of the major circle of the ellipse.

Solution by Robert Gaskell, Brooklyn High School, Michigan.

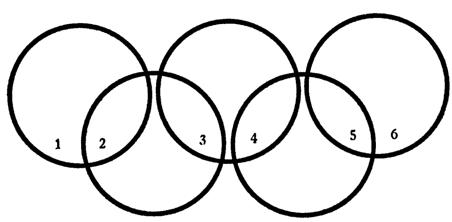
Let O be the center, F and G the foci of the ellipse, and a, b and c the semi-major axis, the semi-minor axis, and the distance OF, respectively. Then the diameters, f and g, or FP and GP, are found by the law of cosines to be such that

$$f^2 = b^2 + c^2 - 2bc \cos \angle POF,$$
$$g^2 = b^2 + c^2 + 2bc \cos \angle POF.$$

Consequently, the sum of the areas of the circles on f and g as diameters is $(\pi/4)(f^2+g^2)=(\pi/2)(b^2+c^2)=(\pi/2)a^2$, since $b^2+c^2=a^2$. Thus the theorem is proved.

Also solved by Katherine S. Arnold, W. E. Buker, Wm. Douglas, E. L. Harp, G. L. Hill, L. M. Kelly, C. A. Murray, C. E. Springer, E. P. Starke, Simon Vatriquant and the proposer.

E 213 [1936, 304]. *Proposed by Franz Denk, Erlangen, Germany.*



Redistribute the six digits in the same areas in the accompanying diagram so that after the rearrangement: (a) no two digits now in one circle will be in one circle; (b) no three digits now in two linked circles will be in two linked circles; (c) no four digits now in three linked circles will be in three linked circles; and finally, (d) no five digits now in four linked circles will be in four linked circles.

Solution by E. P. Starke, Rutgers University.

We seek a permutation of the digits 123456 such that no successive k digits therein shall be a permutation of k consecutive integers, where (a) $k=2$, (b) $k=3$, (c) $k=4$ and (d) $k=5$. Now (a) is satisfied if no adjacent positions are filled by consecutive integers. Furthermore, (d) is satisfied when neither 1 nor 6 occupies an end position.

If (a) and (d) are satisfied, so is (b), for there is no permutation of three consecutive integers without at least two adjacent consecutive integers. If (a) and (d) are satisfied, so also is (c), as may be seen by considering the digits which occupy the remaining two places.

If any satisfactory permutation is reversed, another is obtained. Hence we may restrict our attention to those in which the digit 1 precedes 6. Place 1 and 6 in any one of the six ways allowed by (a) and (d). Next place 2 in one of the two or three places not adjacent to 1. Then the remaining digits can easily be put into the remaining places in one or more ways. (Only in one case, - - - 162, is there no solution.) Thus we find forty-six solutions, the twenty-three following permutations and their opposites: 315264, 415263, 514263, 415362, 314625, 413625, 513624, 513642, 316425, 416352, 416253, 351462, 531462, 251364, 251463, 351642, 351624, 531642, 531624, 241635, 253164, 425163, 524163.

Also solved by W. E. Buker, Mary L. Constable, Simon Vatriquant and the proposer.

E 214 [1935, 305]. *Proposed by D. L. MacKay, Evander Childs High School, New York.*

Two non-congruent, similar triangles have two sides of one respectively equal to two sides of the other. Between what limits must the ratio of similitude lie?

Solution by W. E. Buker, Leetsdale High School, Pa.

Let the sides of the smaller and larger triangles be a, b, c and ra, rb and rc respectively, with r the ratio of similitude, and $c<b<a$. The conditions of the problem then require that

$$a = rb, \tag{1}$$

$$b = rc. \tag{2}$$

Thus the side c, which lies in value between $a-b$ and b, may be replaced in (2) by these values, when the simultaneous solution of (1) and (2) for r gives the limits sought, $1<r<(1+\sqrt{5})/2$.

Also solved by E. P. Starke, Simon Vatriquant and the proposer.

E 215 [1936, 305]. *Proposed by H. T. R. Aude, Colgate University.*

Show that for positive integers written in the scale of five, the sum of any odd number and the subsequent number gives a number whose digit-sum is three, or a number whose digit-sum is a number whose digit-sum is three, or . . . etc.

Solution by Simon Vatriquant, l'Athênê Royale d'Ixelles, Brussels, Belgium.

An odd number has the form $2n+1$. The subsequent number is $2n+2$, and their sum is $4n+3$, or three more than a multiple of four. But in the scale of five, the digit four plays the role of the digit nine in the ordinary decimal system. Every number in the scale of five is congruent, modulo four, to its digit-sum. The statement is now obvious.

Also solved by W. E. Buker, Daniel Finkel and E. P. Starke.

E 216 [1936, 305]. *Proposed by J. Rosenbaum, Hartford Federal College.*

Find

$$\lim_{x\to\infty}\left[x\sin\frac{1}{x}+\frac{1}{x}\right]^{x}.$$

Solution by Daniel Finkel, New York City.

As $x\to\infty$, $1/x$ and $\sin 1/x$ are infinitesimals of the same order, and so we may substitute the former for the latter. We then note

$$\operatorname*{Limit}_{x\to\infty}\left[x\cdot\frac{1}{x}+\frac{1}{x}\right]^{x}=\operatorname*{Limit}_{x\to\infty}\left[1+\frac{1}{x}\right]^{x}=e.$$

Also solved by Sisters M. Claudia and M. Cilini, E. P. Starke, Simon Vatriquant and the proposer.

Advanced Problems

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3811. *Proposed by N. A. Court, University of Oklahoma.*

A plane parallel to the base ABC of the tetrahedron $DABC$ meets the edges DA, DB, DC in the points P, Q, R; the same edges are met in the points U, V, W by a plane antiparallel to the plane ABC. Show that, if the plane PQR varies, the radical axis of the three spheres $AQRV$, $BRPW$, $CPQU$ remains fixed.

Note. For a definition of antiparallel see Court, Modern Pure Solid Geometry, p. 247, The Macmillan Co., 1935. The corresponding problem in the plane was considered in Educational Times, Reprints, vol. 60, 1894, p. 107, Q. 12001.

3812. *Proposed by N. A. Court, University of Oklahoma.*

The vertex of a variable cone is fixed, the base being the circle of intersection of a fixed sphere with a variable plane passing through a fixed straight line. The cone cuts the sphere in a second circle: show that the plane of this second circle passes through a fixed line.

3813. *Proposed by V. Thébault, Le Mans, France.*

Given a circle (O) and two orthogonal axes xx' and yy'; a variable point P on (O) is projected orthogonally into M and N on xx' and yy'; then, on the perpendicular from P to MN, two points V and W are taken so that the lengths of VP and PW are each in a given ratio k with the distance MN. (1) On VW as diameter a circle (Σ) is described. Show that the radical axis of any two circles (Σ) passes through a fixed point. (2) Determine the envelope of the circles (Σ). (3) Examine the special case $k=1$.

3814. *Proposed by I. J. Schoenberg, Colby College.*

Let $f(t)$ be a one-valued and complex-valued function which satisfies the functional relation

$$e^{iks}f(t) = f(t+s) - f(s) \tag{1}$$

for all real values of t and s, where k is a real constant $\neq 0$. Prove that

$$f(t) = C(e^{ikt} - 1)/ik \qquad (C \text{ a constant}). \tag{2}$$

Remark. It should be noticed that for $k=0$ (1) and (2) reduce to (1') $f(t+s)=f(t)+f(s)$ and (2') $f(t)=Ct$ respectively. It is known that (1') implies (2') only if some additional assumption is made on $f(t)$, for instance boundedness in the neighborhood of some point (see G. Hamel, Mathematische Annalen, vol. 60, 1905, pp. 459–462). No such assumption is needed if $k\neq 0$.

SOLUTIONS

3714 [1934, 635]. *Proposed by J. M. Feld, New York.*

Prove that, if the functions $x_i(t)$, $i=1, 2, 3$, possess second derivatives, and, if

$$(x_1x_2' - x_2x_1')^2 + (x_2x_3' - x_3x_2')^2 + (x_3x_1' - x_1x_3')^2 = 0, \tag{1}$$

$$x_1^2 + x_2^2 + x_3^2 \neq 0, \qquad x_2x_3' - x_3x_2' \neq 0, \tag{2}$$

then $(x_1x_2'-x_2x_1')/(x_2x_3'-x_3x_2')=\text{constant}$, and $(x_3x_1'-x_1x_3')/(x_2x_3'-x_3x_2')=\text{constant}$.

Solution by the Proposer.

The problem arose in geometry, t being real and the functions x_i being complex variables. That there exist functions satisfying the given conditions is shown by the examples

$$x_1 = -it^3, \qquad x_2 = t^2, \qquad x_3 = t^3,$$

and

$$(1) \qquad x_1 = mh(t) + i\sqrt{1+m^2}g(t), \qquad x_2 = -g(t), \qquad x_3 = h(t),$$

where m is a constant, and $g(t)$, $h(t)$ are real functions possessing continuous derivatives.

To prove that $(x_1x_2' - x_2x_1')/(x_2x_3' - x_3x_2') = \text{constant}$ it is sufficient to show that the derivative of this fraction is zero. Differentiating this ratio we obtain

$$\frac{(x_2x_3' - x_3x_2')(x_1x_2'' - x_2x_1'') - (x_1x_2' - x_2x_1')(x_2x_3'' - x_3x_2'')}{(x_2x_3' - x_3x_2')^2},$$

which by a simple computation can be shown to be equal to

$$-\frac{x_2}{(x_2x_3' - x_3x_2')^2}\begin{vmatrix} x_1 & x_2 & x_3 \\ x_1' & x_2' & x_3' \\ x_1'' & x_2'' & x_3'' \end{vmatrix}.$$

We proceed to prove that the determinant $\Delta = (x_1x_2'x_3'')$ vanishes identically. Let a repeated subscript indicate summation over 1, 2, 3; furthermore let

$$A = \begin{vmatrix} x_ix_i & x_ix_i' \\ x_i'x_i & x_i'x_i' \end{vmatrix}.$$

Then by an identity of Lagrange and by our hypothesis

$$A = (x_ix_i)(x_i'x_i') - (x_i'x_i)^2 = \sum_{i,j}(x_ix_j' - x_jx_i')^2 = 0.$$

By a theorem on determinants (O. Perron, *Algebra*, vol. 1, 1927, p. 117) we have also

$$A = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}^2 + \begin{vmatrix} x_1 & x_3 \\ x_1' & x_3' \end{vmatrix}^2 + \begin{vmatrix} x_2 & x_3 \\ x_2' & x_3' \end{vmatrix}^2 = 0.$$

Therefore

$$A' = \frac{dA}{dt} = 2\begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}\begin{vmatrix} x_1 & x_2 \\ x_1'' & x_2'' \end{vmatrix} + 2\begin{vmatrix} x_1 & x_3 \\ x_1' & x_3' \end{vmatrix}\begin{vmatrix} x_1 & x_3 \\ x_1'' & x_3'' \end{vmatrix}$$

$$+ 2\begin{vmatrix} x_2 & x_3 \\ x_2' & x_3' \end{vmatrix}\begin{vmatrix} x_2 & x_3 \\ x_2'' & x_3'' \end{vmatrix} = 2\begin{vmatrix} x_ix_i & x_ix_i' \\ x_i''x_i & x_i''x_i' \end{vmatrix} = 0.$$

Evidently

$$\Delta^2 = \begin{vmatrix} x_ix_i & x_ix_i' & x_ix_i'' \\ x_i'x_i & x_i'x_i' & x_i'x_i'' \\ x_i''x_i & x_i''x_i' & x_i''x_i'' \end{vmatrix}.$$

Since $A = 0$, we have for c_1 and c_2 not both zero,

$$c_1x_ix_i + c_2x_ix_i' = 0, \qquad c_1x_i'x_i + c_2x_i'x_i' = 0.$$

Moreover, since $A' = 0$ and $x_ix_i \neq 0$, we have also

$$c_1 x_i'' x_i + c_2 x_i'' x_i' = 0.$$

Therefore

$$\begin{vmatrix} x_i' x_i & x_i' x_i' \\ x_i'' x_i & x_i'' x_i' \end{vmatrix} = 0$$

and consequently, since all the minors of the elements in the third column of Δ^2 vanish, $\Delta^2=0$, which is what we wished to prove. In the same manner it can be proved that $(x_3x_1'-x_1x_3')/(x_2x_3'-x_3x_2')=$constant.

3729[1935, 177]. *Proposed by J. Rosenbaum, Hartford Federal College, Hartford, Conn.*

In an orthocentric n-dimensional simplex the centroid G divides the line segment joining the orthocenter H with the circumcenter O so that $HG/GO=2/(n-1)$.

This is a generalization of problem 3697 [1934, 453].

Solution by A. S. Householder, Washburn College, Topeka, Kansas.

Let the vertices A_i, $(i=1, 2, \cdots, n+1)$, be determined by the vectors $\mathbf{a}_i$. The simplex is orthocentric if

(1) $$\mathbf{a}_i\cdot\mathbf{a}_k + \mathbf{a}_j\cdot\mathbf{a}_l = \mathbf{a}_i\cdot\mathbf{a}_l + \mathbf{a}_j\cdot\mathbf{a}_k,$$

where no two of the four subscripts are equal. Let the origin be at the circumcenter 0, then for every i

(2) $$\mathbf{a}_i^2 = R^2,$$

where R is the circumradius. Let $\mathbf{g}$ be the vector of the centroid G, and $\mathbf{h}$, the vector of the orthocenter H. It is to be shown that

(3) $$(n-1)\mathbf{h} = (n+1)\mathbf{g} = \sum_{l=1}^{n+1}\mathbf{a}_l,$$

or that

(4) $$(\mathbf{a}_j - \mathbf{a}_k)\cdot\sum_{l=1}^{n+1}\mathbf{a}_l - (n-1)(\mathbf{a}_j - \mathbf{a}_k)\cdot\mathbf{a}_i = 0,$$

for any i and all j, $k\neq i$. But this is true, since in the first group of terms appears $\mathbf{a}_j^2-\mathbf{a}_k^2$ which is zero by (2); $\mathbf{a}_j\cdot\mathbf{a}_k-\mathbf{a}_k\cdot\mathbf{a}_j$ vanishes identically; and $(\mathbf{a}_j-\mathbf{a}_k)\cdot\mathbf{a}_i$ combining with the next term to make $-(n-2)(\mathbf{a}_j-\mathbf{a}_k)\cdot\mathbf{a}_i$. The result is identical with the result of fixing i, j, k in (1), transposing all terms to the left, and summing for $l\neq i, j, k$.

Solved also by Frank Ayres, Jr.

Editorial Note. The argument in regard to (4) may be stated in more detail in the following form. Let $\mathbf{k}$ be the vector of the point K on OG such that G divides OK in the ratio $(n-1)$: 2. Then $\mathbf{k}=(n+1)\mathbf{g}/(n-1)$. It will be shown that A_iK is perpendicular to every edge of the simplex not passing through A_i.

This will prove that A_iK is on the altitude from A_i. It then follows that K is the intersection of the altitudes, or that $K \equiv H$, and the proof is complete. The vector of A_kA_j is $\mathbf{a}_j - \mathbf{a}_k$: consider the scalar product

$$(5) \qquad (\mathbf{a}_j - \mathbf{a}_k) \cdot \sum_{l=1}^{n+1} \mathbf{a}_l = (\mathbf{a}_j - \mathbf{a}_k) \cdot (\mathbf{a}_j + \mathbf{a}_k) + (\mathbf{a}_j - \mathbf{a}_k) \cdot \sum_{l \neq j,k} \mathbf{a}_l,$$
$$= (n-1)(\mathbf{a}_j - \mathbf{a}_k) \cdot \mathbf{a}_i,$$

where no two of i, j, k are equal. This follows from (1) and (2), since $\mathbf{a}_j^2 - \mathbf{a}_k^2 = R^2 - R^2 = 0$, and $(\mathbf{a}_j - \mathbf{a}_k) \cdot \mathbf{a}_l = (\mathbf{a}_j - \mathbf{a}_k) \cdot \mathbf{a}_i$, where nc two of the four subscripts are equal. Hence

$$(6) \qquad \begin{gathered} (\mathbf{a}_j - \mathbf{a}_k) \cdot [(n+1)\mathbf{g} - (n-1)\mathbf{a}_i] = 0, \\ (\mathbf{a}_j - \mathbf{a}_k) \cdot [\mathbf{k} - \mathbf{a}_i] = 0 \end{gathered}$$

where the values of $\mathbf{k}$ and $\mathbf{g}$ above have been inserted. Since $\mathbf{k} - \mathbf{a}_i$ is the vector of A_iK, the last result is the proof that A_iK is perpendicular to A_kA_j.

Ayres' solution appears in his paper "On $n+2$ mutually orthogonal hyperspheres," National Mathematics Magazine, vol. 10, no. 7, 1936, p. 254.

3726[1935, 177]. *Proposed by Mannis Charosh, New Utrecht High School, Brooklyn, N. Y.*

The vertices of a triangle inscribed in a given circle are the points of tangency of a triangle circumscribed about the circle. Prove that the product of the perpendiculars from any point on the circle to the sides of the inscribed triangle is equal to the product of the perpendiculars from the same point to the sides of the circumscribed triangle.

Solution by C. W. Trigg, Cumnock College, Los Angeles.

Lemma: The perpendicular from a point on a circle to a chord is the mean proportional between the perpendiculars from the point to the tangents at the extremities of the chord.

Given P a point on the circle of which AB is a chord, PM perpendicular to AB, and PD and PE perpendicular to the tangents at A and B respectively. Draw chords PA and PB. $\angle PBE = \frac{1}{2}\widehat{PB} = \angle PAM$ and $\angle DAP = \frac{1}{2}\widehat{PA} = \angle PBM$. Hence the right triangles PBE and PAM are similar as are also the right triangles DAP and PMB. Therefore, $PE:PM::PB:PA::PM:PD$. Otherwise, $PM = \sqrt{PD \cdot PE}$.

Given inscribed triangle ABC and circumscribed triangle XYZ tangent to the circle at the vertices of ABC. From P, a point on the circle, PM, PN, PR, PD, PE and PF are perpendiculars to AB, BC, CA, XY, YZ, and ZX, respectively. By the lemma, $PM = \sqrt{PD \cdot PE}$, $PN = \sqrt{PE \cdot PF}$ and $PR = \sqrt{PF \cdot PD}$. Multiplying, $PM \cdot PN \cdot PR = PD \cdot PE \cdot PF$, which proves the proposition. It is evident that if P coincides with one of the vertices of ABC the proposition still holds, since both sides of the last equality vanish.

Note by S. Vatriquant, Brussels, Belgium.

The theorem is easily extended to any inscribed polygon. The lemma may be proved by considering it a limit case of Pappus' theorem: The product of the distances from a point on a circle to a pair of opposite sides of an inscribed quadrilateral is equal to the product of the distances from the same point to the second pair of sides. This theorem is proved by joining the point to the vertices of the quadrilateral, and by writing for the four triangles thus obtained that the product of the sides issuing from the point is equal to the product of the altitude, corresponding to the third side, by the diameter of the circle. If one side of a pair of opposite sides approaches its mate, the other pair of sides approach the positions of tangents at the extremities of the fixed side. The theorem of the lemma then results.

Solved also by Frank Ayres, Jr., M. G. Boyce, W. E. Buker, J. W. Clawson, J. M. Feld, H. D. Grossman, A. S. Householder, R. A. Johnson, R. L. Korgen, D. L. MacKay, A. Pelletier, H. A. Pool, Leon Recht, J. Rosenbaum, F. Underwood, R. C. Yates, and the proposer.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items of interest to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

Temporary members of the Institute for Advanced Study during the first term of 1936–37 included the following forty-six persons: J. H. Bartlett, Reinhold Baer, P. G. Bergmann, L. P. Bouckaert, Herbert Busemann, S. S. Cairns, Pei-Yuan Chou, A. H. Clifford, George Comenetz, D. M. Dribin, W. L. Duren, Eugene Feenberg, Aaron Fialkow, A. I. Flores, N. H. Frank, J. W. Givens, Marshall Hall, G. H. Hardy, Banesh Hoffmann, I. M. Hostetter, Witold Hurewicz, Leopold Infeld, P. W. Ketchum, Tsai-Han Kiang, Morris Kline, Tullio Levi-Civita, Norman Levinson, E. J. Moulton, F. D. Murnaghan, Melba N. Phillips, Walter Prenowitz, M. H. L. Pryce, J. F. Randolph, W. T. Reid, Moses Richardson, O. F. G. Schilling, Frank Smithies, M. H. Stobbe, M. H. Stone, E. D. Tagg, H. M. Terrill, J. M. Thomas, C. B. Tompkins, L. R. Wilcox, Rupert Wildt, Y. K. Wong.

Harvard University appoints two Benjamin Peirce Instructors in Mathematics annually. These instructorships are intended to give promising young men who already have their doctorates the opportunity to combine their research activities with a moderate amount of teaching, including some advanced instruction. Applications for 1937–38 should be sent to the Chairman of the Division of Mathematics, Professor W. C. Graustein, by February 15, 1937.

Associate Professors E. F. Allen and W. V. N. Garretson of the Oklahoma Agricultural and Mechanical College have been promoted to professorships.

Assistant Professor Lois W. Griffiths of Northwestern University is on leave of absence for the present year and is studying at Cambridge, England.

Z. L. Smith has been appointed assistant professor in mathematics at the University of Chicago with duties in the University High School and the General Physical Science Survey Course in the college.

L. S. Stephens of the Oklahoma Agricultural and Mechanical College has been promoted to an assistant professorship.

Dr. Bengt Stromgren of the University of Copenhagen has been appointed to an assistant professorship of astrophysics at the University of Chicago.

The following appointments to instructorships in mathematics are announced:

Armour Institute of Technology: Dr. G. C. Webber
Baylor University: Miss Kathryn Cardwell
Iowa State University: Dr. O. K. Sagen
Northwestern University: Dr. W. C. Randels
Ohio University: Mr. Carl Denbow
University of Illinois: Dr. Ruth C. Mason
University of Michigan: Dr. S. B. Myers, Dr. M. E. Shanks
University of Tennessee: F. A. Valentine

Dr. W. D. MacMillan, professor emeritus of astronomy at the University of Chicago, is lecturer in mathematics at Northwestern University during the first semester. He is giving a course on Dynamics of Rigid Bodies based on his recently published volume on that subject.

Dr. Webster G. Simon, Professor of mathematics in Adelbert College and the Graduate School of Western Reserve University, has been appointed Dean of the Faculties of Arts and Sciences for a term of three years. He will act as a responsible liaison officer for the Faculties of Adelbert College, Flora Stone Mather College, and the Graduate School, and in the absence or temporary disability of the President, will discharge the necessary duties of that office.

Ellen Fitz Pendleton, president emeritus of Wellesley College, and formerly associate professor of mathematics, died on July 26, 1936.

The placement bureau of the Association, which has been under the direction of Professor E. J. Moulton of Northwestern University, has been discontinued. It is believed that the greatest stress of unemployment has passed and that candidates for positions will be able to secure aid more effectively through other agencies.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, E. J. MOULTON, Northwestern University, Evanston, Ill.

BOOKS FOR REVIEW should be addressed to REVIEW EDITOR, American Mathematical Monthly, 531 West 116th Street, New York, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of addresss to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-first Summer Meeting, Pennsylvania State College, Sept. 6–7, 1937.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1937 and reported to the Secretary.

ALLEGHENY MOUNTAIN.
ILLINOIS, DeKalb, May.
INDIANA, Greencastle, May.
IOWA, Dubuque, April 16–17.
KANSAS, Witchita, April 3.
KENTUCKY.
LOUISIANA-MISSISSIPPI.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Lynchburg, Va., May 8.
MICHIGAN, Ann Arbor, March 20.
MINNESOTA.
MISSOURI.
NEBRASKA.
OHIO, Columbus, April 1.
OKLAHOMA, Oklahoma City, Feb.
PHILADELPHIA, Haverford, Nov. 27.
ROCKY MOUNTAIN.
SOUTHEASTERN, Nashville, Tenn., April.
SOUTHERN CALIFORNIA, Los Angeles, March 6.
SOUTHWESTERN, State College, N.M., April 23.
TEXAS.
WISCONSIN, Milwaukee, May.

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SOME UNSOLVED PROBLEMS OF TOPOLOGY*

By R. L. WILDER, University of Michigan

Introduction. I wish to emphasize at the outset that this address was not designed for *topologists*. A report on the unsolved problems of topology intended for specialists in that field would presumably carry with it the implication that the material presented had been selected from the standpoint of its basic importance in the field; and in preparing this address I have not used any such criterion. Perhaps many of the problems I have selected would deserve inclusion even had I used such a guide, but my aim has been solely to select certain† problems which combine ease of elucidation with the possibility of attracting general interest.

Like other and more aged branches of mathematics, topology has its unsolved problems that are deserving of general knowledge. The four color problem is a good example; it is easy to explain and has attracted quite general interest, and is so well known that no further mention of it will be made here. There are other problems, however, which have not been popularized and yet are quite as easy to explain, and which have consumed a considerable amount of topological energy (in the human and not physics sense) in the attempts at their solution.

Inasmuch as all of these problems are still being worked upon, I have to admit the possibility of some of them being deleted from the "unsolved" category before this address appears in print. Like the majority of their mathematical brethren, most topologists have great faith in the solvability of their problems. "Behold the problem; seek the solution. You can find it by pure reasoning. Never, indeed, will mathematicians be reduced to saying 'Ignoramibus'" [**1**, p. 69, italics‡]. Perhaps this faith is not justified in all instances, yet it is nice to possess it. It is interesting to note, at any rate, the evolution of this thought in Hilbert's later writings§ as well as the view of Brouwer that the principle of the solvability of every mathematical problem is identical with the unrestricted law of the excluded middle. The activity in the foundations of mathematics relative to this matter of solvability in general has been increasing during the past few years.

Whatever we think of these unsettled matters, it is certainly true that when we propose a mathematical problem for solution, we imply that the universe

* With slight change in wording, this is the substance of an address delivered by invitation before the Mathematical Association of America, Sept. 10, 1935, at Ann Arbor, Mich.

† Due to time limitations, no attempt was made at completeness.

‡ Bold faced numbers in square brackets refer to references listed at the end of the paper.

§ For instance, in [**2**], he poses the principle of solvability, especially that of decidability in a finite number of operations, as a problem in itself to be settled. And in [**3**], he mentions the problem of the freedom of contradiction of the axiom of solvability of an optional mathematical problem as an example of a "question of a fundamental nature in the domain of mathematical thought."

to which the problem appertains itself contains the elements necessary for a solution—this aspect of the matter of solvability merits attention. However, it seems a priori possible that the universe in question may contain the necessary elements, in a certain naive sense, and yet the problem be unsolvable—as for instance when conceivably the solution can be obtained only through an infinity of steps, there being no finite algorithm for a solution obtainable. Perhaps more light will be shed on these matters before long as a result of the investigations now being carried on in the foundations of mathematics [**39**].

Definitions. As you know, topology, in terms of the classical Klein formulation of geometry, concerns itself with the study of the invariants under homeomorphic transformations, or *homeomorphisms*. Two configurations, A and B, are said to be homeomorphic when there exists a one-to-one correspondence between their points which preserves limit points. Someone, thinking of plane topology, has aptly called it "rubber sheet geometry"—if a plane configuration is represented on a rubber sheet, then the sheet may be stretched in any way without altering the configuration in the topological sense. To be sure, not all possible homeomorphisms are obtainable by such stretchings.

One of the simplest invariants under homeomorphisms is that of connectedness: A point set M is said to be *connected** if, no matter how it be separated into two non-empty subsets, at least one of these contains a limit point of the other. For the purposes of the present paper, we shall also imply by "connected" that any set so modified contains more than one point.

Most of the examples which we shall give will refer to point sets in euclidean n-spaces, and in such spaces we call a point set *bounded* if it lies wholly within some $(n-1)$-sphere. For that subgroup of the set of all homeomorphisms of an n-space into itself formed by those which transform bounded sets into bounded sets, an important invariant is that of closure—a set is said to be *closed* if it contains all its limit points. If a point set is both closed and connected, it is called a *continuum*.

The simplest configurations are the *arc* and *simple closed curve*, which are respectively the homeomorphs of the closed linear interval and of the circle.

Problems. 1. The first problem which I wish to mention has come to be known as the "biconnected set problem." By way of introduction, consider any of the ordinary connected configurations of geometry; for example, consider a straight line segment ab. If x is an interior point of ab, then the segment ax and the segment xb without x form a pair of disjoint† connected sets whose sum is the original segment ab. In general, all of the usual configurations possess the property of being decomposable in various ways into pairs of disjoint connected sets. However, it was shown in 1921 by Knaster and Kuratowski [**5**, p. 241] that there exist connected sets which do not have this property. The example

* This is the so-called Lennes-Hausdorff notion of connectedness. See [**4**].

† Two sets are called *disjoint* if they have no elements in common.

given by them is the following: In the coordinate plane, let us denote by C the Cantor ternary perfect set in the interval $[0, 1]$ of the x-axis, by E the set of those points of C that are endpoints of intervals complementary to C, by R the set $C-E$, and by P the point $(\frac{1}{2}, \frac{1}{2})$. If a point lies in E we denote it by e, and if in R by r. Let M_1 be the set of all points whose ordinates are rational and which lie on straight line intervals Pe; and let M_2 be the set of all points whose ordinates are irrational and which lie on straight line intervals Pr. Then the set $M=M_1+M_2$ is a connected set [5]. However, every connected subset of M contains the point P, so that M is obviously not decomposable into disjoint connected sets. Elsewhere [6] I have called a point such as the point P of this example a *dispersion point*. Knaster and Kuratowski call *biconnected* any connected set not decomposable into two disjoint connected subsets. In all examples that have been given [5; also see 7] of biconnected sets, there exist dispersion points. The following problem was proposed by Kuratowski [8]: *Does there exist in every biconnected set a dispersion point?* I am sure that much time has been spent upon this problem, but to date it has not been solved.* Certain interesting results concerning biconnected sets have been found by P. M. Swingle [9]. For instance, he found that in order that a space known to be euclidean should be n-dimensional, $n>1$, it is necessary and sufficient that it be the sum of $n+1$, but not of n, biconnected sets.

Problem 2. Suppose that a continuum M has the property that, given any two points x and y of M, there exists a homeomorphism of M into itself in which x and y are corresponding points; then M is called *homogeneous* [10]. If we try to picture in our minds a homogeneous continuum in the plane, we shall probably think first of the circle; if x and y are points of the circle, a simple rotation about the center of the circle carrying x into y provides a homeomorphism of the type desired. Moreover, the straight line, and the plane itself, are obviously homogeneous. Of these three examples, we notice that only the first is bounded. In 1920 Knaster and Kuratowski raised the question [11], *is the simple closed curve the only bounded, homogeneous continuum in the plane?* This question received a partial answer by Mazurkiewicz, who showed [12] that among the plane, bounded, *locally connected* continua, the simple closed curves are the only homogeneous continua.† A set M is locally connected if for each point P of M and $\epsilon>0$ there exists a $\delta_{P\epsilon}>0$ such that a connected subset of M in $S(P, \epsilon)$‡

* At the time (November, 1936) when this paper was finally presented to the editor, it appeared that a negative solution of this problem (assuming the "continuum hypothesis") had been furnished by Dr. E. W. Miller of the University of Michigan.

† In [12] Mazurkiewicz also showed that an unbounded, locally connected, homogeneous continuum in the plane must either be the plane itself or the homeomorph of a straight line.

‡ By $S(P, \epsilon)$ we denote the set of all points whose distances from P are less than ϵ. The most quoted example of a continuum that is not locally connected is the set of points obtained by adding to the curve $y=\sin 1/x$, $0<x\leqq 1/\pi$, the set of all points $(0, y)$ such that $-1\leqq y\leqq 1$. This continuum is not locally connected at any of its points $(0, y)$. The term "locally connected" is now used in a more general sense than that assigned here, but only the latter is of significance for the purposes of this exposition.

contains all points of M in $S(P, \delta_{P\epsilon})$. It is this property that characterizes the continua that are *continuous curves* (sometimes called *Peano*, or *Jordan continua*), as shown by Hahn and Mazurkiewicz. (For references and discussion of equivalent characterizations see [**13**]). For the general case, the problem concerning homogeneous continua remains unsolved. For sets not in the plane, the problem is usually modified by prescribing other restrictions. Thus, if the set is imbedded in a higher dimensional space, configurations like the sphere and torus are homogeneous, and it was shown by van Dantzig [**14**] that even if the continuum is one-dimensional, it is not necessarily a simple closed curve. It has been pointed out to me by H. E. Vaughan that if the homogeneous continuum is a *rational* continuum, that is, if every point of it possesses arbitrarily small relative neighborhoods with at most denumerable boundaries, then it is a simple closed curve. This is readily shown from certain results of Whyburn and Frankl [**15**, **16**].

Problem 3. Another problem which we can state without additional definitions is the following [**17**, p. 192]: *Suppose M is a continuum which is homeomorphic with each of its proper subcontinua; is M an arc?* It can be shown that if M contains more than one prime part it is an arc.* For continua which consist of only one prime part the problem is not, so far as I know, solved. G. T. Whyburn has shown that if M is in the plane, it cannot separate the plane.†

Although this talk is supposed to be devoted to unsolved problems, I am going to digress at this point to call attention to a problem that is of interest in connection with the last two problems discussed above, but which was cleared up not long ago. I refer to a problem posed by Zaranciewicz [**19**] in 1925: *Does every bounded acyclic continuous curve, that is, a bounded continuous curve which contains no simple closed curve, have a proper subset with which it is homeomorphic?* All of the acyclic curves that one can ordinarily call to mind do have this property: Thus, a curve shaped like the letter T, or whatever continuum one can obtain from it by arbitrarily adding arcs in finite number without, of course, attaching more than one endpoint at each step, certainly has a proper subcontinuum with which it is homeomorphic, obtained simply by cutting off part of one of the arcs. However, in 1931 E. W. Miller [**20**] published an example of an acyclic curve which is not homeomorphic with any of its proper subsets; his example is too involved to reproduce here.

The problems thus far discussed may be considered as relating in general to configurations in the plane. When we pass to three-dimensional space, an interesting new class of problems appears. One of the fundamental theorems of Schoenflies [**21**, **22**] was to the effect that if J is an arbitrary simple closed curve in the plane, and D its "interior," then any given homeomorphism be-

* By a *prime part* of a continuum M is meant a subcontinuum K of M such that M fails to be locally connected at every point of K, and which is not a proper subset of another subcontinuum of M having the same property.

† His paper [**18**] contains an example of a plane continuum K which, in addition to other properties, has the property that *every subcontinuum of it contains a topological image of K itself.*

tween J and a circle K may be extended to a homeomorphism between $J+D$ and $K+$its interior. One would naturally expect that, in 3-space, the analogous statement would hold for simple closed surfaces (that is, the homeomorphs of the ordinary spherical surface). However, J. W. Alexander [**23**] published in 1924 an example to show that this is not the case—indeed, not even the domains bounded by the surfaces need be homeomorphic, and it appears that the possible distinct topological types of domains bounded by simple closed surfaces in 3-space are infinite in number. This introduces the next two unsolved problems:

Problem 4. *When may the homeomorphism between two simple closed surfaces be extended to the bounded domains of which they form the boundaries* [**17**]?

Problem 5. *Under what conditions will a domain bounded by a simple closed surface be homeomorphic with the set of points $x^2+y^2+z^2<1$ in cartesian 3-space?*

Imbedding Problems. A fascinating set of problems relates to the matter of *imbedding* a configuration in a given space. The general problem of this type has the following form: Given a topological configuration C, and a space S, can C be "pictured" in S—more precisely, does S contain a homeomorph of C? If C is a tetrahedron, and S is the plane, C cannot be imbedded in S; and, in general, if E_m and E_n are euclidean spaces such that $m>n$ it follows from the invariance of dimensionality under topological transformations that E_m cannot be imbedded in E_n. Although the particular result just stated may seem intuitively evident, it was not proved until 1911 [**24, 25**].

Many positive results regarding imbedding have been obtained, two of the most useful of which are the theorem of Urysohn [**26**] to the effect that the most general separable metric space in the abstract sense* may be imbedded in the fundamental parallelopiped of Hilbert space,† and the Menger-Nöbeling theorem that every n-dimensional compact metric space may be imbedded in a euclidean space of $2n+1$ dimensions.‡ In view of the Menger-Nöbeling theorem, all possible types of one-dimensional continua, or what we intuitively think of as curves (with as many multiple points as we please) can be pictured topologically in 3-space.

Problem 6. One of the chief imbedding problems is that of finding topological invariants characterizing imbeddability in a given space S—for instance, in a line or plane. The solution of this problem for the case of the euclidean one-

* An abstract set E is called a *metric space* if there exists a single-valued real function $d(x, y)$ defined over all pairs x, y of elements (points) of E such that (1) $d(x, y)=0$ if and only if $x=y$, and (2) $d(x, y)\leqq d(x, z)+d(y, z)$ for all elements x, y, z of E, distinct or not; a point P is called a *limit point* of a set of points M if for every positive number ϵ there exists a point x of M distinct from P such that $d(P, x)<\epsilon$. If E contains a denumerable set M such that every element of E either belongs to M or is a limit element of M, then E is called *separable*.

† That is, the metric space whose elements consist of all sequences $(x_1, x_2, \cdots, x_n, \cdots)$ of positive real numbers <1 and whose metric is defined by the relation $d(x, y)=\sum_{n=1}^{\infty}|x_n-y_n|/2^n$, where $x=(x_1, x_2, \cdots, x_n, \cdots)$ and $y=(y_1, y_2, \cdots, y_n, \cdots)$.

‡ For references to proofs, etc., see [**27**]. A space is *compact* (Fréchet) if every infinite subset of it has at least one limit point.

dimensional space, $S=E_1$, was obtained by L. W. Cohen [**28**] in 1929: A necessary and sufficient condition that a separable metric space M be imbeddable in E_1 is that (1) the maximal connected portions (i.e., *components*) of M be either points or arcs (which may be lacking in one or both end-points); (2) if P is a point component of M, then $\dim_P M=0$;* (3) if P is an end-point of an arc t of M, then $\dim_P(M-t+P)=0$; and (4) if P is an interior point of an arc t of M, then P is not a limit point of $M-t$. That these conditions must be fulfilled by every subset of E_1 is, of course, quite evident.

For $S=E_2$, the euclidean plane, the corresponding problem is partially solved, the history of the solution to date being as follows: In 1921 Mazurkiewicz ventured the opinion [**29**] that any continuous curve† which is acyclic is imbeddable in the plane, and this opinion was confirmed in 1923 by Wazewski, and later (independently) by Menger and Gehman [**30, 31, 32**]. In 1929 Ayres [**33**] extended this result, showing that a continuous curve which contains no curve of the form of the letter θ is imbeddable in the plane. In 1930 Kuratowski [**34**] showed that a continuous curve is imbeddable in E_2 if it contains at most a finite number of simple closed curves and does not contain any *primitive skew curve*‡. By a primitive skew curve is meant a homeomorph of one of the following two configurations (each of which is obviously not imbeddable in the plane): (1) A point set consisting of two sets A and B of three points each and nine arcs t_i $(i=1, 2, \cdots, 9)$ such that each pair of points selected one from A and the other from B is the pair of end-points of an arc t_i, and such that no two arcs t_i meet except possibly in their end-points; (2) a point set consisting of a set A of five points and ten arcs t_i such that each pair of points in A forms the end-points of an arc t_i. Substantially the same result was also obtained by Whitney [**35**; see also **36**], whose basic curve was, however, a combinatorial graph.§

In 1933 Claytor showed, in his thesis [**37**], that those cyclic|| continuous curves that are imbeddable in a simple closed surface, H_2, are characterized by the fact that they contain no primitive skew curves. An example or two here will probably be illuminating: Consider the point set M in the plane constituted by a circle together with its interior, and let P and Q be two distinct interior points of M. Let t be an arc in 3-space meeting M only in its end-points, which are P and Q. Then the point set $M+t$ is a cyclic continuous curve, obviously not imbeddable in H_2; it contains a configuration of type (1) described above. Now consider the configuration obtained in 3-space by letting M denote the plane circle together with its interior as above, but letting t be an arc meeting M

* By $\dim_P M$ we mean the (Menger-Urysohn) dimension of M at P; see [**27**].

† Hereafter we shall mean by "continuous curve" a compact metric space that is connected and locally connected.

‡ Illustrations of the primitive skew curves (term introduced by S. Claytor, [**37**]) will be found on p. 272 (Figs. 1, 2) of [**34**].

§ That is, a curve constructed from a finite number of arcs having at most end-points in common.

|| A continuous curve M is called *cyclic* if each pair of its points lies on a simple closed curve of M.

only in one end-point, say P, which is an interior point of M. This configuration is clearly not imbeddable in H_2, but on the other hand is not cyclicly connected, since no two interior points of t, for instance, lie on a simple closed curve of $M+t$.

In order to obtain a general characterization of continuous curves imbeddable in H_2, applicable to the non-cyclic case, Claytor introduced a special notion of *boundary point* which he characterized as follows: M being a continuous curve and J a simple closed curve of M, J is called a *boundary curve* of M provided there do not exist in $M-J$ two distinct components A and B such that (1) a pair of points of $F(A)$* separates a pair of points of $F(B)$ on J or (2) $F(A)=F(B)=$ three distinct points. An arc t of M is called a *boundary arc* of M provided that if C is a cyclic element† of M containing more than one point of t, then $t\cdot C$ is a subset of some boundary curve of C. And, finally, P is a *boundary point* of M if it lies on some boundary arc of M. The reader can more properly digest these definitions if he employs them to confirm the fact that in the second example stated in the paragraph above the point P of $M+t$ is not a boundary point of M. In terms of these notions Claytor showed that a necessary and sufficient condition that a continuous curve M be imbeddable in the two-dimensional euclidean sphere is that (1) M contain no primitive skew curve, and (2) each cut-point P of M be a boundary point of the closure of every component of the set $M-P$. In the example just referred to, $M+t$ is not imbeddable in the 2-sphere since, although it satisfies (1), it does not satisfy (2).‡

The question as to the characterization of imbeddability of the most general closed set or continuum in the 2-sphere is unsolved. As already mentioned, the Menger-Nöbeling condition is sufficient but by no means necessary. It should be mentioned here that Zippin [**38**] has recently extended Claytor's result to a more general class of configurations which he calls semi-peanian spaces.

Problem 7. In closing this brief summary, I am going to say a few words concerning some of the oldest objects of investigation of the so-called combinatorial topology, namely the manifolds.§ The simplest types of manifolds are the euclidean n-spheres; indeed, for $n=1$, the only closed manifold is the simple closed curve. For $n>1$, the number of distinct manifolds for each n is infinite; thus, for $n=2$, we have, besides the ordinary 2-sphere, the torus, the projective plane, etc., and it is well-known that in order to characterize a closed 2-dimensional manifold completely it is sufficient to know its connectivity and whether

* If L is a subset of M, $F(L)$ is the set of points common to the closure of L and the closure of $M-L$.

† Except for certain cases not of significance here, a *cyclic element* of a continuous curve M is a maximal cyclic subcontinuum of M.

‡ At the time of presenting this paper to the editor, I learned that Dr. Claytor has greatly simplified his result by showing that non-imbeddability of a continuous curve M in H_2 is equivalent to M containing at least one of *four* types of "primitive skew curves," of which two are those described above, and the other two of which are those described by Kuratowski [**34**, footnote 2, bottom of page 272].

§ For a simple exposition relating to manifolds, see the article by J. W. Alexander entitled *Manifolds* in the 14th edition of the Encyclopaedia Britannica.

it is orientable or not ("two-sided" or "one-sided"). Thus, the torus and the 2-sphere are both orientable but the sphere is of connectivity 0 and the torus of connectivity 2. On the other hand, the projective plane is characterized by the fact that its connectivity is one and it is not orientable. Concerning 3-dimensional manifolds, much work has been done, but no complete set of topological invariants has ever been found, and at the present time it appears quite possible that none may ever be found [**39**].

Closing Remarks. In the brief space of half an hour I cannot, of course, give a complete survey of the unsolved problems of topology. As indicated in my opening remarks, I have had in mind selecting only a few problems which are capable of being explained with a minimum of definitions to an audience of non-specialists in the field. It seems likely that several of the problems I have described will ultimately receive satisfactory solutions, and in this connection it is interesting to note new results concerning two problems which have been open for a long time.

As you are probably aware, the classical combinatorial method was based on the idea of regarding a configuration not as made up of points, but as pieced together from euclidean elements called *cells.* Thus, the 2-sphere can be regarded as composed of two 2-cells (the upper and lower hemispheres), mutually bounded by a simple closed curve (equator) which is itself composed of two 1-cells mutually bounded by a pair of 0-cells (the 0-sphere). However, the 2-sphere can be further subdivided into any number of 0-, 1- and 2-cells, and in an infinite variety of ways. The result of each of these ways of subdividing may be exhibited by a pair of matrices, called the *incidence matrices* [**40**], which exhibit the pattern of the cellular construction. Now if two subdivisions of the sphere, although not geometrically coincident, have the same pattern, they are called combinatorially equivalent. It has been called the "Hauptvermutung" of the combinatorial topology to assume that, given two subdivisions S_1 and S_2 of an n-manifold, it is possible, by further subdivisions of S_1 and S_2, to reduce them to the same pattern. Also it has long been suspected, and proved some time ago by Radó [**41**] for $n=2$, that if a separable space has the property that every point of it has a neighborhood in the space homeomorphic to a euclidean n-space, then the space may be divided into cells in the combinatorial manner; that is, "triangulated." On the basis of a new "Glättungssatz" G. Nöbeling has just published [**42**]* proofs of the "Hauptvermutung" and of the "triangulation theorem" for the general case. There appears, however, to be a defect in the proof of the "Glättungssatz," which the author has indicated an intention of clearing up in a later paper. This work indicates the determination on the part of present-day topologists to spare no efforts in the mastering of those difficulties which beleaguer this youthful branch of mathematics, and I hope the meager glimpse which I have tried to give of unsolved problems will lead to a greater appreciation of those difficulties.

* Also, see the paper of Furch cited in [**42**] for the case $n=3$.

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NEW FOUNDATIONS FOR MATHEMATICAL LOGIC*

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In Whitehead and Russell's Principia Mathematica we have good evidence that all mathematics is translatable into logic. But this calls for the elucidation of three terms: translation, mathematics, and logic. The units of translation are statements; also statement forms, i.e., expressions abstracted from statements by supplanting constants by variables. Thus it is not held that every symbol or combination of symbols of mathematics, say "∇" or "d/dx," can be equated directly to an expression of logic. But it is held that every such expression can be translated in context, i.e., that all statements and statement forms containing such an expression can be systematically translated into other statements and statement forms which lack the expression in question and contain no new expressions beyond those of logic. These other statements and statement forms will be translations of the original ones in the sense of agreeing with them in point of truth or falsehood for all values of the variables.

Given such contextual translatability of all mathematical signs, it follows

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that every statement or statement form consisting solely of logical and mathematical notation is translatable into a statement or statement form consisting solely of logical notation. In particular, thus, all principles of mathematics reduce to principles of logic—or to principles, at least, whose formulation needs no extra-logical vocabulary.*

Mathematics, in the sense here intended, may be understood as embracing everything which is traditionally classed as pure mathematics. In the Principia Whitehead and Russell present the constructions of the essential notions of set theory, arithmetic, algebra, and analysis from the notions of logic. Geometry is thereby provided for as well, if we think of geometrical notions as identified with algebraic ones through the correlations of analytical geometry. The theory of abstract algebras is derivable from the logic of relations which is developed in the Principia.

It must be admitted that the logic which generates all this is a more powerful engine than the one provided by Aristotle. The foundations of the Principia are obscured by the notion of propositional function, but, if we suppress these functions in favor of the classes and relations which they parallel, we find a three-fold logic of propositions, classes, and relations. The primitive notions in terms of which these calculi are ultimately expressed are not standard notions of traditional logic; still they are of a kind which one would not hesitate to classify as logical.

Subsequent investigations have shown that the array of logical notions required is far more meager than was supposed even in the Principia. We need only these three: *membership*, expressed by interposing the sign "ϵ" and enclosing the whole in parentheses; *alternative denial*, expressed by interposing the sign "$|$" and enclosing the whole in parentheses; and *universal quantification*, expressed by prefixing a variable enclosed in parentheses. All logic in the sense of the Principia, and hence all mathematics as well, can be translated into a language which consists only of an infinity of variables "x," "y," "z," "x'," etc., and these three modes of notational composition.

The variables are to be regarded as denoting any objects whatever; and among these objects we are to reckon classes of any objects, hence also classes of any classes.

"$(x\epsilon y)$" states that x is a member of y. *Prima facie*, this makes sense only where y is a class. However, we may agree on an arbitrary supplementary meaning for the case where y is an *individual* or non-class: we may interpret "$(x\epsilon y)$" in this case as stating that x *is* the individual y.†

The form "(—|---)," with any statements written in the blanks, may be read "Not both — and ---," i.e. "Either not — or not ---," i.e. "If — then not ---." The first reading is best, being least subject to ambiguities of English usage.

* For a fuller account see **[6]**; here and elsewhere in this paper bold type numbers in brackets refer to references listed at the end of the paper.

† This interpretation, along with the subsequent postulate P1, results in the fusion of every individual with its unit class; but this is harmless.

The compound statement is false if and only if both constituent statements are true.

The quantifier "(x)," finally, may be read "For all x," better "Whatever x may be." Thus "$(x)(x\epsilon y)$" means "Everything is a member of y." The total statement "(x)---" is true if and only if the formula "---" to which the quantifier is prefixed is true for all values of the variable "x."

Now the *formulas* of this rudimentary language are describable recursively thus: if any variables are put for "α" and "β" in "$(\alpha\epsilon\beta)$," the result is a formula; if any formulas are put for "ϕ" and "ψ" in "$(\phi|\psi)$," the result is a formula; and if a variable is put for "α" and a formula for "ϕ" in "$(\alpha)\phi$," the result is a formula. Formulas, so described, are the statements and statement forms of the language.

If all mathematics is translatable into the logic of the Principia, and this logic is to be translatable into the present rudimentary language, then every statement and statement form constructed wholly of mathematical and logical devices must be translatable ultimately into a *formula* in the sense just now defined. I will make the translatability of the Principia apparent, by showing how a series of cardinal notions of that logic can be constructed from the present primitives. The construction of the mathematical notions, in turn, may then be left to the Principia.

Definitions, which are the medium of all such construction of derivative notions, are to be viewed as extraneous conventions of notational abbreviation. The new notations which they introduce are to be regarded as foreign to our rudimentary language; and the only justification of our introducing such notations, unofficially as it were, is the assurance of their unique eliminability in favor of primitive notation. The form in which a definition is expressed is immaterial, so long as it indicates the manner of elimination. The purpose of definitions, in general, is perhaps brevity of notation; but in the present instance the purpose is to signalize certain derivative notions which play important rôles in the Principia and elsewhere.

In stating the definitions, the letters, "α," "β," "γ," "ϕ," "ψ," "χ," and "ω" will be used to denote expressions; "ϕ," "ψ," "χ," and "ω" will denote any formulas, and "α," "β," and "γ" will denote any variables. When such Greek letters are imbedded among signs belonging to the logical language itself, the whole is to denote the expression formed by so imbedding the expressions denoted by those Greek letters. Thus "$(\phi|\psi)$" will denote the formula which is formed by putting the formulas ϕ and ψ, whatever they may be, in the respective blanks of "$(\ \ |\ \ )$." The expression "$(\phi|\psi)$" itself is not a formula, but a noun denoting a formula; it is short for the description "the formula formed by writing a left parenthesis, followed by the formula ϕ, followed by a stroke, followed by the formula ψ, followed by a right parenthesis." The analogous applies to "$(\alpha\epsilon\beta)$," "$(\alpha)\phi$," "$((\alpha)(\alpha\epsilon\beta)|\phi)$," etc. Such use of Greek letters has no place in the language under discussion, but provides a means of discussing that language.

The first definition introduces the customary notation for *denial:*

D1. $\sim\phi$ for $(\phi|\phi)$.

This is a convention whereby the prefixture of "$\sim$" to any formula ϕ is to constitute an abbreviation of the formula $(\phi|\phi)$. Since in general the alternative denial $(\phi|\psi)$ is false if and only if ϕ and ψ are both true, an expression $\sim\phi$ as defined will be false or true according as ϕ is true or false. The sign "$\sim$" may thus be read "not," or "it is false that."

The next definition introduces *conjunction*:

D2. $$(\phi \cdot \psi) \quad \text{for} \quad \sim(\phi\,|\,\psi).$$

Since $(\phi|\psi)$ is false if and only if ϕ and ψ are true, $(\phi \cdot \psi)$ as defined will be true if and only if ϕ and ψ are true. The dot may thus be read "and."

The next definition introduces the so-called *material implication*:

D3. $$(\phi \supset \psi) \quad \text{for} \quad (\phi\,|\sim\psi).$$

$(\phi \supset \psi)$, as defined, is false if and only if ϕ is true and ψ false. The connection may thus be read "if-then," provided that we understand these words merely in a descriptive or factual sense, and do not infer any necessary connection between the antecedent and the consequent.

The next definition introduces *alternation*:

D4. $$(\phi \vee \psi) \quad \text{for} \quad (\sim\phi \supset \psi).$$

It is readily seen that $(\phi \vee \psi)$, as defined, is true if and only if ϕ and ψ are not both false. We may thus read "$\vee$" as "or," provided that this word is understood in the sense which permits joint truth of the alternatives.

The next definition introduces the so-called *material equivalence*:

D5. $$(\phi \equiv \psi) \quad \text{for} \quad ((\phi\,|\,\psi)\,|\,(\phi \vee \psi)).$$

A little study shows that $(\phi \equiv \psi)$, as defined, is true if and only if ϕ and ψ agree in point of truth or falsehood. The sign "$\equiv$" may thus be read "if and only if," provided that we understand this connection merely in a descriptive sense as in the case of D3.

The devices defined so far are called *truth functions*, because the truth or falsehood of the complex statements which they generate depends only on the truth or falsehood of the constituent statements. The use of alternative denial as a means for defining all truth functions is due to Sheffer **[9]**.

The next definition introduces *particular quantification*:

D6. $$(\exists\alpha)\phi \quad \text{for} \quad \sim(\alpha)\sim\phi.$$

$(\exists\alpha)\phi$ will thus be true if and only if it is not the case that the formula ϕ is false for all values of the variable α: hence if and only if ϕ is true for some values of α. The sign "$\exists$" may thus be read "for some"; "$(\exists x)(x \epsilon y)$" means "For some x, $(x\epsilon y)$," i.e., "y has some members."

The next definition introduces *inclusion*:

D7. $$(\alpha \subset \beta) \quad \text{for} \quad (\gamma)((\gamma\epsilon\alpha) \supset (\gamma\epsilon\beta)).$$

Thus "$(x \subset y)$" means that x is a subclass of y, or is included in y, in the sense that every member of x is a member of y.

The next introduces *identity*:

D8. $(\alpha = \beta)$ for $(\gamma)((\alpha \epsilon \gamma) \supset (\beta \epsilon \gamma))$.

Thus "$(x = y)$" means that y belongs to every class to which x belongs. The adequacy of this defining condition is clear from the fact that if y belongs to every class to which x belongs, then in particular y belongs to the class whose sole member is x.

Strictly, D7 and D8 violate the requirement of unique eliminability; thus, in eliminating the expression "$(x \supset y)$" or "$(z = w)$," we do not know what letter to choose for the γ of the definition. The choice is indifferent to the meaning, of course, so long as the letter chosen is distinct from the variables otherwise involved; but this indifference must not be smuggled in by the definitions. Let us then suppose some arbitrary alphabetical convention adopted to govern the choice of such a distinct letter in the general case.*

The next device to be introduced is *description*. Given a condition "---" satisfied by just one object x, the description "(ιx)---" is meant to denote that object. The operator "(ιx)" may thus be read "the object x such that." A description $(\iota\alpha)\phi$ is introduced formally only as part of contexts which are defined as wholes, as follows:

D9. $((\iota\alpha)\phi \epsilon \beta)$ for $(\exists\gamma)((\gamma \epsilon \beta) \cdot (\alpha)((\alpha = \gamma) \equiv \phi))$.

D10. $(\beta \epsilon (\iota\alpha)\phi)$ for $(\exists\gamma)((\beta \epsilon \gamma) \cdot (\alpha)((\alpha = \gamma) \equiv \phi))$.

Let "---" be a condition on x. Then "$(x)((x = z) \equiv$---$)$" means that any object x is identical with z if and only if the condition holds; in other words, that z is the sole object x such that ---. Then "$((\iota x)$---$\epsilon y)$," defined as it is in D9 as "$(\exists z)((z \epsilon y) \cdot (x)((x = z) \equiv$---$))$," means that y has a member which is the sole object x such that ---; hence that y has as a member *the* x such that ---. D9 thus gives the intended meaning. Correspondingly D10 is seen to explain "$(y \epsilon (\iota x)$---$)$" as meaning that y is a member of *the* x such that ---. If the condition "---" is not satisfied by one and only one object x, the contexts "$((\iota x)$--- $\epsilon\ y)$" and "$(y \epsilon (\iota x)$---$)$" both become trivially false.

Contexts such as $(\alpha \subset \beta)$ and $(\alpha = \beta)$, defined for variables, now become accessible also to descriptions; thus $((\iota\alpha)\phi \subset \beta)$, $((\iota\alpha)\phi \subset (\iota\beta)\psi)$, $(\beta = (\iota\alpha)\phi)$, etc., are reduced to primitive terms by the definitions D7–8 of inclusion and identity, together with the definitions D9–10, which account for $(\iota\alpha)\phi$ etc. in the contexts upon which D7–8 depend. Such extension of D7–8 and similar definitions to

* Thus we may stipulate in general that when a definition calls for variables in the definiens which are suppressed in the definiendum, the one occurring earliest is to be rendered as the letter which stands next alphabetically after all letters of the definiendum; the one occurring next is to be rendered as the ensuing letter of the alphabet; and so on. The alphabet is "a," "b," $\cdots$, "z," "a'," $\cdots$ "z'," "a''," $\cdots$. In particular, then, "$(x \subset y)$" and "$(z = w)$" are abbreviations for "$(z)((z \epsilon x) \supset (z \epsilon y))$" and "$(a')((z \epsilon a') \supset (w \epsilon a'))$."

descriptions calls merely for the general convention that definitions adopted for variables are to be retained also for descriptions.

Under this convention, D9 itself applies also when β is taken as a description; we thus get expressions of the form $((\imath\alpha)\phi \in (\imath\beta)\psi)$. But here the requirement of unique eliminability calls for a further convention, to decide whether D9 or D10 is to be applied first in explaining $((\imath\alpha)\phi \in (\imath\beta)\psi)$. We may arbitrarily agree to apply D9 first in such cases. The order happens to be immaterial to the meaning, except in degenerate cases.

Among the contexts provided by our primitive notation, the form of context $(\alpha)\phi$ is peculiar in that the variable α lends it no indeterminacy or variability; on the contrary, the idiom "for all x" involves the variable as an essential feature, and replacement of the variable by a constant or complex expression yields nonsense. The defined forms of context $(\exists\alpha)\psi$ and $(\imath\alpha)\psi$ share this character, for D6 and D9–10 reduce such occurrences of α to the form of context $(\alpha)\phi$. A variable in such a context is called *bound*; elsewhere, *free*.

Free variables are thus limited, so far as primitive notation is concerned, to contexts of the form $(\alpha\epsilon\beta)$. The definitions D9–10 provide use of descriptions in just such contexts. Descriptions are thereby made susceptible also to all further forms of context which may be devised for free variables by definition, as in D7–8. Our definitions thus provide for the use of a description in any position which is available to a free variable. This serves our purpose completely, for, as just observed, descriptions or other complex expressions are never wanted in the position of bound variables.

The theory of descriptions which I have presented is Russell's in its essentials [**10**], but considerably simpler in detail.

The next notion to be introduced is the operation of *abstraction*, whereby, given a condition "---" upon x, we form the class $x\ni$---* whose members are just those objects x which satisfy the condition. The operator "$x\ni$" may be read "the class of all objects x such that." The class $x\ni$--- is definable, by description, as *the* class y to which any object x will belong if and only if ---; symbolically,

D11. $\qquad \alpha\ni\phi$ for $(\imath\beta)(\alpha)((\alpha\epsilon\beta) \equiv \phi)$.

By means of abstraction, the notions of the Boolean class algebra are now definable just as in the Principia: the negate $-x$ is $y\ni\sim(y\epsilon x)$, the sum $(x \cup y)$ is $z\ni((z\epsilon x) \vee (z\epsilon y))$, the universal class V is $x\ni(x=x)$, the null class Λ is $-$V, and so on. Further the class $\{x\}$ whose sole member is x, and the class $\{x, y\}$ whose sole members are x and y, are definable thus:

D12. $\qquad \{\alpha\}$ for $\beta\ni(\beta = \alpha)$,

D13. $\qquad \{\alpha, \beta\}$ for $\gamma\ni((\gamma = \alpha) \vee (\gamma = \beta))$.

Relations can be introduced simply as classes of ordered couples, if we can contrive to define ordered couples. Clearly any definition will serve this purpose if it makes for the distinctness of couples (x, y) and (z, w) in all cases except

* Peano's notation is easier to print than the circumflex accent used in the Principia.

where x is z and y is w. A definition which is readily seen to fulfill this requirement has been devised by Kuratowski [3].

D14. $$(\alpha, \beta) \quad \text{for} \quad \{\{\alpha\}, \{\alpha, \beta\}\}.$$

I.e., the couple (x, y) is a class which has two classes as members; one of these classes has x as sole member, and the other has x and y as sole members.

Next we can introduce the operation of *relational abstraction*, whereby, given a condition "---" upon x and y, we form the relation $xy\ni$--- which anything x bears to anything y if and only if x and y satisfy the condition. Since relations are to be taken as classes of ordered couples, the relation $xy\ni$--- is describable as the class of all those couples (x, y) such that ---; symbolically,

D15. $$\alpha\beta\ni\phi \quad \text{for} \quad \gamma\ni(\exists\alpha)(\exists\beta)((\gamma = (\alpha, \beta)) \cdot \phi).$$

The notion "x bears the relation z to y" needs no special definition, for it becomes simply "$((x, y)\epsilon z)$".*

Enough definitions have here been presented to make the further notions of mathematical logic accessible by means directly of the definitions in the Principia. Let us now turn to the question of theorems. The procedure in a formal system of mathematical logic is to specify certain formulas which are to stand as initial theorems, and to specify also certain inferential connections whereby a further formula is determined as a theorem given certain properly related formulas (finite in number) as theorems. The initial formulas may either be listed singly, as postulates, or characterized wholesale; but this characterization must turn solely upon directly observable notational features. Also the inferential connections must turn solely upon such features. Derivation of theorems then proceeds by steps of notational comparison of formulas.

The formulas which are wanted as theorems are of course just those which are *valid* under the intended interpretations of the primitive signs: valid in the sense of being either true statements or else statement forms which are true for all values of the free variables. Inasmuch as all logic and mathematics is expressible in this primitive language, the valid formulas embrace in translation all valid statements and statement forms of logic and mathematics. Gödel [1] has shown, however, that this totality of principles cannot be comprehended among the theorems of any formal system, in the sense of "formal system" just now described, unless that system be inconsistent. Adequacy of our systematization must then be measured by some standard short of the totality of valid formulas. A fair standard is afforded by the Principia: for the basis of the Principia is presumably adequate to the derivation of all codified mathematical theory, except for a fringe requiring the axiom of infinity and the axiom of choice as additional assumptions.

* The above treatment of dyadic relations is immediately extensible to relations of any higher degree. For, a triadic relation of x, y, and z can be treated as a dyadic relation of x to the couple (y, z); a tetradic relation of x, y, z, and w can next be treated as a triadic relation of x, y, and the couple (z, w); and so on. See [7].

The system here to be presented is adequate to the adopted standard. It embraces one postulate, viz. the *principle of extensionality*

P1. $$((x \subset y) \supset ((y \subset x) \supset (x = y))),$$

according to which a class is determined by its members. It embraces also three rules specifying whole sets of formulas which are to stand as initial theorems:

R1. $((\phi | (\psi | \chi)) | ((\omega \supset \omega) | ((\omega | \psi) \supset (\phi | \psi))))$ is a theorem.

R2. If ψ is formed from ϕ by putting β for α wherever α occurs as a free variable, then $((\alpha)\phi \supset \psi)$ is a theorem.

R3. If "x" does not occur in ϕ, $(\exists x)(y)((y \epsilon x) \equiv \phi)$ is a theorem.

These rules are to be understood as applying to all formulas ϕ, ψ, χ, and ω, and to all variables α and β. R2 as presented has a slight defect which would be removed in a rigorous formulation.*

Finally, the system embraces two rules specifying inferential connections:

R4. If ϕ and $(\phi | (\psi | \chi))$ are theorems, so is ψ.

R5. If $(\phi \supset \psi)$ is a theorem, and α is not a free variable of ϕ, then $(\phi \supset (\alpha)\psi)$ is a theorem.

R1 and R4 are an adaptation of the propositional calculus as systematized by Nicod [5] and Łukasiewicz [4]. Together, R1 and R4 provide as theorems all and only those formulas which are valid merely by virtue of their structure in terms of the truth functions.

R2 and R5 contribute the technique for manipulating the quantifier.† The rules R1, R2, R4, and R5 provide as theorems all and only those formulas which are valid by virtue of their structure in terms of the truth functions and quantification.

P1 and R3, finally, are concerned specifically with membership. R3 may be called the *principle of abstraction*; it provides that, given any condition "---" upon y, there is a class x (viz. $y\ni$---) whose members are just those objects y such that ---. But this principle is readily seen to lead to contradiction. For, R3 gives the theorem

$$(\exists x)(y)((y \epsilon x) \equiv \sim (y \epsilon y)).$$

Now let us take y in particular as x. This step, immediate for intuitive logic, could be accomplished formally by proper use of R1, R2, R4, and R5. We thus have the self-contradictory theorem

$$(\exists x)((x \epsilon x) \equiv \sim (x \epsilon x)).$$

This difficulty, known as Russell's paradox, was overcome in the Principia

* A given occurrence of α is said to be *covered* by β in ϕ if a formula $(\beta)\chi$ forms part of ϕ and contains that occurrence of α. An occurrence of α is *free* in ϕ if it is not covered by α in ϕ. In these terms the correct form of R2 is this: If no free occurrence of α in ϕ is covered by β, and ψ is formed by putting β for every free occurrence of α in ϕ, then $((\alpha)\phi \supset \psi)$ is a theorem.

† R5 answers to the first part of Bernays' rule (γ), in Hilbert and Ackermann [2], and R2 supplants (e) and (α).

by Russell's theory of types. Simplified for application to the present system, the theory works as follows. We are to think of all objects as stratified into so-called types, such that the lowest type comprises individuals, the next comprises classes of individuals, the next comprises classes of such classes, and so on. In every context, each variable is to be thought of as admitting values only of a single type. The rule is imposed, finally, that $(\alpha\epsilon\beta)$ is to be a formula only if the values of β are of next higher type than those of α; otherwise $(\alpha\epsilon\beta)$ is reckoned as neither true nor false, but meaningless.*

In all contexts the types appropriate to the several variables are actually left unspecified; the context remains systematically ambiguous, in the sense that the types of its variables may be construed in any fashion conformable to the requirement that "ϵ" connect variables only of consecutively ascending types. An expression which would be a formula under our original scheme will hence be rejected as meaningless by the theory of types only if there is no way whatever of so assigning types to the variables as to conform to this requirement on "ϵ." Thus a formula in our original sense of the term will survive the theory of types if it is possible to put numerals for the variables in such a way that "ϵ" comes to occur only in contexts of the form "$n \epsilon n+1$."

Formulas passing this test will be called *stratified*. Another way of describing them is the following. An *ϵ-chain* of ϕ is an expression $\alpha_1\epsilon\alpha_2\epsilon\alpha_3 \cdots \epsilon\alpha_n$ $(n>1)$ such that each segment $\alpha_i\epsilon\alpha_{i+1}$ occurs in ϕ. Now ϕ is *stratified* if it has no ϵ-chains with like initial and like terminal variables but unlike lengths. This definition of stratification is equivalent to the other, and has the advantage of affording an immediate criterion, since the ϵ-chains of a stratified formula are readily exhausted.

The formula "$(x\epsilon x)$" is unstratified, for its ϵ-chains "$x\epsilon x$" and "$x\epsilon x\epsilon x$" have like initial and like terminal variables but unlike lengths. Also "$((y\epsilon x) \mid ((z\epsilon y) \mid (z\epsilon x)))$" is unstratified, because of its ϵ-chains "$z\epsilon x$" and "$z\epsilon y\epsilon x$." On the other hand "$(x\epsilon y)$" and "$((x\epsilon z) \mid (y\epsilon z))$" are stratified. It is to be remembered that definitional abbreviations are extraneous to the formal system, and hence that we must expand an expression into primitive notation before testing for stratification. Thus "$(x \supset x)$" turns out to be stratified, but "$((x\epsilon y) \cdot (x \subset y))$" not.†

Imposition of the theory of types upon our system consists in expurgating the language of all unstratified formulas; hence construing ϕ, ψ, etc., in R1–5 as stratified formulas, and adding the uniform hypothesis that the expression to be inferred as a theorem is likewise stratified. This course eliminates Russell's and related paradoxes, by precluding the disastrous use of unstratified formulas such as "$\sim(y\epsilon y)$" for ϕ in R3.

But the theory of types has unnatural and inconvenient consequences. Be-

* In particular, then, β in the context $(\alpha\epsilon\beta)$ cannot denote an individual. The considerations occasioning the third footnote are thus swept away by the theory of types.

† It has been simplest to explain stratification without special allowance for cases where the same letter appears both as a bound and as a free variable in a given formula, or as bound in several quantifiers. The unnecessary restrictions which this formulation seems to impose can be circumvented by suitable choice of letters for the variables. But this calls for a revision of the convention in the fourth footnote.

cause the theory allows a class to have members only of uniform type, the universal class V gives way to an infinite series of quasi-universal classes, one for each type. The negation $-x$ ceases to comprise all non-members of x, and comes to comprise only those non-members of x which are next lower in type than x. Even the null class Λ gives way to an infinite series of null classes. The Boolean class algebra no longer applies to classes in general, but is reproduced rather within each type. The same is true of the calculus of relations. Even arithmetic, when introduced by definitions on the basis of logic, proves to be subject to the same reduplication. Thus the numbers cease to be unique; a new 0 appears for each type, likewise a new 1, and so on, just as in the case of V and Λ. Not only are all these cleavages and reduplications intuitively repugnant, but they call continually for more or less elaborate technical manoeuvres by way of restoring severed connections.

I will now suggest a method of avoiding the contradictions without accepting the theory of types or the disagreeable consequences which it entails. Whereas the theory of types avoids the contradictions by excluding unstratified formulas from the language altogether, we might gain the same end by continuing to countenance unstratified formulas but simply limiting R3 explicitly to stratified formulas. Under this method we abandon the hierarchy of types, and think of the variables as unrestricted in range. We regard our logical language as embracing all formulas, in the sense originally defined; and the ϕ, ψ, etc. of our rules may be taken as any formulas in this sense. But the notion of stratified formula, explained in terms of ϵ-chains and divorced of any connotations of type, survives at one point: we replace R3 by the weaker rule:

R3′. If ϕ is stratified and does not contain "x," $(\exists x)(y)((y\epsilon x)\equiv\phi)$ is a theorem.

I have no proof that this system is consistent; all I can say is that I find no way of getting a contradiction. The lack of a consistency proof is no special ground for misgivings, for there is likewise none for the systematization involving the theory of types.

In the new system there is just one general Boolean class algebra; the negate $-x$ embraces *everything* not belonging to x; the null class Λ is unique; and so is the universal class V, to which absolutely everything belongs, including V itself.* The calculus of relations reappears as a single general calculus treating of relations without restriction. Likewise the numbers resume their uniqueness, and arithmetic its general applicability as a single calculus. The special technical manoeuvres necessitated by the theory of types accordingly become superfluous.

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* Since everything belongs to V, all subclasses of V can be correlated with members of V, viz. themselves. In view then of Cantor's proof that the subclasses of a class k cannot all be correlated with members of k, one might hope to derive a contradiction. It is not clear, however, that this can be done. Cantor's *reductio ad absurdum* of such a correlation consists in forming the class h of those members of the original class k which do not belong to the subclasses to which they are correlated, and then observing that the subclass h of k has no correlate. Since in the present instance k is V and the correlate of a subclass is that subclass itself, the class h becomes the class of all those subclasses of V which do not belong to themselves. But R3′ provides no such class h. Indeed, h would be $\hat{y}\sim(y\epsilon y)$, whose existence is disproved by Russell's paradox.

Indeed, since this new system differs from the original inconsistent one only in the replacement of R3 by R3′, the only restriction which distinguishes the new system from the original one is the lack of any general guarantee of the existence of classes such as $y\ni(y\epsilon y)$, $y\ni\sim(y\epsilon y)$, etc., whose defining formulas are unstratified.* In the case of some unstratified formulas, the existence of corresponding classes is actually still demonstrable by devious means; thus R3′ gives

$$(\exists x)(y)((y\epsilon x) \equiv ((z\epsilon y) \mid (y\epsilon w))),$$

and from this by the other rules we can accomplish the substitutional inference

$$(\exists x)(y)((y\epsilon x) \equiv ((z\epsilon y) \mid (y\epsilon z))), \tag{1}$$

which affirms the existence of a class $y\ni((z\epsilon y)\mid(y\epsilon z))$ whose defining formula is unstratified. But presumably we cannot prove the existence of classes corresponding to certain unstratified formulae, including those from which Russell's paradox or similar contradictions proceed. Within the system, of course, those contradictions can be used for explicitly disproving the existence of the classes concerned, by *reductio ad absurdum.*

The demonstrability of (1) shows that the deductive power of this system outruns that of the Principia. A more striking instance, however, is the axiom of infinity, with which the Principia must be supplemented if certain accepted mathematical principles are to be derived. This axiom asserts that there is a class with infinitely many members. But in the present system such a class is forthcoming without help of the axiom, namely the class V, or $x\ni(x=x)$. The existence of V is provided by R3′; and so is the existence of infinitely many members of V, viz. Λ, $\{\Lambda\}$, $\{\{\Lambda\}\}$, $\{\{\{\Lambda\}\}\}$, and so on.

Note added January 5, 1937. Let a formula be called *acyclic* if it has no ϵ-chain beginning and ending with the same variable. If the foregoing system is consistent we might well inquire into the consistency of the still stronger system obtained by putting "acyclic" for "stratified" in R3′. This suggestion is due in part to Mr. Garrett Birkhoff.

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* R3 can be adequately restricted in certain alternative ways, not involving the notion of stratification; see [11] and [8]. But these methods entail most of the awkward limitations which are entailed by the theory of types. The present method of avoiding the contradictions, if indeed it avoids them, would seem to be the least restrictive method yet suggested.

ON NUMBERS OF THE FORM $a^2+\alpha b^2$

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Introduction. It is believed that the two theorems given in §1 are of sufficient interest on account of their simplicity and elegance and the elementary character of their proofs to justify us in presenting them and in calling attention to some of their applications.

Both proofs are based on certain number-theoretic identities which may be presented as follows. Let a, b, α, p, q, x, y, λ, μ denote integers, and write

$$a^2 + \alpha b^2 = pq, \tag{1}$$

$$Aq = (q\lambda + ax - \alpha by)^2 + \alpha(q\mu + bx + ay)^2. \tag{2}$$

Then by aid of the relation

$$pq(x^2 + \alpha y^2) = (a^2 + \alpha b^2)(x^2 + \alpha y^2) = (ax - \alpha by)^2 + \alpha(bx + ay)^2$$

it is easily verified that

$$A = q(\lambda^2 + \alpha\mu^2) + p(x^2 + \alpha y^2) + 2a(\lambda x + \alpha\mu y) + 2\alpha b(-\lambda y + \mu x), \tag{3}$$

$$Ap = (px + a\lambda + \alpha b\mu)^2 + \alpha(py - b\lambda + a\mu)^2. \tag{4}$$

In the presence of (1) equations (2) and (4) imply each the other. These *recriprocal relations* will yield easy proofs of the two theorems in §1, which follows.

1. *Proofs of Two Theorems.* We first prove the following theorem:

THEOREM I. *Let α be a positive integer. Let p be a prime factor of a number of the form $m^2+\alpha n^2$ where m and n are both prime to p. Let pq, $q>0$, be the least positive integral multiple of p for which integers a and b prime to p exist such that $a^2+\alpha b^2=pq$. Then if $p>(\frac{4}{3}\alpha)^{1/2}$ we have $q\leqq(\frac{4}{3}\alpha)^{1/2}$.*

If in the named relation $a^2+\alpha b^2=pq$, the numbers a and b have a common prime factor p_1 then we can divide the equation through by p_1^2 and obtain a new equation of the same form with q replaced by a smaller positive integer q_1. Since this is contrary to the hypothesis on q it follows that a and b are relatively prime.

Since b is prime to p an integer b' exists such that $bb'\equiv 1 \bmod p$. Let a_1 be the residue, modulo p, of ab' of least absolute value. Then from the relation $(ab')^2+\alpha(bb')^2=pqb'^2$ we have $a_1^2+\alpha=pq_1$ where q_1 is a positive integer. But $a_1^2+\alpha\leqq\frac{1}{4}p^2+\alpha$. Thence, by aid of the relation $p^2>\frac{4}{3}\alpha$, we have the inequality

$$0 < q_1 \leqq \tfrac{1}{4}p + \alpha/p = \tfrac{1}{4}p + \alpha p/p^2 < \tfrac{1}{4}p + \alpha p/(\tfrac{4}{3}\alpha) = p.$$

Hence $q<p$ since from the relation $a_1^2+\alpha\cdot 1^2=pq_1$ it follows that q_1 is not less than q owing to the named minimal character of q.

Now let x and y be integers such that $bx+ay=1$, this being possible since a and b are relatively prime. Take $\mu=0$. Choose λ so that $|q\lambda+ax-\alpha by|\leqq\frac{1}{2}q$.

Then from (2) we see that

$$0 < Aq \leqq \tfrac{1}{4}q^2 + \alpha, \quad \text{whence} \quad 0 < A \leqq q(\tfrac{1}{4} + \alpha/q^2).$$

Let us now suppose that $q^2 > \frac{4}{3}\alpha$. Then we have

$$0 < A < q(\tfrac{1}{4} + \tfrac{3}{4}) = q.$$

Hence $0 < A < q < p$. Now from (4) we have a relation of the form $pA = a_2^2 + \alpha b_2^2$. Since $pA < p^2$ it follows that a_2 and b_2 do not both have the factor p and hence that each of them is prime to p. Since $A < q$ we have a contradiction with our hypothesis concerning the minimal character of q. This has arisen by supposing that $q^2 > \frac{4}{3}\alpha$. Hence $q^2 \leqq \frac{4}{3}\alpha$, and the theorem is proved.

COROLLARY. *In the named relation* $a^2+\alpha b^2 = pq$, *the integers* a *and* b *are relatively prime.*

We shall also prove the following theorem:

THEOREM II. *Let* β *be a positive integer which is not a square. Let* p *be a prime factor of a (positive or negative) number* $m^2 - \beta n^2$ *where* m *and* n *are both prime to* p. *Let* pq, $q > 0$, *be the least positive integral multiple of* p *for which integers* a *and* b *prime to* p *exist such that* $pq = \pm(a^2 - \beta b^2)$. *Then if* $p > \beta^{1/2}$ *we have* $q < \beta^{1/2}$.

If in the named relation $pq = \pm(a^2 - \beta b^2)$ the numbers a and b have a common prime factor p_1 then we may divide the equation through by p_1^2 and obtain a new equation of the same form with q replaced by a smaller positive integer q_1. Since this is contrary to the hypothesis on q it follows that a and b are relatively prime.

Since b is prime to p an integer b' exists such that $bb' \equiv 1$ modulo p. Let a_1 be the residue, modulo p, of ab' of least absolute value. Then we readily have $pq_1 = \pm(a_1^2 - \beta)$, where q_1 is a positive integer since β is not a square, the choice from the double sign $\pm$ in the last equation being not necessarily the same as in the equation for pq. Since $a_1^2 \leqq \frac{1}{4}p^2$ and $\beta < p^2$ it follows that $|a_1^2 - \beta| < p^2$ and hence that $q_1 < p$. But from the definition of q in the theorem it follows that q_1 is not less than q. Hence we have $q < p$.

We now use equations (1) to (4) with α replaced by $-\beta$ and with B in place of $|A|$ and with the double sign $\pm$ prefixed to the first member of each equation with the understanding that the ambiguity is so resolved in each case as to make the members not negative. Let x and y be such that $bx + ay = 1$, this being possible since a and b are relatively prime. Take $\mu = 0$. Choose λ so that $|q\lambda + ax + \beta by| \leqq \frac{1}{2}q$. Then from (2) and the fact that β is not a square we see that $0 < Bq <$ (the greater of $\frac{1}{4}q^2$ and β), whence we have $0 < B <$ (the greater of $\frac{1}{4}q$ and β/q).

If we now suppose that $q^2 > \beta$ we see that $0 < B < q$. But from (4) we have a relation of the form $pB = \pm(a_1^2 - \beta b_1^2)$. Now $0 < pB < p^2$ since $0 < B < q < p$. Therefore neither a_1 nor b_1 has the factor p. Since $B < q$ we have a contradiction with our hypothesis concerning the minimal character of q. This has arisen by

supposing that $q^2>\beta$. Hence, since $q^2\neq\beta$ owing to the fact that β is not a square, we see that $q^2<\beta$, and the theorem is proved.

COROLLARY. *In the named relation* $pq=\pm(a^2-\beta b^2)$, *the integers* a *and* b *are relatively prime.*

2. *Applications of Theorem I.* There are certain cases in which it is easy to see that the integer q in Theorem I has the value 1. This follows at once from the theorem when $\alpha=1$ or $\alpha=2$. When $\alpha=3$ we have $q=1$, since otherwise we would have $q=2$ and this is impossible since a^2+3b^2 is divisible by 4 when it is even and since in this case we have $p>2$. When $\alpha=4$ we have $q=1$, since otherwise we would have $q=2$ and this is impossible since a^2+4b^2 is divisible by 4 when it is even and since in this case we have $p>2$. When $\alpha=7$ we have $q=1$, since the theorem implies that $p>3$ and that q does not exceed 3 and since a^2+7b^2 is divisible by 4 when it is even and is divisible by 9 when it is a multiple of 3. Hence we have $q=1$ when $\alpha=1, 2, 3, 4$, or 7. This conclusion yields the following well-known proposition: *If* p *is an odd prime divisor of a number of one of the forms* m^2+n^2, m^2+2n^2, m^2+3n^2, m^2+7n^2, *where* m *and* n *are relatively prime, then* p *itself may be written in that form.*

If an odd prime number p has the form $p=a^2+b^2$ then one of the numbers a and b is odd and the other even. Hence p is of the form $4k+1$. Therefore no prime of the form $4k+3$ is a divisor of the sum m^2+n^2 of two relatively prime squares. But every prime p of the form $4k+1$ is a divisor of such a sum of two squares, since from Wilson's theorem $(p-1)!+1\equiv 0 \bmod p$ we have, for $p=4k+1$, the relation $\{(2k)!\}^2+1\equiv 0 \bmod p$ as one sees by replacing factors in the foregoing congruence by their residues, modulo p, of least absolute value. Hence we have the following well-known theorems. *Every prime of the form* $4k+1$ *may be written as a sum of two integral squares. The number* -1 *is a quadratic residue of every prime of the form* $4k+1$ *and is a quadratic non-residue of every prime of the form* $4k+3$.

It just so happens that the quadratic character of -1 is an easy consequence of our methods. But, in general, quadratic character is not readily determined by our theorems. But if we assume the elements of the theory of quadratic residues, we may easily draw additional conclusions from the results implied by Theorem I.

If an odd prime p has the form $p=a^2+2b^2$, then a is odd and p has one of the forms $8k+1$ and $8k+3$. Hence from Theorem I and the fact that $q=1$ when $\alpha=2$ it follows that -2 is a quadratic non-residue of every prime of either of the forms $8k+5$ and $8k+7$. This of course is known from the theory of quadratic residues, but it is here established without an appeal to that theory. But from the latter theory it is also known that -2 is a quadratic residue of any prime of either of the forms $8k+1$ and $8k+3$—and this fact is not readily implied by our theorem. However, if we use this fact, we see that for every prime p of either of the forms $8k+1$ and $8k+3$ there exists an integer m such that $m^2+2\cdot 1^2$ is divisible by p. Thence, by aid of Theorem I and the fact that we

now have $q=1$, we conclude to the following well-known proposition. *A prime number p of either of the forms $8k+1$ and $8k+3$ may be written in the form $p=a^2+2b^2$.*

In a similar way the following classic proposition may be proved. *A prime number p of the form $6k+1$ may be written in the form $p=a^2+3b^2$.*

In a similar manner (with $\alpha=7$) one may also prove the following long-known theorem. *Every prime number $14k+1$ or $14k+9$ or $14k+11$ may be written in the form a^2+7b^2.*

When $\alpha=5$ or $\alpha=6$ our Theorem I admits for q either of the values 1 and 2. That neither of these values of q is to be excluded in either case is readily shown as follows. We have $41=6^2+5\cdot 1^2$, a case where $q=1$, and $2\cdot 23=46=1^2+5\cdot 3^2$, a case where $q=2$ since 23 is not of the form a^2+5b^2. Likewise we have $7=1^2+6\cdot 1^2$ and $2\cdot 5=10=2^2+6\cdot 1^2$, whence it follows that both cases for q arise when $\alpha=6$. When $\alpha=8$ it may be shown that 1 and 3 are the possible values for q. For $\alpha=61$ the possible values for q are 1, 2, 5, 7. The reader will readily work out other cases.

Returning to the case $\alpha=5$, with $q=1$ or 2, and applying Theorem I we obtain the following long-known proposition. *If p is an odd prime factor of a number of the form m^2+5n^2 where m and n are relatively prime, then either p itself or else $2p$ may be written in the form a^2+5b^2.* Employing this result and the theory of quadratic residues, one may establish the following known theorem. *Every prime $20n+1$ or $20n+9$ and the double of any prime $20n+3$ or $20n+7$ is of the form a^2+5b^2.*

It is easy to work out additional special results for particular values of α.

We return to the general case of Theorem I. From the relation $a^2+\alpha b^2=pq$ and the fact that a and b are relatively prime it follows that $-\alpha$ is a quadratic residue of every prime factor of q which is not contained in α. This fact, and the theory of quadratic residues, will often enable one to exclude values of q which are not excluded by the theorem. Another useful observation is the following. If we suppose that q is divisible by 2, then it may be proved that:

(1) q is an odd multiple of 2 if α is of the form $4k+1$;

(2) q is an odd multiple of 4 if α is of the form $8k+3$;

(3) q is divisible by 8 if α is of the form $8k+7$.

From Theorem I and the first part of the foregoing paragraph we have at once the following proposition: *The number q of Theorem I is a power of 2 when α is an odd prime such that $-\alpha$ is a quadratic non-residue of every odd prime less than $(\frac{4}{3}\alpha)^{1/2}$.* The following are instances of such odd primes α: 3, 5, 7, 13, 37, 43, 67, 163. (Whether this is a complete list I do not know; it is easily shown to contain all cases in which $\alpha<1000$. It may be of some interest to seek an answer to the question thus raised. See Dickson's History of the Theory of Numbers, vol. I, pp. 420–421.) For $\alpha=13$ or 37 we have $q=1$ or 2; for $\alpha=43$ or 67 or 163 we have $q=1$ or 4.

The case $\alpha=163$ leads to some interesting observations. Since $4\mu^2+163$ is odd and since the relation $1^2+163\cdot 1^2=4\cdot 41$ shows that 41 is the smallest prime

whose quadruple is of the form m^2+163n^2, it follows that $4\mu^2+163$ is a prime for each value of μ for which $4\mu^2+163<1681=41^2$: hence $4\mu^2+163$ is a prime for each of the twenty consecutive values $0, 1, \cdots, 19$ of μ. But for $\mu=20$ we have $40^2+163\cdot 1^2=1763=41\cdot 43$. Also, we see that $(2\mu-1)^2+163$ is divisible by 4 with the quotient $\mu^2-\mu+41$. By an argument similar to the foregoing it may be shown that $\mu^2-\mu+41$ is a prime when less than 1681, and hence for each of the forty consecutive values $1, 2, \cdots, 40$ of μ. Such results have been noted by several writers, beginning with Euler (see Dickson, l.c.).

The cases $\alpha=67$ and $\alpha=43$ may be treated in a precisely similar way. It turns out, for instance with $\alpha=67$, that $\mu^2-\mu+17$ is a prime for each of the sixteen consecutive values $1, 2, \cdots, 16$ of μ, while $\mu^2-\mu+11$ is a prime for each μ of the set $1, 2, \cdots, 10$.

For $\alpha=37$ we have a slightly different case since $q=1$ or 2. It turns out that $2\mu^2-2\mu+19$ is a prime for each of the eighteen consecutive values $1, 2, \cdots, 18$ of μ, while $4\mu^2+37$ is a prime when $\mu=0, 1, 2, \cdots, 7$, or 8. The case $\alpha=13$ gives similar results of less interest.

For the cases of composite α there is at least one value which is interesting in this connection. For $\alpha=58$ it is easy to show, by the methods already frequently employed, that $q=1$ or 2. Hence, *if p is an odd prime factor of a number m^2+58n^2 where m and n are prime to p, then either p or $2p$ is of the form a^2+58b^2.* Now $4s^2+58t^2=2(2s^2+29t^2)$. A prime factor p of an odd number $2s^2+29t^2$ is such that $2p\geqq 0^2+58\cdot 1^2=58$ or $p\geqq 29$. Hence $2s^2+29$ is prime when it is less than 29^2 or 841 and hence when $0\leqq s<21$. Hence $2s^2+29$ is prime for $s=0, 1, \cdots, 20$. Actual check for additional numbers shows it to be prime for $s=0, 1, 2, \cdots, 28$, a fact already observed by Legendre (see Dickson, l.c.).

3. *Applications of Theorem* II. Let us consider the case when $\beta=2$. Since $a^2-2b^2=2(a+b)^2-(a+2b)^2$ and $2b^2-a^2=(a+2b)^2-2(a+b)^2$ it follows that any number m^2-2n^2 can be expressed in the form $2\nu^2-\mu^2$, and vice versa. Applying Theorem II and observing that $q=1$, we have the following known proposition. *Any prime factor p of a number of either of the forms m^2-2n^2 or $2n_1^2-m_1^2$ can be written in either of the forms $p=a^2-2b^2$ and $p=2a_1^2-b_1^2$.* By employing the theory of quadratic residues, according to the method used in the preceding section, one may obtain the following theorem. *A prime p of either of the forms $8k+1$ and $8k+7$ may be represented in either of the forms a^2-2b^2 and $2a_1^2-b_1^2$.*

Let us next consider the case $\beta=3$. From the relation

$$\left(\frac{3}{p}\right)\left(\frac{p}{3}\right) = (-1)^{1/2(3-1)\cdot 1/2(p-1)} = (-1)^{1/2(p-1)},$$

which belongs to the theory of quadratic residues, and from the fact that a prime p greater than 3 is a quadratic residue modulo 3 when and only when it is of the form $6k+1$, we see that the prime divisors greater than 3 of the form m^2-3n^2, with m and n relatively prime, are the primes of the forms $12k+1$ and $12k+11$. When $\beta=3$ we have $q=1$ in Theorem II. From that theorem it

follows therefore that a prime $12k+1$ or $12k+11$ can be written in one of the forms a^2-3b^2 and $3a_1^2-b_1^2$. But if we write $12k+1=3a_1^2-b_1^2$ we have b_1^2+1 divisible by 3, and this is impossible. Again if we write $12k+11=a^2-3b^2$ we have a^2+1 divisible by 3, and this is impossible. Thus, by aid of Theorem II and the theory of quadratic residues, we have the following results. *A prime* $12k+1$ *may be written in the form* a^2-3b^2; *a prime* $12k+11$ *may be written in the form* $3a_1^2-b_1^2$.

Since $5b^2-a^2=-(a^2-5b^2)=(5b-2a)^2-5(2b-a)^2$, a number of the form $-(a^2-5b^2)$ is also of the form $a_1^2-5b_1^2$. Now when $\beta=5$ we have $q=1$ or 2 in accordance with Theorem II. But if a^2-5b^2 is even it is divisible by 4. Hence, since $p>2$, we cannot have $q=2$. Therefore, $q=1$. Applying Theorem II, we now conclude to the following known proposition. *If p is an odd prime factor of a number of the form m^2-5n^2, where m and n are prime to p, then p may be written in the form $p=a^2-5b^2$.* The odd primes of which 5 is a quadratic residue are those of the forms $10k+1$ and $10k+9$. Hence we have the following result. *A prime of either of the forms* $10k+1$ *and* $10k+9$ *may be represented in the form* a^2-5b^2.

Other particular results may also be readily obtained from Theorem II, including cases in which the value of q is not uniquely determined by the value of β alone. Thus for $\beta=10$ there are three possible values for q, namely, 1, 2, 3, depending on the nature of the prime p.

FINITE PROJECTIVE GEOMETRIES, $PG(k, p^n)$

By E. R. OTT, University of Buffalo

1. *Introduction.* In 1905, Veblen noted that many axioms of ordinary geometry are satisfied by certain finite sets of elements, and constructed an example of a finite geometry. In 1906, Veblen and Bussey [1]* gave the first formal definitions, both synthetic and analytic, and discussed general properties of finite projective spaces.

We reproduce, briefly, the analytic definition of a finite projective k-space, $PG(k, p^n)$. If $x_1, x_2, \cdots, x_{k+1}$ are marks [2] of a Galois field of order p^n, there are $p^{nk}+p^{n(k-1)}+\cdots+p^n+1$ elements (points) of the form $(x_1, x_2, \cdots, x_{k+1})$ provided that $(x_1, x_2, \cdots, x_{k+1})$ and $(\mu x_1, \mu x_2, \cdots, \mu x_{k+1})$ denote the same element when μ is any mark of the field other than zero, and provided that $(0, 0, \cdots, 0)$ be excluded from consideration. These elements constitute a finite projective geometry of k-dimensions when arranged according to the following scheme. The equation $u_1x_1+u_2x_2+\cdots+u_{k+1}x_{k+1}=0$ is said to be an equation of a $(k-1)$-space; the points of the $(k-1)$-space being those points of the finite geometry whose coordinates satisfy its equation. A $(k-n)$-space is represented by n linear equations of this type.

Defining assumptions for both a "finite projective geometry" and for a

* Numbers in square brackets refer to references at the end of the paper.

$PG(k, p^n)$, (k, p, n all integers and p a prime) were given by Veblen and Bussey. They prove that the elements of a $PG(k, p^n)$ constitute a "finite projective geometry" and that for given $k>2$, p, and n there is one and only one "finite projective geometry" and that it is the $PG(k, p^n)$. For $k=2$, there exist both finite desarguean and non-desarguean plane geometries under their synthetic definition, but there is exactly one finite plane desarguean geometry, viz., the $PG(2, p^n)$. For this case it is obviously necessary, then, to make some further restrictions of the synthetic definition in order to completely define the $PG(2, p^n)$. Such restricted definitions have been given, and are reproduced in (2.1). In this paper we shall be concerned only with the $PG(k, p^n)$, and, primarily, the $PG(2, p^n)$.

E. H. Moore proved that every finite field is a Galois field, and that any two Galois fields containing the same number of elements are abstractly identical. By using these results, Veblen and Bussey were able to exhibit the collineation groups of the finite geometries. The order of the group leaving a $PG(2, p^n)$ invariant is $(\sigma-1)(\sigma-\lambda)\sigma(p-1)^2$, where $\sigma=p^{2n}+p^n+1$ and $\lambda=p^n+1$. There are several papers dealing with the group theory of finite geometries, and we have elected to omit any discussion of this important phase [**3**] of the subject.

The reader needs only an elementary knowledge of number theory to understand the references made to Galois fields. One interested in algebra, geometry, finite groups, or the axiomatic basis of mathematics can, with this topic, find important applications of his speciality.

2. *Definitions and Representations of Finite Projective Planes*, $PG(2, p^n)$.

(2.1). *Synthetic Definition.* Synthetically, a finite projective plane may be defined [**4, 5**] as a set of elements, which for suggestiveness are called points, arranged in subsets, called lines, and subject to the following defining assumptions:

(2.11). The set contains a finite number, greater than one, of lines; *and* each line contains $\lambda=p^n+1$ points, (p and n integers and p a prime).

(2.12). If A and B are distinct points, there is one and only one line that contains A and B.

Definition. If A, B, C are three points not on the same line, and a is the line joining B and C, the class S_2 of all points on the lines joining A to the points of a is called the *plane* determined by A, B, and C.

(2.13). All lines considered are in the same plane.

The defining conditions may easily be proved equivalent to those given by Veblen and Young, except for the condition that there are λ points on each line. From either set it follows [**5**, pp. 18–29; cf. exercise 3, p. 25] that each pair of distinct lines of the plane so defined is on one and only one point, and that the principle of duality is valid in the plane. Then, by (2.11), there are λ lines on each point. To obtain the number of points in a plane, consider a line m and a point M not on m. Then M and each point of m determine a line on each of which there are $(\lambda-1)$ points in addition to M. Furthermore, every point of the plane

is on one of these λ lines on M. Then the number of points in the plane is $\lambda(\lambda-1)+1=\sigma=p^{2n}+p^n+1$. By duality the number of lines in the plane is also σ.

The triple system

$$(2.14)\qquad \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 1 \\ 4 & 5 & 6 & 7 & 1 & 2 & 3 \end{array}$$

is one representation of a $PG(2, 2)$ satisfying the defining assumptions. The set of integers from 1 to 7, inclusive, may be considered as representing the points of a plane, and the subsets occurring in the columns may be considered as the lines of the plane. This arrangement is exhibited geometrically by means of the complete quadrangle, provided its diagonal points are considered to be collinear [**5**, p. 5].

(2.2). *Analytic Definition.* If x_1, x_2, x_3 are marks of a Galois field of order p^n, $GF[p^n]$, there are σ elements of the form (x_1, x_2, x_3), provided that (x_1, x_2, x_3) and $(\mu x_1, \mu x_2, \mu x_3)$ denote the same element when μ is any mark of the field other than zero, and provided that $(0, 0, 0)$ be excluded from consideration. These elements constitute a finite projective plane if the equation $u_1x_1+u_2x_2+u_3x_3=0$, the domain for coefficients and variables being the $GF[p^n]$, be taken as the equation of a line, except when $u_1=u_2=u_3=0$. The line is denoted by the symbol (u_1, u_2, u_3), and this symbol and $(\mu u_1, \mu u_2, \mu u_3)$, where μ is any mark other than zero, denote the same line. The points of a line are those points whose coordinates (x_1, x_2, x_3) satisfy its equation.

A representation of a finite projective geometry $PG(2, p^n)$ can always be found by using certain collineations of the form

$$(2.21)\qquad \rho x_j' = \sum_{i=1}^{3} a_{ji}x_i, \qquad \tau u_j' = \sum_{i=1}^{3} A_{ji}u_i, \qquad (j = 1, 2, 3),$$

where the coefficients a_{ji} are marks of the $GF[p^n]$, the coefficients A_{ji} are the cofactors of the elements a_{ji} in the determinant $|a_{ji}|$ of the transformation, and $|a_{ji}|\neq 0$. More specifically, the σ points of the $PG(2, p^n)$ may be transformed by the collineation

$$(2.22)\qquad \rho x_1' = \sum_{i=1}^{3} a_{1i}x_i, \qquad \rho x_2' = x_1, \qquad \rho x_3' = x_2.$$

This transformation, of determinant $|a_{ji}|=a_{13}\neq 0$, has the most general characteristic equation

$$(2.23)\qquad \Delta(\rho) \equiv -\rho^3 + a_{11}\rho^2 + a_{12}\rho + a_{13} = 0.$$

To obtain a representation of a $PG(2, p^n)$ such as (2.14) by means of a collineation [6] it is necessary and sufficient to choose the coefficients of (2.21) or (2.22) in such a way that the transformation be of order σ; so that the σ points of the plane are permuted cyclicly. For $n=1$, and p small, such transformations can

be found easily by experiment. In general, the a_{ji} of (2.22) may be chosen in the $PF[p^n]$ so that a root, ρ, of $\Delta(\rho)=0$ is a primitive root of the equation $\rho^\sigma=q$, where q is either unity or a primitive root of unity in the $GF[p^n]$. The order [2, p. 245] of (2.22) is then the least integer, x, for which $\rho^x=\rho^{x(\lambda-1)}=\rho^{x(\sigma-\lambda)}$, that is, for which $x(\lambda-2)\equiv x(p^n-1)$ is a multiple of σ, if $q=1$; or is equal to $\sigma(\lambda-2)\equiv\sigma(p^n-1)$ if q is a primitive root of unity in the $GF[p^n]$. If $p^n=3^n$ or is of the form $(3m-1)$, q may be chosen either as unity or as a primitive root of unity. But if p^n is of the form $(3m+1)$, then q must be a primitive root in the field. In either case, the a_{1i} may be so chosen that the order of (2.22) is σ. The problem of determining the a_{1i} to satisfy the preceding requirements is the problem of determining irreducible and primitive irreducible quantics of degree three in the $GF[p^n]$. A table of primitive irreducible quantics, $PIQ[m, p^n]$, for $p^n\leqq 169$, has been given by Bussey [7].

A representation of a $PG(2, 2)$, equivalent to the triple system (2.14), may be obtained by the use of the collineation

$$(2.24)\qquad \begin{aligned} \rho x_1' &= \phantom{x_1+{}} x_2+x_3, & \tau u_1' &= \phantom{u_1+u_2+{}} u_3,\\ \rho x_2' &= x_1 \phantom{{}+x_2+x_3}, & \tau u_2' &= u_1 \phantom{{}+u_2+u_3},\\ \rho x_3' &= \phantom{x_1+{}} x_2 \phantom{{}+x_3}, & \tau u_3' &= \phantom{u_1+{}} u_2+u_3. \end{aligned}$$

Arranging the points and lines horizontally and vertically, respectively, in the orders determined by (2.24), and indicating their incidences by an i, we have:

(2.25)

	1	2	3	4	5	6	7
·	(1, 0, 0)	(0, 1, 0)	(1, 0, 1)	(1, 1, 0)	(1, 1, 1)	(0, 1, 1)	(0, 0, 1)
(0, 0, 1)	i	i	.	i			
(1, 0, 1)		i	i		i		
(1, 1, 1)			i	i		i	
(1, 1, 0)				i	i		i
(0, 1, 1)	i				i	i	
(1, 0, 0)		i				i	i
(0, 1, 0)	i		i				i

(2.3). *Other Defining Conditions.* We let the σ points in the finite projective plane be designated by the integers of a complete residue system, modulus σ. We look for a representation of the plane configuration in a form similar to (2.14) with the σ columns of λ points each as the lines of the plane. If such an

arrangement can be found, the determination will obviously depend only upon finding a proper set of integers $x_1, x_2, \cdots, x_\lambda$ for the first column (or line). Each subsequent column is obtained by the addition of unity to the elements of the preceding column and then reducing, modulus σ, when necessary.

(2.31). THEOREM. *A necessary and sufficient condition that a set of integers* $x_1, x_2, \cdots, x_\lambda$ *may be used as the elements of a column in the representation of a* $PG(2, p^n)$ *such as* (2.14) *is that the* $\lambda(\lambda-1)=\sigma-1$ *differences* (x_i-x_j), $[i, j=1, 2, \cdots, \lambda]$, $i\neq j$, *be incongruent each to each, modulus* σ; *and, therefore, congruent in some order to the integers* $1, 2, \cdots, (\sigma-1)$, *modulus* σ.

For if two of the differences were congruent to each other, modulus σ, three of the elements of the column might be written as x_i, $x_i+\alpha$, $x_i+2\alpha$. Then the elements in the $(\alpha+1)$st column, and the same three rows, would be $x_i+\alpha$, $x_i+2\alpha$, $x_i+3\alpha$; and the two elements $x_i+\alpha$ and $x_i+2\alpha$, at least, would be contained in *two* different columns (lines) contrary to (2.12) of the defining relations. Hence the necessity of the theorem.

On the other hand, if the elements $x_1, x_2, \cdots, x_\lambda$ of a column satisfy the conditions of the theorem, no pair of elements can appear in two different columns. For if the pair x_i, x_j, did appear, for example, in both the $(\alpha+1)$st and $(\beta+1)$st columns, the corresponding elements in the first column would be $x_i-\alpha$, $x_i-\beta$, $x_j-\alpha$, $x_j-\beta$, and at least two of their differences would be congruent to each other, modulus σ, contrary to assumption. Since an element appears in λ different columns with $(\lambda-1)$ other elements in each, and no one twice, it then appears with each of the $\lambda^2-\lambda=\sigma-1$ other elements exactly once as required by (2.12). Defining assumption (2.11) is automatically satisfied. The points of the plane so determined are those lying on lines joining a point A of one line (column) to the $(\lambda-1)$ points (other than A) of any other line. The plane then consists of the $\lambda(\lambda-1)+1=\sigma$ distinct points, so that the lines satisfy assumption (2.13). Hence the theorem.

(2.32). THEOREM. *If the set of integers* $x_1, x_2, \cdots, x_\lambda$ *satisfies the conditions of theorem* (2.31), *so does the set* $kx_1, kx_2, \cdots, kx_\lambda$, *when* k *is prime to* σ.

For if a pair of the $(\sigma-1)$ differences kx_i-kx_j, $i\neq j$, were congruent to each other, modulus σ, since k is prime to the modulus, we would have a pair of the differences x_i-x_j congruent to each other, contrary to hypothesis.

3. *Conics*. For a $PG(2, p^n)$, a *point conic* is defined ([1], p. 256) as the locus of points of intersection of two projective non-perspective pencils of lines. A *line conic* is defined as the locus of lines joining corresponding points of two projective non-perspective ranges of points. The term *conic* is used to denote the self dual figure consisting of a point conic and its tangents. The number of points (lines) of a point (line) conic is equal to λ, the number of lines (points) in a pencil (range). That is, a conic consists of λ points and λ lines.

A point which is the intersection of two tangents is said to be an *outside*

point.* A point which is neither a point of a conic nor an outside point is said to be an *inside* point. The number of outside points is $(\lambda^2-\lambda)/2$, the number of combinations of tangents two at a time. Subtracting from $\sigma=\lambda^2-\lambda+1$, the number of inside points is found to be $(\lambda-1)(\lambda-2)/2$. By duality, the number of *secants* (meeting conic in two points) is $(\lambda^2-\lambda)/2$, and the number of lines not meeting the conic is $(\lambda-1)(\lambda-2)/2$. Each tangent meets each of the other $(\lambda-1)$ tangents in an outside point, by definition; so that each tangent has one point on the conic, and the remaining $(\lambda-1)$ points on it are outside points. A secant has $(\lambda-2)$ points not on the conic and meets $(\lambda-2)$ tangents in points not on the conic. Half of them are inside and half outside since two tangents meet in each outside point. As in ordinary projective geometry, it may be proved that a point (line) conic is the totality of points (lines) whose coordinates satisfy a homogeneous equation of the second degree.

4. *Some applications.* It has been proved [8] that the group theory of the $PG(3, 2)$ furnishes a complete solution to Kirkman's fifteen schoolgirl problem. Veblen [9] used finite geometries in considering the problem of color mapping.

We shall mention one further application. In the game of Nim [10, 11] a "safe" or winning combination is determined by first expressing the numbers of the counters in each of the three piles in the binary system of notation; $a_0 2^n + a_1 2^{n-1} + \cdots + a_n$. The number may be represented by the symbol $(a_0, a_1, \cdots, a_n)$, and the integers a_i are called the *binary digits* of the number. Then the combination is called "safe" provided that each of the $(n+1)$ sums of corresponding binary digits is congruent to zero, modulus 2. The k binary digits, of a number less than 2^k, may be considered as the homogeneous coordinates of a point in $PG(k, 2)$, provided some $a_i \neq 0$. Since three points are collinear if and only if their sets of coordinates are linearly dependent, a "safe" combination is one corresponding to the three points of a line in $PG(k, 2)$. The number of "safe" combinations with more than zero and less than 2^k counters in any one pile is thus seen to be the number of lines in the $PG(k, 2)$.

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* We here assume $p>2$. The case for $p=2$ requires particular treatment as might be expected for the even prime; see [4]. The possibility of there being three tangents through a point is excluded by the assumption that the two ranges of points are non-perspective.

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A NOTE ON THE PROBLEM OF ESTIMATION

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In a recent note, Keeping [1]* has considered the question of the best estimate of the standard deviation, σ, of an infinite population of variates. He states, "By the 'best' estimate is meant, as usual, that which gives a minimum standard deviation from the true value." With s representing the standard deviation in a sample of size n, he then points out that

$$\overline{(s-k\sigma)^2} < \overline{(s-\sigma)^2} < \overline{\left(\frac{s}{k}-\sigma\right)^2}, \text{ where } k = \sqrt{\frac{2}{n}}\left(\frac{n-2}{2}\right)! \Big/ \left(\frac{n-3}{2}\right)!,$$

and concludes that s is a better estimate of $k\sigma$ than of σ, but that s is a better estimate of σ than is s/k.

On the basis of the author's definition, of which more will be said later, the best estimate of σ is easily obtained by methods of elementary calculus. Let ms represent the best estimate of σ. Then $u=\overline{(ms-\sigma)^2}$ is to be a minimum. Using the author's method, we expand the left member, obtaining

$$\begin{aligned} u &= \overline{m^2s^2} - \overline{2ms\sigma} + \overline{\sigma^2} \\ &= \frac{m^2(n-1)\sigma^2}{n} - 2mk\sigma^2 + \sigma^2 \\ &= \sigma^2\left[\frac{m^2(n-1)}{n} - 2mk + 1\right]. \end{aligned}$$

Then

$$\frac{du}{dm} = \sigma^2\left[\frac{2m(n-1)}{n} - 2k\right],$$

whence we conclude that $m=nk/(n-1)$ gives a minimum value of u, and the best estimate of σ is $nks/(n-1)$. Likewise, the best estimate of $k\sigma$ turns out to be $nk^2s/(n-1)$, as we might expect.

Returning now to the author's definition of "best" estimate, we note that he omits the restriction, included by Neyman [2] and others, that the form of

* Numbers in brackets refer to references listed at the end of this paper.

the estimating function must be so chosen that it will have for its mean value the parameter to be estimated. He says [**2**, p. 564] "Suppose θ is a certain collective character of a population π and $x_1, x_2, \cdots x_n \cdots$ is a sample from this population. We shall say that a function of these x's, say $\theta' = \theta'(x_1, x_2, \cdots x_n) \cdots$, is a 'mathematical expectation estimate' $\cdots$ of θ, if the mean value of θ' in repeated samples is equal to θ. Further we shall say that the estimate θ' is the best linear estimate of θ if it is linear with regard to the x's, i.e. $\theta' = \lambda_1 x_1 + \lambda_2 x_2 + \cdots \lambda_n x_n + \lambda_0 \cdots$ and if its standard error is less than the standard error of any other linear estimate of θ.

"Of course, in using the words 'best estimate' I do not mean that the estimate defined has unequivocable advantages over all others. This is only a convention and, as long as the definition is borne in mind, will not cause any misunderstanding." This added restriction, regarding the mean value, would rule out both s and $nks/(n-1)$ but would allow s/k.

If θ' is normally distributed with mean θ, and θ_1' is a sample value of θ', then the best estimate of θ is θ_1'. But if we are dealing with a skew distribution, there seems to be no generally accepted definition of best estimate [**3**, **4**, **5**]. If the mean value of θ' is θ, the modal value $a\theta$, the median $b\theta$, and the square root of the mean value of the squares is $c\theta$, and the distribution of θ' is skewed—then, if a sample value, θ_1', is taken, we might prefer to select θ_1'/a or θ_1'/b or θ_1'/c rather than θ_1' as an estimate of θ. For example, if the mean value of θ' is θ, the mode 1.2θ, the median 1.1θ, and the square root of the mean variance is 1.04θ; and if a sample value of θ' is $\theta_1' = 4$, then we might conclude that $\theta_1' \doteq \theta$ and $\theta \doteq 4$; or $\theta_1' \doteq 1.2\theta$ and $\theta \doteq 4/1.2$; or $\theta_1' \doteq 1.1\theta$ and $\theta \doteq 4/1.1$; or $\theta_1' \doteq 1.04\theta$ and $\theta \doteq 4/1.04$. Some one of these would be used by the observer but no one of them would give the exact value of θ.

In his book, An Introduction to Mathematical Probability, J. L. Coolidge devotes several pages (pp. 101–113) to a discussion of the determination of "best" value. He is concerned with the question of what value to take as the best estimate of a quantity after a series of measurements of the quantity have been made. He concludes, "When all of the measurements are equally trustworthy, the best value is their average." But he observes, later, " $\cdots$ there arise cases where the observed values group themselves somewhat asymmetrically about the average, and the question arises whether it be not well to take a best value which will minimize some other function of the observed measurements." These remarks have to do with quantities which are directly examined, as opposed to situations where only certain functions of the quantities are open to observation. In this latter case, Coolidge postulates (page 152), "The best values for the unknowns are those which will give a maximum value to the probability of obtaining just this series of measurements." This postulate may be understood to be the concept of "maximum likelihood," as stated by R. A. Fisher [**6**] in 1922, and developed and extended by Fisher and others [**7**, **8**] in subsequent articles.

It is needless to give further references to show lack of uniformity. Whether

we agree to define estimates obtained by the method of maximum likelihood as being "best" estimates, or prefer some other convention, it appears that, for some time to come, all estimates labeled "best" estimates, need some explanation of how they have been obtained. Otherwise, the reader may quite misunderstand the author's meaning. In any case it might be well to consider the desirability of relegating "best," along with "satisfactory" and "appropriate," to the limbo of non-statistical terms.

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"There is an astonishing imagination, even in the science of mathematics .... We repeat, there was far more imagination in the head of Archimedes than in that of Homer." Voltaire. A Philosophical Dictionary, Boston, 1881, vol. 3, p. 40. Article "Imagination."

"Pure mathematics consists entirely of such asseverations as that, if such and such a proposition is true of *anything*, then such and such another proposition is true of that thing. It is essential not to discuss whether the first proposition is really true, and not to mention what the anything is of which it is supposed to be true. ... If our hypothesis is about *anything* and not about some one or more particular things, then our deductions constitute mathematics. Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true." Bertrand Russell, International Monthly, vol. 4, 1901, p. 84.

QUESTIONS, DISCUSSIONS, AND NOTES

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The department of Questions, Discussions, and Notes in the Monthly is open to all forms of activity in collegiate mathematics including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A THEOREM ON MATRICES

By W. E. ROTH, University of Wisconsin Extension Division

Sylvester's theorem, which states that the characteristic values of AB are the same as those of BA, is well known.* The present note is concerned with the proof of a related theorem, which follows:

The sufficient condition that the elementary divisors of $AB-\lambda I$ be the same as those of $BA-\lambda I$, where A and B are square matrices of order n, is that either A or B be non-singular.

If A is non-singular,

$$A^{-1}(AB-\lambda I)A = BA - \lambda I;$$

or if B is non-singular,

$$B(AB-\lambda I)B^{-1} = BA - \lambda I.$$

Hence in case either A or B is non-singular the theorem is proved.

If, however, both A and B are singular matrices, then the equation

$$P(AB-\lambda I)Q = BA - \lambda I, \tag{1}$$

where P and Q are non-singular matrices, is not necessarily possible. To show that such is the case it suffices to exhibit a pair of singular matrices A and B such that (1) cannot hold. One such pair is the following:

$$A = \begin{pmatrix} 1, & -1 \\ 0, & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1, & 1 \\ 1, & 1 \end{pmatrix}.$$

Here $AB-\lambda I = \left(\begin{smallmatrix} -\lambda, & 0 \\ 0, & -\lambda \end{smallmatrix}\right)$ has two linear elementary divisors λ, λ; whereas $BA-\lambda I = \left(\begin{smallmatrix} 1-\lambda, & -1 \\ 1, & -1-\lambda \end{smallmatrix}\right)$ has the elementary divisor λ^2. Therefore AB and BA are not equivalent and (1) does not hold for them.

The condition of the theorem above is not also necessary in that singular matrices A and B such that $AB-\lambda I$ and $BA-\lambda I$ have the same elementary divisors do exist; for example, all commutative matrices A and B whether singular or not have the property.

* A proof of this theorem by Thurston appeared in the MONTHLY, vol. 38, 1931, pp. 322–4. His proof in case both A and B are singular is not entirely valid.

The Taylor Series Approximation Curves for the Sine and Cosine

By Norman Miller, Queen's University

The first ten approximation curves for sin x and for cos x, drawn by Helen M. Kammerer and reproduced in the Monthly in May, 1936, suggest some questions regarding the higher approximation curves. Each approximation curve forms a good fit to its limit curve for some distance, after which it diverges rapidly above or below. The real roots of the approximation curves, with the possible exception of the numerically greatest roots of each, lie close to corresponding roots of the limit curve. An idea of the interval within which any approximation curve is a good fit and of the number of its real roots may be obtained in a simple manner.

We shall confine attention to sin x and its approximations,

$$S_n(x) = \sum_{k=1}^{n} (-1)^{k+1} \frac{x^{2k-1}}{(2k-1)!}, \qquad n = 1, 2, 3, \cdots .$$

In the figures for the first ten approximation curves it appears that when x has increased to a point where the difference of the ordinates of S_n and S_{n-1} is as much as 1, S_{n-1} is already diverging rapidly and is no longer a good fit to the sine curve and as x increases further S_n begins to diverge rapidly. These intuitive facts suggest that we take as a tentative definition, that $S_n(x)$ is a good approximation to sin x so long as $|S_n(x)-S_{n-1}(x)|<1$, that is, so long as $x<\bar{x}=\sqrt[2n-1]{(2n-1)!}$. If, for example, $n=50$, $\bar{x}=\sqrt[99]{99!}=37.62$. If we use instead the inequality $|S_{50}-S_{49}|<\epsilon$ we get $x<\sqrt[99]{99!\epsilon}$. If $\epsilon=0.1$ this gives $x<36.76$ and if $\epsilon=10$, $x<38.51$. This means that S_{50} diverges from S_{49} a vertical distance of 0.9 in an interval of 0.86 on the x-axis and diverges 9 in the adjacent interval of 0.89. The example provides further justification of our tentative definition.

Factorial tables are available for computing the value of $\bar{x}$ for values of n up to 500.* Thus, for $n=500$, $\bar{x}=369.1$. For larger values of n we may approximate by means of the limit†

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}.$$

This fact may be written $\sqrt[n]{n!}\cong n/e$, whence $\bar{x}\cong(2n-1)/e$. For $n=500$, $(2n-1)/e=367.5$, which shows that $\bar{x}$ found from the formula $(2n-1)/e$ for $n>500$ will be accurate to within 1.6, and the accuracy increases with n. Again, for large values of n the curve S_{n+1} follows the limit curve closely (within the meaning of the above definition) for a horizontal distance of $2/e$ farther than does S_n. Even for small values of n this estimate is not far out.

* Karl Pearson, Tables for Statisticians and Biometricians, London, Third edition, 1930, Part 1, p. 98.

† Konrad Knopp, Theory and Application of Infinite Series, p. 71.

The roots of sin x are all real and distributed at intervals of π along the x-axis. From the figure it appears that S_{10} has 7 real roots and since S_{10} is of degree 19 it has 12 imaginary roots. In general, since the real roots of S_n approximate closely those of sin x as long as S_n is a good approximation, an estimate for the number of real positive roots of S_n is the nearest integer to $\bar{x}/\pi$. This estimate is probably not in error by more than 2. Again, for large values of n the number of real positive roots is of the order* $[(2n-1)/\pi e+\frac{1}{2}]$ and the total number of real roots is of the order $2[(2n-1)/\pi e+\frac{1}{2}]+1=R$, say. The number of imaginary roots differs little from $2n-1-2[(2n-1)/\pi e+\frac{1}{2}]-1=I$, say. Hence $\lim_{n\to\infty} I/R=\pi e/2-1=3.27$, to three figures. As n becomes infinite the approximation curve comes to have nearly 3.27 times as many imaginary as real roots. In the complex plane the imaginary roots recede farther and farther from the origin as $n\to\infty$ which accords with the fact that sin x has only real roots and has an essential singularity at the point infinity.

When n increases by 1 the total number of real roots of $S_n(x)$ either increases by 2 or decreases by 2. This appears from the graph and also by consideration of the imaginary roots. Since the coefficients of $S_n(x)$ are real and the function is odd the imaginary roots of $S_n(x)$ occur in the complex plane in groups of four which are symmetrical with respect to both the axis of reals and of pure imaginaries. Hence as n increases by 1 there are two possibilities, (a) the number of real roots increases by 2 and the number of imaginary roots remains constant, (b) the number of real roots decreases by 2 and the number of imaginary roots increases by 4. We may determine for large values of n the ratio of the numbers of times that the possibilities (a) and (b) are realized. Take $S_1=x$ and consider $n-1$ unit increases of the subscript. Suppose that in p of these (a) is realized and in $n-1-p$, (b) is realized. We obtain for the number of real roots of S_n, $R=1+2p-2(n-1-p)=-2n+4p+3$ and for the number of imaginary roots, $I=4(n-1-p)$. Hence $I/R=(4n-4p-4)/(-2n+4p+3)$. But $\lim_{n\to\infty} I/R=\pi e/2-1$, whence it follows that $\lim_{n\to\infty} p/n=(\pi e+2)/2\pi e=0.62$ to two figures. Hence for large values of n possibility (a) is realized in approximately 0.62 of all cases.

Notes from a Freshman Algebra Class

By B. D. Roberts, New Mexico Normal University

The writer is one of that group of mathematicians interested primarily in teaching but still having an interest in research, even though little time is available to devote to research. In this locality the general public and incoming students in particular seem so possessed by the idea that mathematics is a cut and dried subject, all contained within the covers of a book or two, that there is a distinct challenge for the teacher to show continually that there is a frontier which may be approached and even pushed back a little from almost any point of departure, even in a Freshman class. This point of view and its concomitant

* The nearest integer to a positive number p is the greatest integer which is $\leqq p+\frac{1}{2}$ and will be denoted by $[p+\frac{1}{2}]$.

method of procedure stimulates interest in the classroom and occasionally develops in a student the attitude and spirit of research which is carried on by him after leaving the undergraduate classroom; thus enabling the teacher to have at least a vicarious part in the extension of his favored field.

The following example of what is meant is submitted as a type of result that can be achieved with even mediocre students.

In the Algebra text used the following two problems are given to be proved by induction:

$$\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots \text{ to } n \text{ terms } = \frac{n}{2(n+2)}. \tag{1}$$

$$\frac{1}{4.5}+\frac{1}{5.6}+\frac{1}{6.7}+\cdots \text{ to } n \text{ terms } = \frac{n}{4(n+4)}. \tag{2}$$

The similarity of these forms was noted and the question asked, whether a series of this type might be started with any term. This led directly to the formulation of the problem:

To prove:

$$\begin{aligned}\frac{1}{m(m+1)}&+\frac{1}{(m+1)(m+2)}+\frac{1}{(m+2)(m+3)}\\ &+\cdots \text{ to } n \text{ terms } = \frac{n}{m(n+m)}.\end{aligned} \tag{3}$$

The verification of this form seemed to be independent of any restriction as to m being an integer. The use of fractional values for m, at first numerical but directly the more general p/q, resulted in the form:

$$\frac{1}{p(p+q)}+\frac{1}{(p+q)(p+2q)}+\cdots \text{ to } n \text{ terms } = \frac{n}{p(p+nq)}. \tag{4}$$

This seemed a bit more general than (3), in that the difference between the two factors of the denominators had been made an arbitrary integer.

At this point the class period had been exhausted, but sufficient interest had been aroused that a bit more work was done individually. Having moved from integers to fractions, the next logical step was to question irrational possibilities for p and q. No new form resulted; (4) being valid, however, for the extension. One daring soul, that rare joy of the teacher's heart, had the courage to take the next logical step and try imaginary values for p and q. Conjugate values, $a\pm bi$, were used and upon equating the real and the imaginary parts of the derived form there resulted, for the particular case $a=b=1$:

$$\frac{1}{1\cdot 2}+\frac{3}{2\cdot 5}+\cdots+\frac{2n-1}{[(n-1)^2+1](n^2+1)}=\frac{n^2}{n^2+1}$$

and

$$\frac{1}{2\cdot 5}+\frac{5}{5\cdot 10}+\cdots+\frac{n^2+n-1}{(n^2+1)[(n+1)^2+1]}=\frac{n^2}{2[(n+1)^2+1]}.$$

The more general foımulae in terms of general a and b were a bit too complex though not impossible. The above particular ones seemed a substantial step from the original forms and, as there was no particulaı need for the end products and the desired interest and attitude had been achieved, no further efforts were elicited. No effort was made to check whether or not these forms were new; to the class and the individuals they were new and the benefits of the work cannot be challenged on the ground that the results here found were already existent elsewhere.

Note by the Editor: The fact that the whole discussion can be simplified by observing that

$$\frac{1}{1\cdot 2}=1-\frac{1}{2},\qquad \frac{1}{2\cdot 3}=\frac{1}{2}-\frac{1}{3},\ \text{etc.}$$

really has no bearing on the pedagogical point involved. R.E.G.

RECENT PUBLICATIONS

EDITED BY W. R. LONGLEY, Yale University

All books for review should be sent directly to the editor of this department, at the American Mathematical Monthly, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Algebra for College Students: By Harley R. Willard and Noah R. Bryan, New York, Scott, Foresman and Company, 1936. 7+383 pages. $2.00.

Professors Willard and Bryan have written an Algebra which should be very satisfactory for the classroom although it conforms closely to custom in selection and treatment of topics. The book contains enough material to enable a particular teacher to select topics for the course which best fits the ability and preparation of his class. The first few chapters cover material frequently included in secondary school work. Exponents and logarithms are wisely discussed in the same chapter. There follow, among others, chapters on progressions, mathematical induction, theory of equations, permutations, determinants, complex numbers, interest and annuities and probability. For reasons unknown to the reviewer mathematical induction is difficult for Freshmen and Professors Willard and Bryan have not achieved the miraculous by eliminating this difficulty. Their statement (p. 192) "the theorem is verified for $n=1$, $n=2$, and perhaps $n=3$" is needlessly confusing and complicating. The statement on p. 55 that "any quantity with a zero exponent shall equal unity" should of course be corrected. The derivative is defined and discussed slightly, and the ellipse and

hyperbola are mentioned by name in connection with systems of quadratic equations. As the authors state the book should be useful in an introductory, an intermediate, or an advanced course.

DEANE MONTGOMERY

Gli Elementi d'Euclide e la Critica Antica e Moderna. Edited by Federigo Enriques col concorso di diversi collaboratori. Libri XI–XIII. Bologna, 1936, 355 pages, 30 lire.

This work is No. 11 of a series, Per la Storia e la Filosofia delle Matematiche edited by Professor Enriques and including translations of such works as those by Heiberg, Newton, Ruffini, Dedekind, Clairaut, and Euclid—a series therefore of great importance.

A brief review of the first volume (libri I–IV) of this edition of Euclid and a more extended one of volumes II and III (libri V–X) appeared in this MONTHLY in 1926 and 1935 (XXXIII, 383, and XLII, 442–3). In these reviews the nature of this publication was set forth and a comparison was made with Sir Thomas Heath's edition of Euclid through Book X. It is therefore unnecessary at this time to dwell upon any details of selection of material, of historical content, or of comments on the conventional propositions, except as they relate to the subject of the contents of the last three Books of the Elements.

These Books relate to what is often called "solid geometry," or more appropriately "geometry of three dimensions." As with other volumes in this series the work is edited by Professor Federigo Enriques of the University of Rome, a scholar of international reputation, but the translation from the Greek is due to Dr. Maria Teresa Zapelloni, with the assistance of Signor Attilio, a well-known engineer and classical scholar. The specific work of editing Books XI, XII, and XIII is the work of Amedeo Agostini, who was also the editor of Book IV, and Professor Enriques also expresses his appreciation of the advice and assistance given by Professor Giovanni Vacca of the University of Rome.

This volume is, in one respect, more important than its predecessors, since it contains a commentary on Books XI–XIII, which are not often included in the textbooks of Euclid, as well as notes on the various definitions and propositions. It begins with an introduction to the historical development of solid geometry (pp. 9–19) by the Greeks, Arabs, and Europeans. This introduction, together with a brief bibliography, will be found helpful to teachers who read Italian, but it is by no means complete. The most important historical material, however, is found in the comments upon definitions, tracing the development of the terms from ancient to modern times, the propositions being treated in a similar way. Such theorems as those on the "three round bodies," for example, are treated with care, although not with the completeness found in Sir Thomas Heath's Euclid, volume III. As in the latter work, the so-called Books XIV and XV are briefly mentioned with notes upon the references made to them by the post-Euclidean writers. Book XIV was apparently due to Hypsicles (*c.* 130 B.C.) and Book XV to Isidorus of Miletus, the architect of the church of Santa Sophia (*c.* 532 A.D.).

While this, as well as the first two volumes, is a valuable contribution to the works of Euclid, it can hardly be ranked as the equal of volume III of Heath's monumental treatise. It closes with an elaborate index of authors, covering all three volumes, and a brief index to the third.

It will not be out of place to say a word about the interest shown for many years by Italian scholars in the history of mathematics. No country has contributed more profusely, and indeed few countries have so much material relating to the mathematics of the middle ages and the renaissance. To such writers as Frisi, Baldi, Cossali, Boncompagni, Libri, Favaro, Loria, Bortolotti, and various others, we are much indebted.

DAVID EUGENE SMITH

MATHEMATICS CLUBS

EDITED BY F. W. OWENS AND HELEN B. OWENS, State College, Pa.

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to F. W. Owens, 462 East Foster Ave., State College, Pa.

CONTEST PAPERS

Lack of space precludes publication in full of the interesting papers submitted to this department. It is hoped the following reviews will bring these papers to the attention of other students of mathematics. Two other papers were reviewed in the January MONTHLY.

III. GRAPHIC REPRESENTATION OF COMPLEX ROOTS

By JOSEPH HILSENRATH, New Jersey State Teachers College

This interesting paper is based largely on the paper by A. F. Frumveller, "The graph of $F(x)$ for complex numbers," this MONTHLY, vol. 24, 1917, pp. 409–420, and one by Sampachi Fukuzawa, Tokio, 1907, "Vier mathematische Abhandlungen," the first of the four entitled "Geometrische Darstellung der Komplexe werte besitzenden Funktionen eines reellen Arguments" bearing on this subject. A graph of $y=kx^2+mx+n$ is devised to illustrate the roots of the quadratic $kx^2+mx+n=0$ when $m^2-4kn\leqq 0$. Similar graphs are made for the cubic $y=x^3-4x+15$ and the quartic $y=x^4+16$, showing the y's for complex values of x, as well as reals, and hence giving graphical interpretation for the complex roots. Finally graphs are developed which show the imaginary intersections of two circles in a plane having no real points of intersection.

IV. DERIVATION OF CERTAIN FORMULAS FOR FINDING THE AREA OF A TRIANGLE OR THE VOLUME OF A TETRAHEDRON

By RICHARD SCHAFTER, University of Buffalo

This is a very clearly written paper in which the author develops by the methods of analytical geometry formulas for the area of a triangle and for the volume of a tetrahedron in terms of the coefficients in the equations of the lines forming the sides of the triangle and the planes of faces of the tetrahedron, respectively. Among the various forms derived are the following:

If the sides of a triangle in a plane are $a_ix+b_iy+c_i=0$, $(i=1, 2, 3)$, the area K of the triangle is given (numerically) by

$$2K = \frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}.$$

If $A_ix+B_iy+C_iz+D_i=0$, $(i=1, 2, 3, 4)$, are the equations of the planes of its faces, the volume V of a tetrahedron is given (numerically) by

$$6V = \frac{\begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix}}{\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_4 & B_4 & C_4 \end{vmatrix} \begin{vmatrix} A_1 & B_1 & C_1 \\ A_3 & B_3 & C_3 \\ A_4 & B_4 & C_4 \end{vmatrix} \begin{vmatrix} A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \\ A_4 & B_4 & C_4 \end{vmatrix}}.$$

The author notes that the latter has been derived by a different procedure by Robert J. T. Bell in his "Coordinate Geometry of Three Dimensions," Macmillan & Co., pp. 66–67. This paper won the Wilfred H. Sherk Memorial Prize.

A NEW GAME

The last census of the United States lists over two thousand occupations. The man on the street will contend that few of these make use of more mathematics than the four fundamental operations of arithmetic. In how many occupations can you give illustrations to convert this man? This is an idea for a new and amusing contest for your club meetings.

Illustration: Housewife buying oranges meets this problem—oranges three inches in diameter are offered at 35 cents a dozen, those four inches in diameter at 50 cents a dozen. Which will give most juice for one cent?

Club Reports
1935–36

Nicholson Mathematics Club, Louisiana State University

President, Maralena White; Vice President, G. Walker; Secretary-Treasurer, Elizabeth Freas. This club, listed last summer as the Louisiana State University Mathematics Club has adopted its new name in honor of Colonel J. W. Nicholson, former president of the University and for forty years the head of its mathematics department. The publication, Math Club News, contains eight to twelve pages of reports of papers read at meetings, activities of the club and questions for general discussion. During the year it was frequently enlivened by portraits and illustrations from the clever pen of F. A. Rickey. A study was made of the history of mathematics in the United States, with special meetings devoted to the lives and works of J. J. Sylvester, B. Peirce, W. F. Osgood, and E. H. Moore. The last number of the year announces the installation at Louisiana State University of a chapter of Kappa Mu Epsilon, the fourteenth chapter of this national mathematics fraternity. This will be distinct from the club but will not supersede it.

The Mathematics Club, Wellesley College

Last spring appeared in this MONTHLY the play "Modern mathematics looks up his ancestors," written by Marion E. Stark for this group last year. In addition to the fun of presenting the play the members held five meetings with discussions of "Women in mathematics"; "27 triangles formed by internal and external trisectors of an angle"; "Cycloids"; "Inverse points"; "Sections of an hyperboloid," and a supper meeting to close a successful year.

White Mathematics Club, University of Kentucky

President, F. Donaldson; Faculty Adviser, Professor M. C. Brown. Monthly meetings were held with the following programs: "Mathematical nuts and short cuts"; "Teaching of mathematics in German and English universities"; "Why mathematics?"; "Extraction of roots and geometrical interpretation of the binomial theorem"; "Astronomical folklore"; "A problem in geo-physics."

Mathematics Club, New York University, Washington Square College

President, N. Steinman; Vice President, Selma Siegel; Secretary, Helen Rice; Treasurer, S. Rothstein; Editor of Magazine "X," S. Mandelbaum; Chairman of coaching, Naomi Rosenstein. The weekly meetings, besides serious mathematics talks, included two dances, some teas, and a boat ride. Besides continuing the unique plan of student coaching, the club published three numbers of the magazine "X" which were sold throughout the school.

Kappa Mu Epsilon of Eastern Illinois State Teachers College

President, Wilma Nuttall; Vice President, C. Elam; Secretary, Clara Balmer; Treasurer, Winifred Gillum; Corresponding Secretaries, H. H. Heller and F. Allen. Each initiate must prepare and read a paper before the Mathematics Club. The chapter held two banquets and a tea. The programs at regular meetings included: "Comparison of mathematics in secondary schools of Germany and the United States"; "The brachistochrone"; "Methods of observing, calculating and charting a meteor"; "The fourth dimension."

Mathematics Club of the University of Kansas

President, J. K. Hitt; Vice President, Millicent Robinson; Secretary-Treasurer, Dorothy Jones; Faculty Adviser, Professor W. Babcock. The club held business meetings, an annual picnic, and eleven meetings with the following topics for discussion: "Trisection of the angle"; "Game of Nim"; "Determination of astronomical distance"; "Various proofs of the Pythogorean theorem"; "Circulating decimals"; "The calendar"; "Approximations"; "Calculating machines"; "Flatland"; "Light polarization"; "Visual aspects of the theory of relativity."

Pi Mu Epsilon of the University of Kansas

The chapter assists in the activities of the Mathematics Club. Besides its initiation meetings it sponsored a lecture by Dean R. W. Babcock of Kansas State College on "Orthogonal functions" and one by Helen Welch on "Inversion."

"It is true that mathematics, owing to the fact that its whole content is built up by means of purely logical deduction from a small number of universally comprehended principles, has not unfittingly been designated as the science of the *self-evident* (Selbstverständlichen). Experience however, shows that for the majority of the cultured, even of scientists, mathematics remains the science of the *incomprehensible* (Unverständlichen). Alfred Pringsheim. Über Wert und angeblichen Unwert der Mathematik, Jahresbericht der Deutschen Mathematiker Vereinigung, 1904, p. 357.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 259. *Proposed by Mannis Charosh, New Utrecht High School, Brooklyn, N. Y.*

If the tangents of the angles of a plane triangle form an arithmetic progression, prove that the Euler Line is parallel to a side of the triangle.

E 260. *Proposed by C. E. Springer, University of Oklahoma.*

Two lines AB and CD of given lengths slide independently along two fixed skew lines. Show that the locus of the center of the sphere through A, B, C and D is a hyperbolic paraboloid.

E 261. *Proposed by V. Thébault, Le Mans, France.*

Find the smallest possible base for a system of enumeration which contains three-digit squares of the form *aaa*, and six-digit squares of the form *bcbcbc*.

E 262. *Proposed by Cezar Coşniţă, Roumanian Mathematical Institute.*

Find the locus of the center of a circle which so varies that its radical axes with two fixed circles pass always respectively through two fixed points.

E 263. *Proposed by D. L. MacKay, Evander Childs High School, New York.*

In the triangle ABC, the bisector of angle A, the median from vertex B, and the altitude from vertex C, are concurrent. Show that the triangle may be constructed with ruler and compasses if the lengths of sides b and c are given.

E 264. *Proposed by N. A. Court, University of Oklahoma.*

Given four spheres, (A), (B), (C) and (D), with non-coplanar centers. The sphere (AB) is constructed coaxal with [i.e. having same radical plane as] the spheres (A) and (B), and passing through the given point P. The spheres (AC), $\cdots$ (CD) are similarly constructed. Show that the six spheres thus obtained have their centers in the same plane.

E 265. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

A right triangle has integer sides without any common factor. When each digit is replaced by a code letter, the sides are $SSWTVU$, $PTWTS$ and $RRWWQ$. Solve the code and show that the solution is unique.

is either two or three. It follows that $f=9$, so the complete division may be reconstructed as follows:

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484)3527876(7289
    3388
    ----
     1398
      968
     ----
      4307
      3872
      ----
       4356
       4356
       ----
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Also solved by T. E. Berry, W. E. Buker, Mary L. Constable, Wm. Douglas, Daniel Finkel, C. E. Kirkwood, Jr., Helen T. Raudenbush, E. P. Starke, Simon Vatriquant, B. C. Zimmerman and the proposer.

E 218 [1936, 373]. *Proposed by V. Thébault, Le Mans, France.*

Find the unique number N whose digit pattern is $aabbc$, where $b=c+1$, and such that N^2 contains just the nine digits, 1, 2, 3, 4, 5, 6, 7, 8 and 9 once each.

Solution by B. C. Zimmerman, Corozal, B. H.

$N=aabbc=aa110+ccc=110(100a+1)+111c$, which must be a multiple of 3 because N^2 contains each digit just once and is thus a multiple of 9. Hence since $N^2<10^9$, $a=2$, and $2<c<b$. Thence $N^2\geqq(22110+333)^2>5\cdot 10^8$. Since N^2 contains no zero, $N^2>51\cdot 10^7$. Because $N^2=(22110+111c)^2>51\cdot 10^7$, $c>4$. Moreover, $c\neq 5$, for then N^2 would both begin and end with a 5. Also $c\neq 9$, since $c<b$. Therefore $c=6$, 7 or 8. The corresponding values of N are 22776, 22887 and 22998. Of these, 22998 is ruled out because its square ends in 04, and zero is taboo. $22887^2=518746176$, which contains duplicate digits. There remains only 22887, whose square, 523814769, satisfies the given conditions uniquely.

Also solved by C. A. Barnhart, W. E. Buker, Mary L. Constable, Daniel Finkel, E. A. Nordhaus, E. P. Starke, W. R. Talbot, C. W. Trigg and the proposer.

E 219 [1936, 373]. *Proposed by Cezar Coşniţă, Roumanian Mathematical Institute.*

Find the locus of a point which so moves that its distances from the fixed point Q and from the fixed skew lines r and s are all three equal.

Solution by C. E. Springer, University of Oklahoma.

Let Q have the coordinates (a, b, c), and let r and s be represented by $x \tan\phi - y = 0 = z-k$, and $x\tan\phi+y=0=z+k$ respectively. Then if (u, v, w) is the moving point, we have

$$(u-a)^2+(v-b)^2+(w-c)^2$$

$$=\begin{vmatrix} v & w-k \\ \sin\phi & 0 \end{vmatrix}^2+\begin{vmatrix} w-k & u \\ 0 & \cos\phi \end{vmatrix}^2+\begin{vmatrix} u & v \\ \cos\phi & \sin\phi \end{vmatrix}^2$$

$$=\begin{vmatrix} v & w+k \\ -\sin\phi & 0 \end{vmatrix}^2+\begin{vmatrix} w+k & u \\ 0 & \cos\phi \end{vmatrix}^2+\begin{vmatrix} u & v \\ \cos\phi & -\sin\phi \end{vmatrix}^2.$$

Omitting in turn the first and second members of this equality and simplifying, we obtain $uv\sin\phi\cos\phi+kw=0$, and $u^2\cos^2\phi+v^2\sin^2\phi-2(au+bv+cw)+a^2+b^2+c^2-k^2=0$.

The desired locus is then the fourth degree curve of intersection of the two quadric surfaces, $xy\sin 2\phi+2kz=0$, and $x^2\cos^2\phi+y^2\sin^2\phi-2(ax+by+cz)+a^2+b^2+c^2-k^2=0$. The first is a hyperbolic paraboloid having the z-axis as its axis of symmetry and its generating lines respectively perpendicular to the x- and y-axes. The second surface is an elliptic paraboloid whose axis of symmetry is parallel to the z-axis, and whose axial planes are respectively perpendicular to the x- and y-axes.

E 220 [1936, 373]. *Proposed by C. W. Trigg, Cumnock College, Los Angeles.*

If circles be constructed on the sides of a triangle as diameters, show that (a) the common tangent to the circles on two of the sides is the mean proportional between the segments into which the third side is divided by the point of contact of the incircle; and (b) the area of the triangle is equal to the square root of the product of the three common tangents and the semi-perimeter.

Solution by Clair T. Grastorf, Colgate University.

Let the sides of the triangle be a, b and c. It is known that the side a is divided by the point of tangency of the incircle into the parts $s-b$ and $s-c$, where s is the semi-perimeter. Denote by t_{bc} the common tangent to the circles constructed on the sides b and c. The problem is now to show that $t_{bc}^2=(s-b)(s-c)$, and similarly for the other tangents. Let M_b and M_c be the mid-points of sides b and c respectively. Assume t_{bc} drawn, draw radii from M_b and M_c to the respective points of tangency. Also join M_b with M_c. Through M_b draw a line parallel to t_{bc}, thus forming a right triangle whose hypotenuse, M_bM_c, equals $a/2$, and whose legs are equal to t_{bc} and to the difference between the radii of the two circles, or $(c-b)/2$. Consequently, $t_{bc}=\frac{1}{2}[a^2-(c-b)^2]^{1/2}=[(s-b)(s-c)]^{1/2}$. It follows from symmetry that $t_{ca}^2=(s-c)(s-a)$ and that $t_{ab}^2=(s-a)(s-b)$, which proves the first part of the proposed problem.

The second part follows immediately, because $st_{bc}t_{ac}t_{ab}=s(s-a)(s-b)(s-c)$ and taking square roots gives Hero's formula for the area of the triangle.

Also solved by W. E. Buker, Mannis Charosh, W. B. Clarke, Wm. Douglas, D. L. MacKay, C. E. Springer, E. P. Starke, Herbert Tate and the proposer.

E 221 [1936, 373]. *Proposed by E. P. Starke, Rutgers University.*

Let the real numbers u_j, $j=1, 2, \cdots n$, be the roots of the nth degree equation $f(u)=0$; and let $i^n f(i)=P+iQ$, $i^2=-1$. Show that

$$\tan \sum_{j=1}^{n} \arctan u_j = \frac{Q}{P}.$$

Solution by J. M. Feld, New York City.

Evidently

$$i^n f(i) = i^n a_0 \prod_{j=1}^{n} (i - u_j) = (-1)^n a_0 \prod_{j=1}^{n} (1 + iu_j) = P + iQ.$$

Assuming, as we may, that a_0 is positive, and equating the logarithms of the last two members of this equality, we obtain

$$ni(2k+1)\pi + \log a_0 + \sum_{j=1}^{n} \log |1 + iu_j| + i \sum_{j=1}^{n} (\arctan u_j + 2\pi c_j).$$
$$= \log |P + iQ| + i(\arctan Q/P + 2\pi c).$$

Now equating the coefficients of i in this equation, we find that

$$m\pi + \sum_{j=1}^{n} (\arctan u_j) = \arctan Q/P,$$

whence, by taking tangents, the required formula at once appears.

Also solved by W. B. Campbell, E. A. Nordhaus, C. E. Springer and the proposer.

E 222 [1936, 373]. *Proposed by E. L. Harp, Jr., Roswell High School, New Mexico.*

A filling station proprietor finds his brand of gasoline gives one mile more per gallon than that of a competitor. Supposing that the competitor's gasoline costs 17 cents per gallon to the owner's price of 18 cents per gallon, what sales talk can the proprietor present to his prospective customers? Generalize.

Solution by W. E. Buker, Leetsdale High School, Pennsylvania.

Suppose that A and B sell gasoline at $x+p$ and x cents per gallon, respectively, and suppose, moreover, that the mileages gotten therefrom are $y+q$ and y miles. The cost of traveling a mile, using A's gasoline, is $(x+p)/(y+q)$ cents, and the cost with B's gasoline is x/y cents. Therefore, if $py<qx$, it will be possible to travel more economically using A's gasoline, while if the inequality is reversed, the use of B's gasoline would be advantageous.

In this problem, x, p and q are respectively 17, 1 and 1, so A should be able to persuade anyone to become his customer who has been getting less than 17 miles with a gallon of B's gasoline.

Also solved by L. J. Adams, E. A. Nordhaus, B. C. Zimmerman and the proposer.

E 223 [1936, 373]. *Proposed by Virgil Claudian, Roumanian Mathematical Institute.*

Solve the equation, $x^9+9px^7+27p^2x^5+30p^3x^3+9p^4x+q=0$.

Solution by E. P. Starke, Rugers University.

The proposed equation is transformed by the substitution, $x=z-p/z$, into $z^9-(p/z)^9+q=0$, or $z^{18}+qz^9-p^9=0$. Hence z is any one of the eighteen values indicated by $[-q/2\pm(q^2+4p^9)^{1/2}/2]^{1/9}$.

Let A be one value of this expression, and let B be $-p/A$. Since $A^9B^9=-p^9$ (the product of the roots of the quadratic in z^9), B is that one of the values of z for which AB is real. Then the roots x are easily seen to be $A+B$ and Ar^j+Br^{9-j}, where $j=1, 2, \cdots 8$; and $r^9=1$.

This is case $k=9$, of an equation discussed by L. E. Dickson in connection with "Waring's Formula" in the text, "First Course in the Theory of Equations" p. 139. $k=3$ gives Cardan's solution of the cubic; $k=5$ solves De Moivre's quintic; analogously, $k=7$, the solution of $x^7+7px^5+14p^2x^3+7p^3x+q=0$ is given by $A+B$, where $2A^7=-q+(q^2+4p^7)^{1/2}$ and $2B^7=-q-(q^2+4p^7)^{1/2}$, subject to the condition that AB is real.

Also solved by C. H. Graves, H. L. Krall and F. A. Reiber.

E 224 [1936, 373]. *Proposed by W. B. Campbell, Ithaca, New York.*

A wheel of unit radius rolls along a straight line on a horizontal plane, with velocity v. Particles (say of snow) are thrown off from each point of the rim. Find the maximum velocity attained by any of the particles, if (a) the wheel is wafer-thin so that it does not interfere with the paths of any of the particles it has thrown off; or if (b) the wheel does halt the progress of any particle which impinges on it.

Solution by the Proposer.

If the wheel has its center at (0, 0) when $t=0$, and rolls to the right at velocity v, on the plane $y+1=0$, the motion of the particle leaving the point ($\cos\theta$, $\sin\theta$), on the rim, is given by

$$v_x = v + v\sin\theta,\quad v_y = -v\cos\theta - gt,\quad u^2 = v_x^2 + v_y^2.$$

Now u^2, the square of the velocity at time t, has its minimum, v_x^2, at the vertex of the trajectory, when $t=-(v/g)\cos\theta$; for larger t, u^2 increases without limit. For a given particle, the largest u^2 actually attained occurs where $y=-1$, when $u^2=2(v^2+g)(1+\sin\theta)$, and the largest of these values of u^2 occurs for $\theta=90°$. Then $u=2(v^2+g)^{1/2}$, this being the maximum velocity attained in the absence of interference, as was called for in (a) of the problem proposed.

For case (b), we must consider the instantaneous distance D of the moving point, ($\cos\theta+\int_0^t v_x dt$, $\sin\theta+\int_0^t v_y dt$), from the moving wheel center, (vt, 0). It is given by

$$D^2 = 1 + t^2(v^2 - g\sin\theta + vgt\cos\theta + g^2t^2/4).$$

When $v^2 \not< g$, for the leading trajectory, from $\theta = 90°$, $1 < D$ for all $0 < t$, and there is no interference. When $v^2 < g$, it can be shown that for θ in quadrant I, with $\sin\theta \leqq v^2/g$, $1 < D$ for all $0 < t$, but for θ in quadrants I or II, with $v^2/g < \sin\theta$, $D < 1$ for sufficiently small values of t. The latter trajectories do not even start if the wheel is impervious, and the maximum value of u^2 is then $2(v^2+g)(1+v^2/g)$ Consequently, in case (b), the maximum value of u is given by

$$[2(v^2 + g)(1 + v^2/g)]^{1/2} \quad \text{when} \quad 0 < v^2 \leqq g$$

and by

$$2(v^2 + g)^{1/2} \quad \text{when} \quad g \leqq v^2.$$

Also solved by C. E. Springer.

Advanced Problems

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3815. *Proposed by P. Turán, Budapest, Hungary.*

Given the function of a real variable x

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{x}{4^n}:$$

show that there is a positive constant c_1, independent of x such that

$$(1) \qquad |f(x)| < c_1 \log\log x, \qquad x > e.$$

Show also that there exists a sequence $x_1 < x_2 < \cdots \to \infty$, and a positive constant c_2, independent of x, such that

$$|f(x_\nu)| > c_2 \log\log x_\nu, \qquad \nu = 1, 2, \cdots.$$

3816. *Proposed by E. Weiszfeld, Budapest, Hungary.*

Given the function of the complex variable z

$$f(z) = \sum_{i=1}^{n} \frac{a_i - z}{|a_i - z|},$$

where a_i, $(i = 1, 2, \cdots, n)$, denotes any given complex number: show that $|f(z)|$ takes on its maximum value in any domain not containing the points a_i at the boundary of the domain.

3817. *Proposed by V. Thébault, Le Mans, France.*

Parallels, with arbitrary direction, drawn through the vertices A, B, C of a triangle cut any given transversal Δ of the plane in α, β, γ. The parallels to BC, CA, AB through α, β, γ determine, by intersections or suitable pairs of the nine lines, a triangle $A_1B_1C_1$ symmetrically equal to triangle ABC (see J. Neuberg, *Wiskundij Tydschrift*, t.X. p. 80). Show that the center of symmetry of ABC and $A_1B_1C_1$ describes the Newton line of the quadrilateral (ABC, Δ) when the direction of $A\alpha$, $B\beta$, $C\gamma$ varies.

3818. *Proposed by V. Thébault, Le Mans, France.*

Consider a triangle ABC, a transversal Δ, passing through the orthocenter H, which cuts BC, CA, AB in α, β, γ, and the line Δ' which joins the orthocenters H, H_a, H_b, H_c of triangles ABC, $A\beta\gamma$, $B\gamma\alpha$, $C\alpha\beta$. Show that: (1) The line Δ and the Newton lines of the quadrilaterals (ABC, Δ) and (ABC, Δ') meet in a point P. (2) The sides of the triangles symmetrically equal to ABC, $A\beta\gamma$, $B\gamma\alpha$, $C\alpha\beta$ with respect to P pass respectively through the orthocenters (H_a, H_b, H_c), $(H. H_b, H_c)$, (H, H_c, H_a), (H, H_a, H_b).

3819. *Proposed by V. Thébault, Le Mans, France.*

If four spheres passing respectively through the vertices of a tetrahedron $ABCD$ intersect in pairs on the edges joining corresponding vertices, they meet in a point P. (S. Roberts, *Mathesis*, 1881, p. 95.) The straight lines from an arbitrary point Q to the vertices cut the corresponding spheres again in E, F, G, H. Show that the points P, Q, E, F, G, H lie upon a sphere.

SOLUTIONS

3730 [1935, 177]. *Proposed by William Chisholm, Quincy, Mass.*

There is said to be a short and simple method of finding one solution x of each of the equations

$$x^5 - (m+n)(5x^2+5x+1) = (m+n)^2, \tag{1}$$

$$x^2 + m^2 + n^2 - 2(mn + mx + nx) = 8\sqrt{2mnx(x+m+n)}. \tag{2}$$

What is the method?

Solution of (1) *by F. Underwood, University College, Nottingham, England.*

We set $m+n=p^5$ and $x=p^2+y$ in (1), and get

$$-p^5(5p^4+5p^2+1) + y(5p^8 - 10p^7 - 5p^5) + \cdots = 0,$$

where higher powers of y are omitted. This suggests as an approximation $y=p$. Setting now $x=p^2+p$ in (1) we have

$$p^5[(p+1)^5 - 5p^2(p+1)^2 - 5p(p+1) - 1 - p^5]$$

and a simple calculation shows that this is zero. Hence one solution is given by

$$x = p^2 + p = (m+n)^{2/5} + (m+n)^{1/5}.$$

Editorial Note. If $m+n\neq 0$ there are five distinct fifth roots of $m+n$, any one of which may be used in the above value for x, giving five roots of (1) but these are not all distinct for certain values of $m+n$.

3732 [1935, 255]. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

Given the focus and directrix of a parabola and a line anywhere in the plane of the parabola; derive a ruler and compass construction for determining the points in which the line cuts the parabola.

I. *Solution by A. Day Bradley, Hunter College.*

Let d be the given directrix, l the given line, and P the given focus. Construct PP' so that l is its perpendicular bisector and R its midpoint. Let T be the intersection of PP' and d. Construct on d, $MT=TM'=\sqrt{TP\cdot TP'}$. At M and M' construct perpendiculars to d intersecting l in O and O', which are the required points. The cases in which P lies on one of the lines simplify the construction, but are not exceptions. If l and d are perpendicular at S, construct the perpendicular bisector of PS, intersecting l in O, which is the required point. The proof is sufficiently indicated by taking the case with P between P' and T. Let $\overline{TP}=a$, $\overline{PP'}=2b$, $\overline{OT}=c$; then,

$$\overline{OM}^2=\overline{OT}^2-\overline{MT}^2=c^2-a^2-2ab, \text{ and}$$

$$\overline{OP}^2=\overline{OR}^2+\overline{RP}^2=\overline{OT}^2-\overline{TR}^2+\overline{RP}^2=c^2-a^2-2ab.$$

II. *Solution by Leon Recht, College of the City of New York.*

Let F be the focus; V, the intersection of the given line AB and the directrix CD; and P, the intersection of AB and the perpendicular to CD at any point H of the latter. With P as center describe a circle of radius PH cutting FV in Q and R. Parallels through F to PQ and PR cut AB in X and Y, respectively, and these are the required points. For, let H' be the projection of X on CD; from two sets of similar triangles, we have $FX/H'X=PQ/PH=1$. The proof for Y is similar.

The construction can be extended to the case of any conic. Here we describe a circle with center P and radius $r=ePH$, where e is the eccentricity; and the construction follows in a similar manner.

Solved also by W. B. Campbell, J. H. Edmonston, L. M. Kelly, J. S. Miller, Ethel I. Moody, A. Pelletier, Otto J. Ramler, E. P. Starke, R. C. Yates, and the proposer.

Editorial Note. Campbell, Edmonston, Starke, and Yates used analytic methods. Moody and Ramler considered the parabola as the locus of the intersections of corresponding rays of two projective pencils. The required points are the self-corresponding points of two projective ranges of points on l cut out by the two pencils. The proposer introduced a cone of revolution which cuts a plane in the parabola. Kelly stated that the construction is equivalent to the well-known construction of a circle through a point F and tangent to the direc-

trix d and to the line d' which is symmetric to d with respect to the given line l. Miller and Pelletier stated that the construction is equivalent to the well known construction of a circle tangent to d and passing through F and its symmetric F' with respect to l.

In solution II, if AB is perpendicular to CD, or if AB passes through F, the proposed construction cannot be made; however, solution I can be modified to meet these cases. The construction for the second case, where AB contains F, or P as given in solution I, leads to two important properties of the parabola. Here l passes through F; the point T on d projects orthogonally in F on l; on d we have $MT=TM'=TF$; and on d the points M and M' are the orthogonal projections of O and O' on l. It is then obvious that MTO and FTO are symmetric triangles, as are also $M'TO'$ and FTO'. Hence TO and TO' are tangents to the parabola at O and O', and OTO' is a right angle. The circle on OO' as a diameter is easily seen to be tangent to d at T. Hence the polar of any point T on the directrix of a parabola is the perpendicular at F to TF. Also the directrix d is the envelope of circles having focal chords of the parabola as diameters.

3733 [1935, 255]. *Proposed by E. P. Starke, Rutgers University.*

(a) For any two rational numbers, p and q, it is known that the triangle whose sides are the rational numbers

$$a = |p^2 - q^2|, \qquad b = 2pq, \qquad c = p^2 + q^2$$

is right angled at C. Derive an analogous rule for a triangle with an angle of 60° or 120° and having rational sides.

(b) Show that, except for the cases noted in (a), no triangle with rational sides can contain an angle commensurate with a straight angle.

Solution by J. Rosenbaum, Hartford Federal College.

In triangle ABC, if angle C is 120°, then by the Law of Cosines,

$$c^2 = a^2 + b^2 + ab. \tag{1}$$

This can be written as

$$(a + b + c)(a + b - c) = ab. \tag{2}$$

Hence if

$$a + b + c = \frac{p}{q}a, \qquad p > q, \tag{3}$$

then

$$a + b - c = \frac{q}{p}b. \tag{4}$$

Since the problem calls for only rational sides (not necessarily integral) the unit of length can be chosen equal to c.

Writing $c=1$ in (3) and (4), and solving them for a and b, we have

$$a = (2pq - q^2)/(p^2 + q^2 - pq), \tag{5}$$

$$b = (p^2 - 2pq)/(p^2 + q^2 - pq). \tag{6}$$

Multiplying each side of the triangle by the denominator above, we find

$$a = 2pq - q^2, \tag{7}$$

$$b = p^2 - 2pq, \tag{8}$$

$$c = p^2 + q^2 - pq. \tag{9}$$

Equations (7)–(9), with the condition that $p>2q$, give the sides of a triangle in which angle $C=120°$. It is also seen that by reversing the sign of the right-hand member of either (7) or (8), with the condition $q<2p<4q$, the resulting equations (7)–(9) will give a triangle with angle C equal to 60°.

Proof of (b). From the law of cosines it is seen that, if the sides of a triangle are rational, the cosines of its angles are also rational. So it will suffice to prove that, if $\alpha=m\pi/n$, where m and n are relatively prime positive integers, $0<m<n$, and $\cos\alpha$ is rational, then m/n is either 1/3, 1/2, or 2/3. If α is any angle, formula (7) for y_n in terms of a in the note to the solution of 3677 [1935, 575] gives

$$\begin{aligned} 2\cos n\alpha &= \sum_{i=0}^{[n/2]} (-1)^i A_i^{(n)} (2\cos\alpha)^{n-2i}, \\ A_i^{(n)} &= \frac{n(n-i-1)(n-i-2)\cdots(n-2i+1)}{i!}, \qquad A_0^{(n)} = 1, \end{aligned} \tag{1}$$

by setting $a=\cos\alpha$, $y_n=\cos n\alpha$. From the relations (3) and a following equation of that note, $y_0=1$, $y_1=a$, $y_{n+2}=2ay_{n+1}-y_n$, it easily follows that $A_i^{(n)}$ is an integer. For $\alpha=m\pi/n$, $y=2\cos\alpha$, the equation

$$\sum_{i=0}^{[n/2]} (-1)^i A_i^{(n)} y^{n-2i} - (-1)^m 2 = 0, \tag{2}$$

must be satisfied. By an elementary theorem of algebra, if this equation has a rational root, it must be an integral divisor of the constant term. If n is odd, the only possible rational roots are ± 1, ± 2. If n is even, m is odd, since n and m are relatively prime. In this case the last two terms of (2) are

$$(-1)^{(n-2)/2}\left(\frac{n}{2}\right)^2 y^2 + 2[(-1)^{n/2} + 1].$$

If $n/2$ is odd, there are two zero roots. The only possible rational roots not exceeding 2 in absolute value for the depressed equation are ± 1. If $n/2$ is even, the only possible rational roots not exceeding 2 in absolute value that equation (2) can have are ± 1, ± 2. Thus the only possible rational values of $\cos\alpha$ are 0, $\pm 1/2$, ± 1. Hence the only possible values of m/n, $0<m<n$, are 1/2, 1/3, 2/3, and the proof is complete.

Editorial Note. The following theorem may be easily derived. If p and q are relatively prime positive integers, $p>2q$, then all triangles with integral sides a, b, c whose H. C. F. is unity, and angle C is 120° are given by

$$a = q(2p - q), \qquad b = p(p - 2q), \qquad c = p^2 - pq + q^2,$$

where p and q do not have the form

$$p = 2n - m, \qquad q = n - 2m,$$

where m and n are relatively prime integers.

For these exceptional values of p and q, a, b, c have 3 for the H. C. F., and the triangles given by these exceptional values have sides which are three times those of the triangle given by $p=n$, $q=m$. Results for triangles in which $C=60°$ are easily obtained from the above.

A note from Roger A. Johnson states that a somewhat similar problem appears in the solution of 195 [1915, 27]. A solution by Johnson of a generalization was printed. Evidently owing to exigencies of space, parts of Johnson's note were omitted from publication, including, curiously enough, the formulation of the problem. The following is the generalized problem:

To determine formulas representing integers (or rational numbers) which are the sides of a triangle having one given angle whose cosine is a rational number m/n.

Since there was also some confusion in the printing of the summary of the results, this summary will be restated.

Let m and n be any two integers without common factor, $n>0$, $|m|<n$. If a, b, c be three integers without a common factor, representing lengths of the sides of a triangle in which $\cos(b, c)=m/n$, then the following formulas always give sets (a, b, c), and they give all possible sets:

I: (applicable only when $m>0$)

$$a = \frac{1}{F}\,[n(p^2 + q^2) - 2mpq], \qquad b = \frac{1}{F}\,[n(p^2 - q^2)], \qquad c = \frac{1}{F}\,[2p(mp - nq)].$$

II: (applicable for all values of m)

$$c = \frac{1}{F}\,[2q(np - mq)], \qquad a \text{ and } b \text{ as in I},$$

where p, q are any two relatively prime positive integers, such that in I, $q/p<m/n$, in II $q<p$, and in each case F is the highest common factor of the three brackets.

We can tabulate all possible common factors of either set of three brackets. The following, and no others, will be common factors: (1) in several cases, 2 is a common factor; (2) any factor common to $(m+n)$ and $(p+q)$; (3) any factor common to $(m-n)$ and $(p-q)$; (4) in I, any factor common to n and p; (5) in II, any factor common to n and q.

3736 [1935, 325]. *Proposed by J. M. Feld, New York City.*

Prove that the two following sets of 12 points in three-dimensional projective space are projective with one another:

(a) The 8 vertices and the center of a cube and the three points at infinity in the direction of the edges;

(b) The 6 vertices of a regular (right) triangular prism, the three middle points of the lateral faces of the prism, and the three points at infinity in the directions of the edges of the bases.

Determine the number of different one-to-one projective correspondences that can be established between the two sets of 12 points.

Solution by Gertrude Blanch, Hunter College, New York.

It is no restriction to take a cube with edge of length 2, symmetric about the origin, whose designated points have the following homogeneous coordinates: Upper base: $P_1\equiv(-1,\ -1,\ 1,\ 1)$; $P_2\equiv(1,\ -1,\ 1,\ 1)$; $P_3\equiv(1,\ 1,\ 1,\ 1)$; $P_4\equiv(-1,\ 1,\ 1,\ 1)$.
Lower base: $P_5\equiv(-1,\ 1,\ -1,\ 1)$; $P_6\equiv(-1,\ -1,\ -1,\ 1)$; $P_7\equiv(1,\ -1,\ -1,\ 1)$; $P_8\equiv(1,\ 1,\ -1,\ 1)$; $P_0\equiv(0,\ 0,\ 0,\ 1)$.

At infinity: $P_9\equiv(1,\ 0,\ 0,\ 0)$; $P_{10}\equiv(0,\ 1,\ 0,\ 0)$; $P_{11}\equiv(0,\ 0,\ 1,\ 0)$.

Similarly, the prism may be taken with edge of length 2, and the points designated for convenience as follows:
Upper base: $Q_0\equiv(1,\ \sqrt{3},\ 1,\ 1)$; $Q_1\equiv(0,\ 0,\ 1,\ 1)$; $Q_2\equiv(-1,\ \sqrt{3},\ 1,\ 1)$;
Lower base: $Q_{11}\equiv(1,\ \sqrt{3},\ -1,\ 1)$; $Q_3\equiv(0,\ 0,\ -1,\ 1)$; $Q_4\equiv(-1,\ \sqrt{3},\ -1,\ 1)$;
Midpoints: $Q_7\equiv(0,\ \sqrt{3},\ 0,\ 1)$; $Q_6\equiv(1,\ \sqrt{3},\ 0,\ 2)$; $Q_{10}\equiv(-1,\ \sqrt{3},\ 0,\ 2)$;
At infinity: $Q_5\equiv(1,\ 0,\ 0,\ 0)$; $Q_8\equiv(1,\ \sqrt{3},\ 0,\ 0)$; $Q_9\equiv(-1,\ \sqrt{3},\ 0,\ 0)$.

We exhibit a transformation T_0 which carries the prism points (x) into the points of the cube (x') by the same designated number.

$$x' = -\sqrt{3}x + 3y - 2\sqrt{3}t; \qquad y' = \sqrt{3}x + y - 2\sqrt{3}t;$$
$$z' = -\sqrt{3}x - y + 2\sqrt{3}z; \qquad t' = \sqrt{3}x + y + 2\sqrt{3}z.$$

Let T_1 be another transformation which carries the prism points into the cube points. Then $T_1^{-1}\cdot T_0 = K$ transforms the set of points of the cube into itself, and $T_1 = T_0\cdot K^{-1}$. Conversely, any two distinct transformations K_i, K_j of the set of points P_i into themselves determine two distinct transformations $T_0\cdot K_i$, $T_0\cdot K_j$ of the prism points into the cube points, and the problem reduces to finding all the transformations K_j. We must therefore study the geometry of the cube points.

The twelve points are arranged in a configuration of twenty-four planes, twelve of which contain six of the points each, and the other twelve just three of the points each. We list the last twelve at the end of our solution.

Each point P_j is associated with three and only three of the other points in this group of planes. Since a projective transformation preserves planes, this group of twelve planes must be carried into itself. The point P_0 in R_1 may be carried into any one of the twelve points (including itself). After that has been

done, the trasnformation of P_{10} may be chosen in only three ways, and there remain two choices for P_{11}. For P_9 in R_2 there remains but one choice—the fourth point associated with the transforms of the points of R_1. These four points are linearly independent, and since a linear transformation in space is determined by five points, we may still transform any other point—say P_1—into any of the eight remaining points. The transformation will then be uniquely determined. Hence there are $12\cdot 3\cdot 2\cdot 8=576$ projective correspondences which carry the prism into the cube.

It may be of interest to note that among the transformations there will be 24 harmonic homologies, each leaving one of the planes R_j fixed, with center at the point of intersection of the lines formed by the points not in that plane. The products of these homologies will generate the group.

Plane	Equation	Points in Plane											
R_1	$x=0$	P_0										P_{10}	P_{11}
R_2	$y=0$	P_0									P_9		P_{11}
R_3	$z=0$	P_0									P_9	P_{10}	
R_4	$t=0$										P_9	P_{10}	P_{11}
R_5	$x+y+z+t=0$		P_1				P_5		P_7				
R_6	$x+y-z+t=0$			P_2		P_4		P_6					
R_7	$x+y-z-t=0$				P_3		P_5		P_7				
R_8	$x+y+z-t=0$			P_2		P_4				P_8			
R_9	$x-y-z-t=0$			P_2				P_6		P_8			
R_{10}	$x-y+z-t=0$		P_1		P_3				P_7				
R_{11}	$x-y-z+t=0$		P_1		P_3		P_5						
R_{12}	$x-y+z+t=0$					P_4		P_6		P_8			

3737 [1935, 325]. *Proposed by J. P. Ballantine, University of Washington.*

Derive the following formulas:

$$\text{(a)}\qquad \pi = \frac{10}{3} - 24\left\{\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{1}{3\cdot 4\cdot 5\cdot 6\cdot 7} + \frac{1}{5\cdot 6\cdot 7\cdot 8\cdot 9} - \cdots\right\},$$

$$\text{(b)}\qquad \pi = 3.15 - 360\left\{\frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8} - \frac{1}{4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10} + \cdots\right\},$$

$$\text{(c)}\qquad \log 2 = \frac{17}{24} - 12\left\{\frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6} - \frac{1}{4\cdot 5\cdot 6\cdot 7\cdot 8} + \cdots\right\},$$

$$\text{(d)}\qquad \pi = \frac{64}{21} + 96\left\{\frac{1}{1\cdot 3\cdot 5\cdot 7\cdot 9} - \frac{1}{3\cdot 5\cdot 7\cdot 9\cdot 11} + \cdots\right\}.$$

Solution by F. Underwood, University College, Nottingham, England.

Let the series inside the brackets be denoted by S_a, S_b, S_c, S_d, respectively; and denote the nth term by u_n. We have

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots,$$

$$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$

(a) $$(-1)^{n-1}24u_n = \frac{1}{2n-1} - \frac{4}{2n} + \frac{6}{2n+1} - \frac{4}{2n+2} + \frac{1}{2n+3},$$

$$24S_a = -4\left(\frac{\pi}{4}\right) + 5 - 2 + \frac{1}{3},$$

$$\pi + 24S_a = \frac{10}{3}.$$

(b) $$(-1)^{n-1}720u_n = \frac{1}{2n} - \frac{6}{2n+1} + \frac{15}{2n+2} - \frac{20}{2n+3} + \frac{15}{2n+4} - \frac{6}{2n+5} + \frac{1}{2n+6},$$

$$720S_b = -8\left(\frac{\pi}{4}\right) + 8 + \frac{1}{2} + \frac{14}{3} + \frac{14}{4} - \frac{6}{5} + \frac{1}{6},$$

$$\pi + 360S_b = \frac{63}{20} = 3.15.$$

(c) $$(-1)^{n-1}24u_n = \frac{1}{2n} - \frac{4}{2n+1} + \frac{6}{2n+2} - \frac{4}{2n+3} + \frac{1}{2n+4},$$

$$24S_c = \frac{5}{2} - \frac{4}{3} + \frac{1}{4} - 4\left(\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots\right),$$

$$\log_e 2 + 12S_c = \frac{17}{24}.$$

(d) $$(-1)^{n-1}384u_n = \frac{1}{2n-1} - \frac{4}{2n+1} + \frac{6}{2n+3} - \frac{4}{2n+5} + \frac{1}{2n+7},$$

$$384S_d = 16\left(\frac{\pi}{4}\right) - 15 + \frac{11}{3} - 1 + \frac{1}{7},$$

$$\pi - 96S_d = \frac{64}{21}.$$

Solved also by O. J. Ramler and the proposer.

Editorial Note. Ramler's solution was similar to the above. The solutions used the identities

$$\frac{k!}{x(x+1)\cdots(x+k)} = \sum_{i=0}^{k} (-1)^i \frac{{}_kC_i}{x+i},$$

$$\frac{k!2^k}{y(y+2)\cdots(y+2k)} = \sum_{i=0}^{k}(-1)^i \frac{{}_kC_i}{y+2i}.$$

The second identity results from the first by the substitution $2x=y$. The first one is easily obtained by the method of partial fractions; or we may derive it by the method of finite differences as follows:

$$\begin{aligned}\Delta^k x^{(-1)} &= (-1)^k k! x^{(-k-1)},\\ &= (U-1)^k x^{(-1)} = \left[\sum_{i=0}^{k}(-1)^{k-i}{}_kC_i\, U^i\right]x^{(-1)},\\ &= (-1)^k \sum_{i=0}^{k}(-1)^i{}_kC_i(x+i)^{(-1)}.\end{aligned}$$

Hence

$$k!x^{(-k-1)} = \sum_{i=0}^{k}(-1)^i{}_kC_i(x+i)^{(-1)}.$$

This is the first identity.

3738 [1935, 325]. *Proposed by V. Thébault, Le Mans, France.*

Consider a convex quadrilateral $ABCD$ inscribed in a circle and the circles (α), (β), (γ), (δ) described on the sides AB, BC, CD, DA as diameters. Show that the diagonal AC is to the diagonal BD as the product of the common tangents to circles (α) and (β), (γ) and (δ) is to that of the tangents to circles (β) and (γ), (δ) and (α).

Solution by C. W. Trigg, Cumnock College, Los Angeles.

Let the common tangents to (α), (β); (β), (γ); (γ), (δ); and (δ), (α) be m, n, p and q, respectively. From E and F, the centers of (α) and (β), respectively, draw the radii to the points of tangency of m. From F draw FK perpendicular to the radius from E, and draw EF. Then

$$\overline{EF}^2 = (r_\alpha - r_\beta)^2 + m^2,\quad EF = \tfrac{1}{2}AC,\quad r_\alpha = \tfrac{1}{2}AB,\quad r_\beta = \tfrac{1}{2}BC,$$

$$\overline{AC}^2 = 4m^2 + \overline{BC}^2 + \overline{AB}^2 - 2BC\cdot AB.$$

In the triangle ABC,

$$\overline{AC}^2 = \overline{BC}^2 + \overline{AB}^2 - 2BC\cdot AB\cos B.$$

Eliminating AC,

$$m^2 = \tfrac{1}{2}BC\cdot AB(1-\cos B) = BC\cdot AB\sin^2\tfrac{1}{2}B,\quad m = \sin\tfrac{1}{2}B\sqrt{BC\cdot AB}.$$

Similarly

$$n = \sin\tfrac{1}{2}C\sqrt{BC\cdot CD},\quad p = \sin\tfrac{1}{2}D\sqrt{CD\cdot AD},\quad \text{and}\quad q = \sin\tfrac{1}{2}A\sqrt{AD\cdot AB}.$$

Now since $ABCD$ is inscriptible, A and C are supplementary, as are B and D. Hence,

$$\frac{m\cdot p}{n\cdot q}=\frac{\sin\frac{1}{2}B\sin\frac{1}{2}D}{\sin\frac{1}{2}C\sin\frac{1}{2}A}=\frac{2\sin\frac{1}{2}B\cos\frac{1}{2}B}{2\sin\frac{1}{2}C\cos\frac{1}{2}C}=\frac{\sin B}{\sin C}.$$

In the triangles ABC and BCD,

$$\frac{AC}{BC}=\frac{\sin B}{\sin BAC}, \qquad \frac{BC}{BD}=\frac{\sin BDC}{\sin C}.$$

Since angles BAC and BDC are equal,

$$\frac{AC}{BD}=\frac{\sin B}{\sin C}.$$

It follows that $AC/BD=(m\cdot p)/(n\cdot q)$.

Solved also by J. W. Clawson, Otto J. Ramler, H. D. Ruderman, F. Underwood, and the proposer.

3739 [1935, 396]. *Proposed by Paul Erdös, The University, Manchester, England.*

Given $n+1$ integers, $a_1, a_2, \cdots, a_{n+1}$, each less than or equal to $2n$, prove that at least one of them is divisible by some other of the set.

I. *Solution by Martha Wachsberger and E. Weiszfeld, Budapest, Hungary.*

We may write

$$a_r = 2^{b_r}c, \qquad c = \text{an odd integer}.$$

Since the number of odd integers less than $2n$ is n, there are at least two a's, e.g., a_i and a_k, having the same c; and it is then evident that one of these is a divisor of the other.

II. *Solution by Mannis Charosh, New Utrecht High School, Brooklyn, N. Y.*

The integers

$$(2i-1)2^k, \qquad 1\leqq i\leqq n, \qquad k=0, 1, 2, \cdots,$$

for a given i may be written in a row. There are then n rows which contain at least every integer from unity to $2n$. In order to select $n+1$ integers from this system, it is necessary to take at least two of them from the same row; and of these one must be divisible by some other in this selection. The theorem is therefore true.

Solved also by W. L. Ayres, W. E. Buker, J. M. Feld, B. P. Gill, A. D. Hestenes, Emma Lehmer, Roy MacKay, L. C. Mathewson, and E. P. Starke.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items of interest to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

At the meeting of the National Academy of Sciences in November, G. A. Bliss presented a paper on "Some recent advances in the theory of the calculus of variations," L. E. Dickson presented one on "Remarkable results in additive number theory," and Dunham Jackson one on "Polynomial approximation on a curve of the fourth degree."

A symposium on Tauberian Theorems and the Theory of Numbers was held at the University of Cincinnati on November 14 under the auspices of the Taft Memorial Fund. The principal speakers were Professors T. Vijayaraghavan, G. H. Hardy, and Otto Szasz.

Professor T. Vijayaraghavan, the visiting lecturer of the American Mathematical Society, has given lectures at Harvard, Washington University, the Universities of Chicago, Cincinnati, Illinois, Wisconsin, and other places.

The Association for Symbolic Logic met in conjunction with the annual meeting of the American Philosophical Association in Cambridge, Massachusetts, on December 28. The following papers were presented: "Some logical problems encountered by a physicist" by Professor P. W. Bridgman; "The formal distinction between asserted and unasserted propositions" by Professor A. A. Bennett; "Systems of 'complete logic' within the scope of Goedel's theorem" by Dr. F. B. Fitch; "A calculus of individuals" by Drs. H. S. Leonard and H. N. Goodman.

At Iowa State College Dr. E. W. Anderson has been promoted to an assistant professorship, and Assistant Professor J. V. Atanasoff has been promoted to an associate professorship.

Assistant Professor H. E. Arnold of Wesleyan University, Middletown, Connecticut has been promoted to an associate professorship.

Professor W. L. Ayres of the University of Michigan has been on leave of absence the past semester and has been visiting at the University of Pennsylvania.

Professor Samuel Beatty of the University of Toronto has been appointed dean of the faculty of arts.

Dean G. D. Birkhoff of Harvard University was awarded an honorary doctorate by the University of Paris. He has also been selected by Pope Pius as a member of the newly formed Pontifical Academy of Science composed of seventy world-famed scientists; members were chosen solely for their outstanding scientific achievement.

Dr. F. A. Butter, formerly at Stanford University, is spending the academic year 1936–37 as research assistant to Professor Szegö at Washington University.

Dr. H. C. Carter of the University of Missouri has been appointed to a professorship at State Teachers College, Fredericksburg, Virginia.

Professor R. V. Churchill of the University of Michigan has been on leave of absence the past semester and has been studying at the University of Freiburg, Freiburg, Germany.

Major A. P. Cowgill has been appointed to a professorship in mathematics at the Indiana Technical College, Fort Wayne.

Dr. H. S. M. Coxeter of Washington, D. C., has been appointed to an assistant professorship at the University of Toronto.

Dr. G. F. Cramer of Tulane University has been promoted to an assistant professorship.

Dr. P. D. Crout of the Massachusetts Institute of Technology has been promoted to an assistant professorship.

Assistant Professor W. L. Duren of Tulane University is on leave of absence for one year and is at the Institute for Advanced Study.

Dr. James Fisher, head of the department of mathematics and physics of the Michigan College of Mining and Technology, has been appointed dean of the faculty of that institution.

Dr. Max Mason has joined the staff at the California Institute of Technology.

Assistant Professor R. C. Yates of the University of Maryland has been promoted to an associate professorship.

Dr. H. M. Bacon of the Stanford University has been promoted to an assistant professorship.

Dr. M. H. Martin of the University of Maryland has been promoted to an assistant professorship.

Dr. C. R. Dines, head auditor for Armour & Co., died on November 14, 1936. He had previously taught mathematics at Dartmouth and at Northwestern University.

We regret to announce the death in England last September of Mrs. Emma Gifford, mathematician. Mrs. Gifford was best known for her work, in collaboration with her husband, on the ten-place tables of trigonometric functions that bear her name.

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BOOKS FOR REVIEW should be addressed to REVIEW EDITOR, American Mathematical Monthly, 531 West 116th Street, New York, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of addresss to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-first Summer Meeting, Pennsylvania State College, Sept. 6–7, 1937.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1937 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Waynesburg, Pa., May 1.
ILLINOIS, DeKalb, May 14–15.
INDIANA, Greencastle, May.
IOWA, Dubuque, April 16–17.
KANSAS, Witchita, April 3.
KENTUCKY.
LOUISIANA-MISSISSIPPI.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Lynchburg, Va., May 8.
MICHIGAN, Ann Arbor, March 20.
MINNESOTA.
MISSOURI.
NEBRASKA.
OHIO, Columbus, April 1.
OKLAHOMA, Oklahoma City, Feb.
PHILADELPHIA, Haverford, Nov. 27.
ROCKY MOUNTAIN.
SOUTHEASTERN, Nashville, Tenn., April.
SOUTHERN CALIFORNIA, Los Angeles, March 6.
SOUTHWESTERN, State College, N.M., April 2–3.
TEXAS.
WISCONSIN, Milwaukee, May.

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THE TWENTY-FIRST ANNUAL MEETING OF THE ASSOCIATION

The twenty-first annual meeting of the Mathematical Association of America was held at Chapel Hill and Durham, North Carolina, on Thursday and Friday, December 31, 1936 and January 1, 1937, in conjunction with the annual meeting of the American Mathematical Society. Three hundred nine were in attendance at the meetings, including the following one hundred sixty-three members of the Association:

C. R. ADAMS, Brown University
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MABEL M. YOUNG, Wellesley College

OSCAR ZARISKI, Johns Hopkins University

The mathematical organizations for a long term of years have held their Christmas meeting in affiliation with the American Association for the Advancement of Science. When, however, the American Association in January 1935 decided to meet at Atlantic City because the hotels in Washington, D. C. would be over-crowded, the officers of the organizations decided that it would not be desirable to meet at Atlantic City so soon after the previous meeting there, and that it would be more important to recognize a new mathematical region by a visit to North Carolina. A number of mathematicians attended the meetings of the American Association at Atlantic City and heard the retiring vice-presidential address by Professor T. H. Hildebrandt on "Recent developments in the theory of integration" and a program of several mathematical papers. The Council of the American Association on December 31 elected Professor G. D. Birkhoff president of the Association for 1937, Professor W. D. Cairns vice-president and chairman of Section A, Professor M. H. Stone member of the committee for Section A, and Professor R. C. Archibald vice-president and chairman of Section L (historical and philological sciences).

Through the generosity of Duke University, rooms in the dormitories on the men's campus were furnished free of charge to the visiting mathematicians and their families. Meals were served in the Men's Union at very reasonable rates. A lounge and social rooms in the Men's Union provided a convenient place for numerous social meetings and committee consultations which contributed so largely to the success of these meetings. A large proportion of the visitors availed themselves of the opportunity to inspect the magnificent chapel and bell tower, the library and other parts of the large set of buildings comprising the men's campus. Opportunity was also afforded to visit the Woman's College campus. Tea was served on Monday, Tuesday and Wednesday afternoons by the wives of the members of the Duke mathematics department. Other pleasant features of the week were a musical entertainment by the Durham Master Singers, a negro choral group, and a delightful piano recital on Thursday evening by Professor Marston Morse in which compositions from Bach, Debussy, Chopin, and Schumann were played.

Dean Eisenhart acted in the happy capacity of toastmaster at the annual dinner in the Men's Union; this was attended by two hundred eighty. President Few of Duke University added expressly to the welcome already made manifest to the visitors and expressed his pleasure in meeting with actual teachers and scholars who are carrying the burden and the heat of the educational day. President Graham of the University of North Carolina described the co-operative efforts of the two universities of providing possible means for enriching the scholarly development of the Piedmont region of North Carolina, and, speaking

as a teacher of history, gave a tribute to mathematics as a means whereby the learning of the earlier centuries had been passed on.

Professor Tamarkin spoke of the way in which his native land, Russia, is developing mathematical work, ascribing to Russia the next place after the United States in mathematical research. In 1934 more than seven hundred mathematicians were in attendance at the Soviet Mathematical Congress, many mathematical books are now being published in topology, the theory of functions, mechanics, etc., and a new journal of considerable importance is now being established. We have now in America excellent journals in mathematics, most recently the excellent Duke Journal, but much more must be done in the publication of mathematical books to a degree comparable with the vast growth of mathematical research in this country.

Professor Dunham Jackson spoke of the extent to which subjects in which there are accurate results are replaced by those in which there is a weighing of argument on this side and on that. It is easy to come to the defense of mathematics as a study; but we can take the offensive and say that the reason why so many criticisms of mathematics are made is that it is so vitally important in so many vital ways that there must be a steady adaptation of mathematical methods to meet the changing conditions. There is thus a need for the best possible teaching. We as mathematicians must recognize both that there are questions as to the place of mathematics and that these questions can be answered.

A resolution offered by Professor Hollcroft was adopted by rising vote rendering our sincere thanks and appreciation to President Few for so generously placing the facilities of the University at our disposal, to Vice-President Flowers, Mr. Dwire, the director of public relations, Professor Thomas and the other members of the committee on local arrangements, and the ladies who assisted at the teas, for the excellent and hospitable arrangements for our comfort and convenience.

According to the schedule of events, the mathematicians went by bus or private automobiles to Chapel Hill on Thursday forenoon for the final session of the Society and the first session of the Association. Luncheon was served at the Carolina Inn, and immediately at the close of the afternoon session a reception at the Carolina Inn was tendered to the visitors by President Graham and Dean House of the University of North Carolina assisted by the faculty ladies. At the close of the tea a resolution was heartily adopted thanking President Graham, Dean House, Professor Archibald Henderson, the members of the local committee, and the ladies who had so graciously shown their hospitality during the day.

The American Mathematical Society held sessions for the reading of papers on Tuesday morning and afternoon and on Wednesday and Thursday mornings. About one hundred papers were read at these sessions. On Tuesday afternoon by invitation of the program committee, Professor J. M. Thomas gave an address on "Differential systems." On Wednesday afternoon following the annual business meeting and election, Professor Solomon Lefschetz gave his retiring presidential address on "The rôle of algebra in topology."

The Mathematical Association held sessions on Thursday afternoon and Friday morning, with President Curtiss in the chair until he called the new president to the chair after the annual election. The program follows, together with abstracts of some of the papers, numbered in accordance with their place on the program.

FIRST SESSION OF THE ASSOCIATION

1. "New foundations in mathematical logic" by Dr. W. V. QUINE, Harvard University.

2. "Undergraduate instruction in mathematics" by Professor J. I. TRACEY, Yale University.

3. "The complex number as a working variable in the plane" by Professor L. R. FORD, Rice Institute.

1. Dr. Quine's paper appeared in full in the February issue of this MONTHLY.

2. Professor Tracey's paper will be printed in an early issue of this MONTHLY.

3. Professor Ford discussed certain advantages of the complex variable in inversive geometry, analytic geometry, and the differential and integral calculus. In those parts of mathematics in which straight lines and circles play the dominant rôles, such as elementary plane geometry, inversion, and the modern geometry of the triangle, the complex variable is frequently of very great use. In the differential calculus its adoption often leads to simple formulas. Thus the curvature of $F(z, \bar{z})=0$ turns out to be $\frac{1}{2}|d^2z/d\bar{z}^2|$

The following integral formula was discussed:

$$\int_S\int \frac{\partial U}{\partial \bar{z}} dS = \frac{1}{2i}\int_C U dz,$$

where U is a function of z and $\bar{z}$. This is of value in finding areas, centers of gravity, volumes of revolution, and moments of inertia, and in evaluating double integrals generally. It is particularly useful in the case of regions—such as the eccentric circular sector—which are bounded by segments of straight lines and arcs of circles.

SECOND SESSION OF THE ASSOCIATION

1. Annual business meeting and election of officers.

2. "Gauss sums" by Professor LEONARD CARLITZ, Duke University.

3. "Thinking versus manipulation" by Professor W. B. CARVER, Cornell University.

4. "The problem of three finite masses" by Professor H. E. BUCHANAN, Tulane University.

1. Report given on pages 129–130.

2. This paper deals with the generalized sum

$$S(H, M, m) = \sum_{A \,(\mathrm{mod}\, M)} \epsilon^{c(A^2H, M)}, \qquad \epsilon = e^{2\pi i/m},$$

where A, H, M are polynomials in a single indeterminate x with coefficients

(mod m), $m>0$ an arbitrary integer, and the summation extends over a complete residue system (mod M); the symbol $c(A, M)$ is defined as the coefficient of x^{k-1} in the residue of A (mod M): M is of degree k with leading coefficient $=1$. The sum $S(1, M, m)$ is first evaluated in a simple way; the general sum is easily obtained. Applications are indicated to the problem of determining the number of solutions of the equation $M=A_1X_1^2+\cdots+A_sX_s^2$, where $M, A_1, \cdots, A_s$ are assigned polynomials.

3. Professor Carver's paper will appear in an early issue of this MONTHLY.

4. Professor Buchanan's paper described in some detail certain recent contributions to the theory of motions of three bodies in the vicinity of the straight line and equilateral triangle solutions of Lagrange. He included not only the case of three attracting bodies but also cases in which repulsive forces existed.

MEETINGS OF THE BOARD OF TRUSTEES

Members of the outgoing and of the incoming Board met on Thursday evening and Friday morning respectively.

The following thirty-four persons and two institutions were elected to membership on applications duly certified:

To Individual Membership

R. H BARDELL, Ph.D.(Chicago) Instr., Math. and Astr., Univ. of Wisconsin, Extension Div., Milwaukee, Wis.

G. F. BARNES, Ph.D.(Indiana) Prof., Math. and Physics, Judson Coll., Marion, Ala.

F. A. BEELER, Asst. Prof., Hillsdale Coll., Hillsdale, Mich.

R. P. BOAS, Jr., A.B.(Harvard) Instr., Harvard Univ., Cambridge, Mass.

R. C. BRIANT, B.S. in C.E.(Lafayette) Industrial fellow, Mellon Inst., Pittsburgh, Pa.

A. M. BRYSON, A.M.(Pittsburgh) Instr., Univ. of Pittsburgh, Pittsburgh, Pa.

D. M. DRIBIN, Ph.D.(Chicago) Natl. Research Fellow, Princeton Univ. and Inst. for Advanced Study, Princeton, N. J.

A. M. GINSBURG, A.M.(Columbia) New York, N. Y.

MARION C. GRAY, Ph.D.(Bryn Mawr) Tech. research, Bell Telephone Labs., New York, N. Y.

ELIZABETH M. HASKINS, M.S.(Mass. Inst. Tech.) Teacher, Sandia School, Albuquerque, N. M.

L. D. HEMENWAY, A.M.(Harvard) Asso. Prof., Math. and Physics, Simmons Coll., Boston, Mass.

TRYPHENA HOWARD, A.M.(Michigan) Asst. Prof., State Teachers Coll., Bowling Green, Ky.

K. S. K. IYENGAR, B.A.(Cantab.) Prof., Central Coll., Bangalore, India

W. A. KLEIN, B.S. Grad. student, Carnegie Inst. of Tech., Pittsburgh, Pa.

R. H. KNOX, Jr., B.S.(Virginia Milit. Inst.) Asst. Prof., Virginia Milit. Inst., Lexington, Va.

E. S. LAVOIE, student, Brooklyn Coll., Brooklyn, N. Y.

H. M. MACNEILLE, Ph.D.(Harvard) Benj. Peirce Instr., Harvard Univ., Cambridge, Mass.

J. J. MAHONEY, Ph.D.(California) Asst. Prof., Loyola Univ., Chicago, Ill.

Sister MARY RESIGNATA, A.M.(Catholic Univ.) Prof., Clarke Coll., Dubuque, Iowa

Mrs. SALLIE P. MEAD, A.M.(Columbia) Technical staff, Bell Telephone Labs., New York, N. Y.

W. R. MURRAY, M.S.(Cornell) Instr., Franklin and Marshall Coll., Lancaster, Pa.

F. C. W. OLSON, B.S.(Chicago) Technologist, American Can Co., Maywood, Ill.

J. K. RECKZEH, A.B.(W. Ky. State Teachers Coll.) Teacher, High School, Rockfield, Ky.

F. A. REIBER, M.S.(Florida) Grad. student, Univ. of New Mexico, Albuquerque, N. M.

W. M. ROBERTSON, Clerk, Pennsylvania Railroad, Philadelphia, Pa.

S. T. SANDERS, Jr., Ph.D.(Iowa) Head of Dept., Nebraska State Normal Coll., Chadron, Nebr.

M. E. SHANKS, Ph.D.(Iowa) Instr., Univ. of Michigan, Ann Arbor, Mich.

ZENIA J. SMOKOWSKI, student of law, Univ. of Buffalo, Buffalo, N. Y.

C. E. SNOKE, Jr., A.M.(Washington and Jefferson) Head of Dept., Buena Vista Coll., Storm Lake, Iowa

C. E. VAN HORN, Ph.D.(Chicago) Head of Dept., Fisk Univ., Nashville, Tenn.

T. VIJAYARAGHAVAN, Reader, Univ. of Dacca, Dacca, India

B. S. WHITNEY, A.M.(Oklahoma) Instr., Capitol Hill Jr. Coll., Oklahoma City, Okla.

R. H. WILSON, Jr., Ph.D.(Pennsylvania) Honorary research fellow, Bartol Foundation, Swarthmore, Pa.

R. T. ZIMMERMAN, B.S.(Westminster) Teacher, Swissvale High School, Pittsburgh, Pa.

To Institutional Membership

BUENA VISTA COLLEGE, Storm Lake, Iowa

MARYMOUNT COLLEGE, Tarrytown, N. Y.

The financial report of the Secretary-Treasurer for the year 1936, having been previously approved by Professor Slaught for the Finance Committee and also by Professors Curtiss and Kempner, was presented and accepted. The Finance Committee was empowered to transfer from current funds to the General Endowment Fund any surplus up to $2000.

Following the election on Friday, Marie J. Weiss, Vassar College, was appointed to the vacancy in the list of Trustees, caused by the election of E. J. Moulton as vice-president, for the term ending January 1939.

The following were appointed associate editors of this MONTHLY for the year 1937, as nominated by Professor Moulton, Editor-in-Chief:

W. B. Carver	R. E. Gilman	Helen B. Owens
W. F. Cheney	W. R. Longley	R. G. Sanger
Otto Dunkel	J. R. Musselman	D. E. Smith
B. F. Finkel	H. L. Olson	F. E. Wood
T. C. Fry	F. W. Owens	

Professors C. S. Atchison and H. L. Rietz were appointed as representatives of the Association on the Council of the American Association for the Advancement of Science for the year 1937.

Other matters were considered which concerned the choice of future meeting places and provision for committee work which is now being organized.

ANNUAL BUSINESS MEETING

The annual business meeting and election of officers was held Friday morning, January 1, 1937. The Secretary announced the names of those who had been elected to membership at the meeting of the Trustees. He also reported the deaths of the following members:

R. J. ALEY, President, Butler University. (November 17, 1935)

C. H. ASHTON, Professor of mathematics, University of Kansas. (August 2, 1936)

C. F. BOWLES, Assistant Professor of mathematics, South Dakota State School of Mines. (February 7, 1936)

H. C. BRADLEY, Associate Professor of mathematics, Massachusetts Institute of Technology. (March 7, 1936)

JULIA T. COLPITTS, Associate Professor of mathematics, Iowa State College. (August 8, 1936)
C. W. CROCKETT, Professor of mathematics and astronomy, Rensselaer Polytechnic Institute. (December 30, 1936)
T. R. EAGLES, Vice-President and Treasurer, Howard College. (June 7, 1936)
TSURUICHI HAYASHI, Professor emeritus of mathematics, Tohoku Imperial University. (October 4, 1935)
C. L. HERRON, Professor of mathematics and Dean, Hillsdale College. (November 13, 1936)
E. M. HORSBURGH, Reader and lecturer, University of Edinburgh. (December 28, 1935)
EMMA L. KONANTZ, Professor of mathematics, Yeu Ching University, Peking, China. (January 3, 1936)
E. B. LYTLE, Associate Professor of mathematics, University of Illinois. (September 5, 1936)
EMILIE N. MARTIN, Professor of mathematics, Mount Holyoke College. (February 8, 1936)
Rev. J. H. MEYER, Spring Hill College, Mobile, Alabama. (August 9, 1935).
I. L. MILLER, Professor of mathematics, South Dakota State College. (February 22, 1936)
E. A. OSSE, Design draftsman, retired, U. S. Navy Dept., Philadelphia, Pa. (May 23, 1936)
N. A. PATTILLO, Professor of mathematics, Randolph-Macon Woman's College. (September 8, 1936)
SALVATORE PINCHERLE, Professor of mathematics, University of Bologna. (July 10, 1936)
G. A. PLIMPTON, Senior member, Ginn and Company. (July 1, 1936)
J. A. REISING, Teacher, Fort Wayne High School. (September 9, 1936)
MARGUERITE STAGNER, Ames, Iowa. (May 7, 1936)
H. I. THOMSEN, Baltimore, Maryland. (October 20, 1936)
F. C. TOUTON, Professor of mathematics and Vice-President, University of Southern California. (June 1, 1936)

On recommendation of the Board of Trustees, Professors T. S. Fiske and G. A. Miller were made honorary life members in recognition of their loyalty to the activities of the Association. A resolution was unanimously adopted requesting and instructing the Secretary to send the cordial greetings of the members of the Association to Professors H. E. Slaught and David Eugene Smith, recording our gratefulness for their many valuable contributions to the activities of the Association.

The annual meeting also adopted unanimously the following resolution:

> "Professor W. B. Carver retires as Editor-in-Chief of the AMERICAN MATHEMATICAL MONTHLY after serving in this capacity for five years. Probably few will realize the great sacrifice in time and energy which the Editorship calls for. The Association desires to express its sincere gratitude to Professor Carver for his faithful services and for his maintenance of the excellent standards of our journal."

Professors H. A. Robinson and W. F. Cheney served as tellers for the election of officers for 1937, the results being as follows:

President for 1937–38: A. J. KEMPNER, University of Colorado.

Vice-Presidents for 1937: T. H. HILDEBRANDT, University of Michigan; E. J. MOULTON, Northwestern University.

Additional members of the Board of Trustees, to serve until January 1940: R. W. BRINK, University of Minnesota; D. R. CURTISS, Northwestern University; DAVID EUGENE SMITH, Columbia University; J. M. THOMAS, Duke University.

Report of the Secretary-Treasurer as Treasurer, December 14, 1936

Receipts

Balance Dec. 12, 1935		$8,062.36
1935 indiv. dues	$562.40	
1935 instit. dues	15.60	
1936 indiv. dues	6,708.03	
1936 instit. dues	705.70	
1936 subscriptions	960.23	
Initiation fees	278.00	
Advertising	428.75	
Authors' reprints	245.65	
Sale copies of MONTHLY	167.49	
Sale First Carus Mon.	20.00	
Sale Second Carus Mon.	13.75	
Sale Third Carus Mon.	16.25	
Sale Fourth Carus Mon.	18.75	
Sale Fifth Carus Mon.	18.75	
Sale Archibald's Outline of Hist. of Math.	149.30	
Contrib. to cost of same.	25.00	
Annals subscriptions	5.00	
Placement bureau fees	15.00	
Sale Rhind Papyrus	142.50	
Anon. gift to Chauvenet Fund	100.00	
Drury Coll. int. Hardy Fund	120.00	
Refund expense Harvard meeting	75.00	
From certif. of deposit No. Trust Co.	885.50	
Sale Union Pacific Bonds	3,072.96	
Int. Peoples Bkg. Co.	22.07	
Int. Cleveland Trust Co.	88.46	
Int. Genl. End. Fund	567.50	
Int. Carus Fund	186.25	
Int. Chace Fund	214.37	
Int. Chauvenet Fund	15.00	
Int. current funds	15.62	
Payment from restricted Carus Fund	99.40	
Payment from restricted Chace Fund	4.40	15,962.68
Total 1936 receipts to date		$24,025.04
Total expenditures		17,625.60
Balance to end of 1936 business		$ 6,399.44
Received on 1937 business		703.62
Book balance Dec. 14, 1936		$ 7,103.06

Expenditures

Publisher's bills (Nov. '35–Oct. '36)		$4,723.62
Reprints		336.15
Register		446.39
Editor-in-chief's office 1936		630.44
Editor-in-chief's office 1937		119.03
Committee on membership		210.90
Comm. on Training & Utilization		97.95
Comm. on Place of Math.		162.79
Committee on Testing		83.26
Carus Monograph Committee		116.85
Secretary-Treasurer's office		
Postage	$414.52	
Bond	11.26	
Safety deposit	4.40	
Office supplies	86.25	
Express, tel., etc.	81.02	
Clerical work	2,088.83	
Printing	162.11	
Bank tax	18.35	2,866.74
Annals subvention		200.00
Duke Journal subvention		250.00
Expense sections from init. fees		157.06
St. Louis meeting		95.76
Harvard meeting		75.00
Forwarded *Annals* subscriptions		5.00
Paid *Annals* subscriptions		12.50
Sust. memb. in Amer. Math. Soc.		100.00
Refund membership dues		4.00
Refund subscriptions		7.65
Storage back copies MONTHLY		60.00
Insurance back copies MONTHLY		17.70
Paid back copies MONTHLY		17.60
Paid B. F. Finkel int. Hardy Fund		120.00
Award Chauvenet Prize		100.00
Printing Archibald's Outline		235.01
Library expense		63.18
Int. and cost above par Penna. R.R. Bonds		19.96
Cost C. and O. Bonds		3,041.06
Transfer to Chace Fund		1,250.00
Transfer to Genl. End. Fund		2,000.00
Total expenditures		$17,625.60
Checking account		$493.16
Oberlin Savgs. Bk. acct. restricted		999.60
Peoples Banking Co. savgs. acct.		1,120.59
Cleveland Trust Co. savgs. acct.		4,489.71
Bank balance Dec. 14, 1936		$ 7,103.06

Exhibit of the Funds of the Association

Carus Monograph Fund

Balance Dec. 12, 1935		$6,286.41
Receipts: Sales	$ 87.50	
Interest	217.98	305.48
		$6,591.89
Expense acct. Carus Committee		116.85
		$6,475.04
Certificate of deposit	$1,717.31	
C. & O. $3\frac{1}{2}\%$ Refunding Mortgage Bonds Series D, 1996	2,000.00	
$3\frac{1}{8}\%$ U. S. Treasury Bond 1946–49	1,000.00	
3% HOLC Bond 1944–52	1,000.00	
Cash in bank, restricted, certificate of participation	596.40	
Cash in bank, unrestricted	161.33	
Balance Dec. 14, 1936		$6,475.04

Arnold Buffum Chace Fund

Balance Dec. 12, 1935		$6,343.36
Receipts: Sale Papyrus	$142.50	
Interest	217.30	
Appreciation in value of Western United Gas and Elec. Co. Bonds.	130.00	489.80
		$6,833.16
$3\frac{1}{8}\%$ U. S. Treasury Bonds 1946–49	$2,000.00	
3% HOLC Bond 1944–52	1,000.00	
U. S. Savings Bonds	750.00	
Western United Gas & Elec. Co. $5\frac{1}{2}\%$ Bonds Series A	2,500.00	
Certificate of deposit	129.06	
Cash in bank, restricted, certificate of participation	26.40	
Cash in bank, unrestricted	427.70	
Balance Dec. 14, 1936		$6,833.16

Chauvenet Prize Fund

Balance Dec. 12, 1935		$642.94
Receipts: Interest	$ 15.00	
Anonymous contribution	100.00	115.00
		$757.94
Award Chauvenet Prize Dec. 30, 1935		100.00
		$657.94
3% HOLC Bond 1944–52	$500.00	
Cash in bank, unrestricted	157.94	
Balance Dec. 14, 1936		$657.94

Life Membership Fund

Liability on life memberships as of Jan. 1, 1936	$752.97
To be transferred to current funds, surplus	19.25
Liability on life memberships as of Jan. 1, 1937	$733.72

GENERAL ENDOWMENT FUND

Balance Dec. 12, 1935		$14,085.00
Transferred from current funds		2,000.00
Appreciation in value of Texas Power & Light Co. Bonds		115.00
		$16,200.00
$3\frac{1}{4}$% U. S. Treasury Bonds 1944–46	$1,000.00	
$3\frac{1}{4}$% U. S. Treasury Bond 1943–45	1,000.00	
3% HOLC Bonds 1944–52	5,500.00	
Land Trust Certificate	700.00	
5% Idaho Power Co. Gold Bonds 1947	2,000.00	
5% Texas Power & Light Co. Gold Bonds 1956	1,000.00	
$3\frac{1}{2}$% C. & O. Refunding Mortgage Bond Series D, 1996	1,000.00	
Pennsylvania R. R. Co. $3\frac{3}{4}$% Bonds Series C, 1970	2,000.00	
Oberlin Savings Bank savings account	2,000.00	
Balance Dec. 14, 1936		$16,200.00

It is a pleasure to announce an anonymous gift of one hundred dollars to the Chauvenet Prize Fund. It may also be pointed out that the Trustees transferred two thousand dollars to the General Endowment Fund in the form of bonds of the Pennsylvania Railroad and that three thousand dollars invested in Union Pacific bonds were replaced by a like amount in Chesapeake and Ohio Railway bonds.

Of the funds on hand, indicated in the first division of this financial report, $161.33 belongs to the Carus Monograph Fund, $427.70 to the Arnold Buffum Chace Fund, $157.94 to the Chauvenet Prize Fund, while $733.72 is held as a Life Membership Fund, representing the liability on life memberships already paid for, as of date Jan. 1, 1937.

When the accounts were closed Dec. 14, 1936, there remained on the total business for 1936 the following items:

BILLS RECEIVABLE

1936 individual dues	$200.00
Advertising	50.00
	$250.00

BILLS PAYABLE

Publisher's bills (Nov.–Dec. 1936)	$1,150.00
President's office	20.00
Editor-in-chief's office	120.00
Secretary-Treasurer's office	250.00
Comm. on Place of Math.	100.00
Subsidy *Duke Journal*	50.00
Carus Monograph Fund	161.33
Chace Fund	427.70
Chauvenet Prize Fund	157.94
Life Membership Fund	733.72
Init. fees due to sections	940.00
Expense acct. Carus Mon. Fund	75.00
	$4,185.69

If to the balance on 1936 business shown in the report, $6,399.44, there be added the estimated bills receivable, $250.00, and there be subtracted the estimated bills payable, $4,185.69, there results an estimated final balance on 1936

business of approximately $2,465. Since $2,000 of the current funds were transferred by vote of the Trustees to the General Endowment Fund during the past year, the result shows that the Association has made a comfortable profit this year. On the other hand, the Association faces an expectation of increased expense for some of the important committee activities during the coming year. Judging by the extent to which members have cleared up arrears in dues, the economic depression is not so severe as previously for most of our members; unfortunately, however, a considerable group still finds it well nigh impossible to afford continuous contact with the stimulating activities of the Association and its official journal.

W. D. CAIRNS, *Secretary-Treasurer*

THE FALL MEETING OF THE MICHIGAN SECTION

The second fall meeting of the Michigan Section of the Mathematical Association of America was held at Albion College on Saturday, November 28, 1936. The chairman of the Section, Professor C. C. Richtmeyer, presided.

The attendance was about seventy, including the following twenty-seven members of the Association: W. D. Baten, J. B. Brandeberry, C. J. Coe, C. C. Craig, S. E. Crowe, Wayne Dancer, C. H. Fischer, K. W. Folley, W. B. Ford, R. E. Gaskell, T. H. Hildebrandt, Ralph Hull, E. E. Ingalls, L. S. Johnston, H. S. Kaltenborn, Roy MacKay, H. H. Pixley, L. C. Plant, J. E. Powell, T. E. Raiford, G. Y. Rainich, Gladdis E. Richards, C. C. Richtmeyer, E. R. Sleight, D. E. South, H. E. Vaughan, Fern Welker.

A new feature of the program was the group of three papers prepared by undergraduates in pursuance of the program adopted by the Section to stimulate the interest of undergraduates in mathematics. These papers will be mimeographed and bound for distribution among the colleges whose members comprise the Michigan Section. A business meeting was held during the noon hour at the luncheon at the Parker Inn. The consensus was that a meeting should be held next fall, particularly to measure the response that may have developed to the program on undergraduate interests. It was felt the results so far obtained justify giving the plan a thorough trial. The invited address was given by Professor G. Y. Rainich of the University of Michigan in commemoration of the 200th anniversary of Lagrange.

The following papers were presented at the morning and afternoon sessions:

1. "A simplification of the calendar" by Professor W. D. Baten, Michigan State College.

2. Papers presented by the Committee on Undergraduate Interests: (a) "A locus problem" by Richard Fowler '36, Albion College; (b) "Complex numbers and triangles" by Paul Nims '37, University of Michigan; (c) "A study of the cardioid by inversion" by Violet Davis '36, University of Toledo.

3. "The generalization of the theorem of Pythagoras to three-dimensional space" by D. K. Kazarinoff, University of Michigan, introduced by the Secretary.

4. "A generalization of a theorem concerning harmonic functions" by Dr. Max Coral, Wayne University, introduced by Professor A. L. Nelson.

5. "Some results of a testing program at Michigan State College" by Dean L. C. Emmons, Michigan State College.

6. General Session: Professor N. H. Anning, University of Michigan; Professor W. D. Baten, Michigan State College; Professor T. H. Hildebrandt, University of Michigan; E. E. Ingalls, Albion College; and D. K. Kazarinoff, University of Michigan, each spoke briefly on mathematical topics of general interest.

7. "Conics and their inverses" by Professor L. S. Johnston, University of Detroit.

8. "Note on the class number function" by Dr. J. D. Elder, University of Michigan, introduced by the Secretary.

9. "J. L. Lagrange, on the 200th anniversary of his birth" by Professor G. Y. Rainich, University of Michigan.

Abstracts of these papers follow, the numbers corresponding to the numbers in the list of titles:

1. This paper presented in detail two simplified, perpetual calendars which the civilized nations of the world are now considering for adoption. The International Fixed Calendar, consisting of 13 months of 28 days each, and the World Calendar, consisting of 12 months with equal quarters, were explained by Professor Baten together with the advantages and disadvantages of each.

2. (a) Mr. Fowler traced the development and generalization of an ordinary textbook problem in analytic geometry. The problem was to find the locus of the midpoint of a polar to a conic as its pole describes a given curve. An account of this paper is given on page 173 of this number of this MONTHLY.

(b) If the vertices of certain types of triangles are represented by complex numbers there exist simple equations relating the vertices. From these equations several theorems about figures composed of a number of such triangles were proved by Mr. Nims. This paper is an extension of an article by J. R. Musselman in this MONTHLY for March 1932.

(c) Making use of well-known theorems on inversion, Miss Davis showed how geometrical properties of the cardioid can be derived from classical properties of the parabola. Defining the inverse of a tangent to the parabola as a *tangent circle*, and the inverse of the directrix of the parabola as the *director circle* of the cardioid, the author proved that the director circle is the locus of centers of the tangent circles.

3. It is known that Pappus of Alexandria extended the Pythagorean theorem in the 4th proposition of his Collectionis, liber IV (Pappi Alexandrini Collectionis quae supersunt, Fr. Hultsch's edition, Vol. I, p. 177). Mr. Kazarinoff extended the same theorem to three dimensional space as follows. The sum of the volumes of any three triangular prisms $SBC\alpha$, $SCA\beta$, $SAB\gamma$ constructed respectively on any three faces SBC, SCA, SAB of any tetrahedron $SABC$ is equal to the volume of the triangular prism constructed on the fourth face ABC and having lateral edges equal and parallel to SS', where S' is the point of intersection of the faces α, β, γ of the prisms parallel respectively to the faces SBC, SCA, SAB of the tetrahedron. Remark: The volume of any prism is considered positive if it is constructed as above outward from the tetrahedron, and negative if inward.

4. In Dr. Coral's paper, the methods developed by Haar for the calculus of variations for double integrals were applied to give a simple proof of the following theorem recently established by J. J. Gergen: if $v(x, y)$ is harmonic and positive in a plane region R and if $u(x, y)$ and its first partial derivatives are continuous in R, then $u(x, y)$ is harmonic in R if the integral of $v(\delta u/\delta n)$ is equal to the integral of $u(\delta v/\delta n)$ on every circle C in R. This theorem is a generalization of another characterization of harmonic functions discovered by Bôcher and Koebe in 1906. The proof employed by Dr. Coral made it apparent that Gergen's theorem is in turn a special case of a more general theorem concerning the solutions of a linear second order partial differential equation which is self-adjoint and of elliptic type.

5. In this paper the results of a four-year study of the grades of engineering students were presented. The students had been divided into quartiles on the basis of their showing on three examinations at the opening of their college careers. What happened to these students in the next four years in the academic studies was pointed out by Dean Emmons. Another study of students in the various Divisions of Michigan State College rating "good" or "poor" upon admission to college on the basis of their scores in the American Council of Education Psychological test was outlined. It was shown that of 93 expected to do well and 92 expected to do poorly 41 of the better group graduated and 5 of the poorer group. Many other points of interest were alluded to in connection with this latter study.

7. A student noting that the inverse of a circle with respect to a circle is a circle (including the special case where the inverse is a straight line) asked Professor Johnston if the inverse of every conic is a conic. This note is in part his answer. By letting the center of inversion move along the principal axis of the conic there is derived a class of curves containing as special cases such familiar higher plane curves as the cissoid of Diocles, the strophoid, the limaçon (all three forms), Cassinian ovals, and the lemniscate of Bernoulli—but never a conic unless the inverted curve is a circle. It is suggested that inversion of quadric surfaces in general may give important analogues of the stereographic projection (inverse) of the spherical surface, and therefore important practical applications of the theory in navigation and cartography.

8. E. T. Bell has shown by transcendental methods that if $H(d)$ is the number of solutions of $d=xy+yz+zx$ in positive integers and $K(4d)$ is the number of classes of binary quadratic forms of discriminant $-4d<0$, then $H(d) = 3[K(4d)-1]$, for d a prime. L. J. Mordell, again by transcendental methods, has generalized this to composite values of d. Dr. Elder obtained Mordell's result in slightly different form: $H(d)=3[K(4d)-SO/2]$ where SO is the number of distinct divisors of d. The additional result that, for d odd, $H_1(d)=3K(d)$ in which $H_1(d)$ is the number of solutions of $d=xy+yz+zx$ in positive odd integers was shown. Both proofs are elementary.

9. Professor Rainich spoke on the life of Joseph Louis Lagrange (1736–1813), characterized his personality by quotations from his letters, and dwelt briefly on the main features of his mathematical work.

C. C. Craig, *Secretary*

THE INDIAN MATHEMATICIAN RAMANUJAN*

By G. H. HARDY, Cambridge University

I have set myself a task in these lectures which is genuinely difficult and which, if I were determined to begin by making every excuse for failure, I might represent as almost impossible. I have to form myself, as I have never really formed before, and to try to help you to form, some sort of reasoned estimate of the most romantic figure in the recent history of mathematics; a man whose career seems full of paradoxes and contradictions, who defies almost all the canons by which we are accustomed to judge one another, and about whom all of us will probably agree in one judgment only, that he was in some sense a very great mathematician.

The difficulties in judging Ramanujan are obvious and formidable enough. Ramanujan was an Indian, and I suppose that it is always a little difficult for an Englishman and an Indian to understand one another properly. He was, at the best, a half-educated Indian; he never had the advantages, such as they are, of an orthodox Indian training; he never was able to pass the "First Arts Examination" of an Indian university, and never could rise even to be a "Failed B.A." He worked, for most of his life, in practically complete ignorance of modern European mathematics, and died when he was a little over 30 and when his mathematical education had in some ways hardly begun. He published abundantly—his published papers make a volume of nearly 400 pages—but he also left a mass of unpublished work which had never been analysed properly until the last few years. This work includes a great deal that is new, but much more that is rediscovery, and often imperfect rediscovery; and it is sometimes still impossible to distinguish between what he must have rediscovered and what he may somehow have learnt. I cannot imagine anybody saying with any confidence, even now, just how great a mathematician he was and still less how great a mathematician he might have been.

These are genuine difficulties, but I think that we shall find some of them less formidable than they look, and the difficulty which is the greatest for me has nothing to do with the obvious paradoxes of Ramanujan's career. The real difficulty for me is that Ramanujan was, in a way, my discovery. I did not invent him—like other great men, he invented himself—but I was the first really competent person who had the chance to see some of his work, and I can still remember with satisfaction that I could recognise at once what a treasure I had found. And I suppose that I still know more of Ramanujan than any one else, and am still the first authority on this particular subject. There are other people in England, Professor Watson in particular, and Professor Mordell, who know parts of his work very much better than I do, but neither Watson nor Mordell knew Ramanujan himself as I did. I saw him and talked with him almost every

* A lecture delivered at the Harvard Tercentenary Conference of Arts and Sciences, August 31, 1936.

day for several years, and above all I actually collaborated with him. I owe more to him than to any one else in the world with one exception, and my association with him is the one romantic incident in my life. The difficulty for me then is not that I do not know enough about him, but that I know and feel too much and that I simply cannot be impartial.

I rely, for the facts of Ramanujan's life, on Seshu Aiyar and Ramachaundra Rao, whose memoir of Ramanujan is printed, along with my own, in his *Collected Papers*. He was born in 1887 in a Brahmin family at Erode near Kumbakonam, a fair-sized town in the Tanjore district of the Presidency of Madras. His father was a clerk in a cloth-merchant's office in Kumbakonam, and all his relatives, though of high caste, were very poor.

He was sent at 7 to the High School of Kumbakonam, and remained there nine years. His exceptional abilities had begun to show themselves before he was 10, and by the time that he was 12 or 13 he was recognised as a quite abnormal boy. His biographers tell some curious stories of his early years. They say for example that, soon after he had begun the study of trigonometry, he discovered for himself "Euler's theorems for the sine and cosine" (by which I understand the relations between the circular and exponential functions), and was very disappointed when he found later, apparently from the second volume of Loney's *Trigonometry*, that they were known already. Until he was 16 he had never seen a mathematical book of any higher class. *Whittaker's Modern Analysis* had not yet spread so far, and Bromwich's *Infinite Series* did not exist. There can be no doubt that either of these books would have made a tremendous difference to him if they could have come his way. It was a book of a very different kind, Carr's *Synopsis*, which first aroused Ramanujan's full powers.

Carr's book (*A synopsis of elementary results in pure and applied mathematics*, by George Shoobridge Carr, formerly Scholar of Gonville and Caius College, Cambridge, published in two volumes in 1880 and 1886) is almost unprocurable now. There is a copy in the Cambridge University Library, and there happened to be one in the library of the Government College of Kumbakonam, which was borrowed for Ramanujan by a friend. The book is not in any sense a great one, but Ramanujan has made it famous, and there is no doubt that it influenced him profoundly and that his acquaintance with it marked the real starting point of his career. Such a book must have had its qualities, and Carr's, if not a book of any high distinction, is no mere third-rate textbook, but a book written with some real scholarship and enthusiasm and with a style and individuality of its own. Carr himself was a private coach in London, who came to Cambridge as an undergraduate when he was nearly 40, and was 12th Senior Optime in the Mathematical Tripos of 1880 (the same year in which he published the first volume of his book). He is now completely forgotten, even in his own college, except in so far as Ramanujan has kept his name alive; but he must have been in some ways rather a remarkable man.

I suppose that the book is substantially a summary of Carr's coaching notes. If you were a pupil of Carr, you worked through the appropriate sections of the

Synopsis. It covers roughly the subjects of Schedule A of the present Tripos (as these subjects were understood in Cambridge in 1880), and is effectively the "synopsis" it professes to be. It contains the enunciations of 6165 theorems, systematically and quite scientifically arranged, with proofs which are often little more than cross-references and are decidedly the least interesting part of the book. All this is exaggerated in Ramanujan's famous note-books (which contain practically no proofs at all), and any student of the note-books can see that Ramanujan's ideal of presentation had been copied from Carr's.

Carr has sections on the obvious subjects, algebra, trigonometry, calculus and analytical geometry, but some sections are developed disproportionally, and particularly the formal side of the integral calculus. This seems to have been Carr's pet subject, and the treatment of it is very full and in its way definitely good. There is no theory of functions; and I very much doubt whether Ramanujan, to the end of his life, ever understood at all clearly what an analytic function is. What is more surprising, in view of Carr's own tastes and Ramanujan's later work, is that there is no elliptic functions. However Ramanujan may have acquired his very peculiar knowledge of this theory, it was not from Carr.

On the whole, considered as an inspiration for a boy of such abnormal gifts, Carr was not too bad, and Ramanujan responded amazingly.

"Through the new world thus opened to him," say his Indian biographers,* "Ramanujan went ranging with delight. It was this book which awakened his genius. He set himself to establish the formulae given therein. As he was without the aid of other books, each solution was a piece of research so far as he was concerned . . . Ramanujan used to say that the goddess of Namakkal inspired him with the formulae in dreams. It is a remarkable fact that frequently, on rising from bed, he would note down results and rapidly verify them, though he was not always able to supply a rigorous proof. . . ."

I have quoted the last sentences deliberately, not because I attach any importance to them—I am no more interested in the goddess of Namakkal than you are—but because we are now approaching the difficult and tragic part of Ramanujan's career, and we must try to understand what we can of his psychology and of the atmosphere surrounding him in his early years.

I am sure that Ramanujan was no mystic and that religion, except in a strictly material sense, played no important part in his life. He was an orthodox high-caste Hindu, and always adhered (indeed with a severity most unusual in Indian residents in England) to all the observances of his caste. He had promised his parents to do so, and he kept his promises to the letter. He was a vegetarian in the strictest sense—this proved a terrible difficulty later when he fell ill—and all the time he was in Cambridge he cooked all his food himself, and never cooked it without first changing into pyjamas.

Now the two memoirs of Ramanujan printed in the *Papers* (and both written

* Quotations (except those from my own memoir of Ramanujan) are from Seshu Aiyar and Ramachaundra Rao.

by men who, in their different ways, knew him very well) contradict one another flatly about his religion. Seshu Aiyar and Ramachaundra Rao say

"Ramanujan had definite religious views. He had a special veneration for the Namakkal goddess. . . . He believed in the existence of a Supreme Being and in the attainment of Godhead by men. . . . He had settled convictions about the problem of life and after . . . ";

while I say

" . . . his religion was a matter of observance and not of intellectual conviction, and I remember well his telling me (much to my surprise) that all religions seemed to him more or less equally true . . . ".

Which of us is right? For my part I have no doubt at all; I am quite certain that I am.

Classical scholars have, I believe, a general principle, *difficilior lectio potior*—the more difficult reading is to be preferred—in textual criticism. If the Archbishop of Canterbury tells one man that he* believes in God, and another that he does not, then it is probably the second assertion which is true, since otherwise it is very difficult to understand why he should have made it, while there are many excellent reasons for his making the first whether it be true or false. Similarly, if a strict Brahmin like Ramanujan told me, as he certainly did, that he had no definite beliefs, then it is 100 to 1 that he meant what he said.

This was no sufficient reason why Ramanujan should outrage the feelings of his parents or his Indian friends. He was not a reasoned infidel, but an "agnostic" in its strict sense, who saw no particular good, and no particular harm, in Hinduism or in any other religion. Hinduism is, far more for example than Christianity, a religion of observance, in which belief counts for extremely little in any case, and, if Ramanujan's friends assumed that he accepted the conventional doctrines of such a religion, and he did not disillusion them, he was practising a quite harmless, and probably necessary, economy of truth.

This question of Ramanujan's religion is not itself important, but it is not altogether irrelevant, because there is one thing which I am really anxious to insist upon as strongly as I can. There is quite enough about Ramanujan that is difficult to understand, and we have no need to go out of our way to manufacture mystery. For myself, I liked and admired him enough to wish to be a rationalist about him; and I want to make it quite clear to you that Ramanujan, when he was living in Cambridge in good health and comfortable surroundings, was, in spite of his oddities, as reasonable, as sane, and in his way as shrewd a person as anyone here. The last thing which I want you to do is to throw up your hands and exclaim "here is something unintelligible, some mysterious manifestation of the immemorial wisdom of the East!" I do not believe in the immemorial wisdom of the East, and the picture I want to present to you is that of a man who had his peculiarities like other distinguished men, but a man in whose

* The Archbishop.

society one could take pleasure, with whom one could take tea and discuss politics or mathematics; the picture in short, not of a wonder from the East, or an inspired idiot, or a psychological freak, but of a rational human being who happened to be a great mathematician.

Until he was about 17, all went well with Ramanujan.

"In December 1903 he passed the Matriculation Examination of the University of Madras, and in the January of the succeeding year he joined the Junior First in Arts class of the Government College, Kumbakonam, and won the Subrahmanyam scholarship, which is generally awarded for proficiency in English and Mathematics . . . ",

but after this there came a series of tragic checks.

"By this time, he was so absorbed in the study of Mathematics that in all lecture hours—whether devoted to English, History, or Physiology—he used to engage himself in some mathematical investigation, unmindful of what was happening in the class. This excessive devotion to mathematics and his consequent neglect of the other subjects resulted in his failure to secure promotion to the senior class and in the consequent discontinuance of the scholarship. Partly owing to disappointment and partly owing to the influence of a friend, he ran away northward into the Telugu country, but returned to Kumbakonam after some wandering and rejoined the college. As owing to his absence he failed to make sufficient attendances to obtain his term certificate in 1905, he entered Pachaiyappa's College, Madras, in 1906, but falling ill returned to Kumbakonam. He appeared as a private student for the F. A. examination of December 1907 and failed . . .".

Ramanujan does not seem to have had any definite occupation, except mathematics, until 1912. In 1909 he married, and it became necessary for him to have some regular employment, but he had great difficulty in finding any because of his unfortunate college career. About 1910 he began to find more influential Indian friends, Ramaswami Aiyar and his two biographers, but all their efforts to find a tolerable position for him failed, and in 1912 he became a clerk in the office of the Port Trust of Madras, at a salary of about £30 a year. He was then nearly 25. The years between 18 and 25 are the critical years in a mathematician's career, and the damage had been done. Ramanujan's genius never had again its chance of full development.

There is not much to say about the rest of Ramanujan's life. His first substantial paper had been published in 1911, and in 1912 his exceptional powers began to be understood. It is significant that, though Indians could befriend him, it was only the English who could get anything effective done. Sir Francis Spring and Sir Gilbert Walker obtained a special scholarship for him, £60 a year, sufficient for a married Indian to live in tolerable comfort. At the beginning of 1913 he wrote to me, and Professor Neville and I, after many difficulties, got him to England in 1914. Here he had three years of uninterrupted activity, the results of which you can read in his *Papers*. He fell ill in the summer of 1917,

and never really recovered, though he continued to work, rather spasmodically, but with no real sign of degeneration, until his death in 1920. He became a Fellow of the Royal Society early in 1918, and a Fellow of Trinity College, Cambridge, later in the same year (and was the first Indian elected to either society). His last mathematical letter on "Mock-Theta functions", the subject of Professor Watson's presidential address to the London Mathematical Society last year, was written about two months before he died.

The real tragedy about Ramanujan was not his early death. It is of course a disaster that any great man should die young, but a mathematician is often comparatively old at 30, and his death may be less of a catastrophe than it seems. Abel died at 26 and, although he would no doubt have added a great deal more to mathematics, he could hardly have become a greater man. The tragedy of Ramanujan was not that he died young, but that, during his five unfortunate years, his genius was misdirected, side-tracked, and to a certain extent distorted.

I have been looking again through what I wrote about Ramanujan 16 years ago, and, although I know his work a good deal better now than I did then, and can think about him more dispassionately, I do not find a great deal which I should particularly want to alter. But there is just one sentence which now seems to me indefensible. I wrote

"Opinions may differ about the importance of Ramanujan's work, the kind of standard by which it should be judged, and the influence which it is likely to have on the mathematics of the future. It has not the simplicity and the inevitableness of the very greatest work; it would be greater if it were less strange. One gift it shows which no one can deny, profound and invincible originality. He would probably have been a greater mathematician if he could have been caught and tamed a little in his youth; he would have discovered more that was new, and that, no doubt, of greater importance. On the other hand he would have been less of a Ramanujan, and more of a European professor, and the loss might have been greater than the gain . . ."

and I stand by that except for the last sentence, which is quite ridiculous sentimentalism. There was no gain at all when the College at Kumbakonam rejected the one great man they had ever possessed, and the loss was irreparable; it is the worst instance that I know of the damage that can be done by an inefficient and inelastic educational system. So little was wanted, £60 a year for five years, occasional contact with almost anyone who had real knowledge and a little imagination, for the world to have gained another of its greatest mathematicians.

Ramanujan's letters to me, which are reprinted in full in the *Papers*, contain the bare statements of about 120 theorems, mostly formal identities extracted from his note-books. I quote fifteen which are fairly representative. They include two theorems, (14) and (15), which are as interesting as any but of which one is false and the other, as stated, misleading. The rest have all been verified

since by somebody; in particular Rogers and Watson found the proofs of the extremely difficult theorems (10)–(12).

(1) $$1-\frac{3!}{(1!2!)^3}x^2+\frac{6!}{(2!4!)^3}x^4-\cdots$$
$$=\left(1+\frac{x}{(1!)^3}+\frac{x^2}{(2!)^3}+\cdots\right)\left(1-\frac{x}{(1!)^3}+\frac{x^2}{(2!)^3}-\cdots\right).$$

(2) $$1-5\left(\frac{1}{2}\right)^3+9\left(\frac{1.3}{2.4}\right)^3-13\left(\frac{1.3.5}{2.4.6}\right)^3+\cdots=\frac{2}{\pi}.$$

(3) $$1+9\left(\frac{1}{4}\right)^4+17\left(\frac{1.5}{4.8}\right)^4+25\left(\frac{1.5.9}{4.8.12}\right)^4+\cdots=\frac{2^{3/2}}{\pi^{1/2}\{\Gamma(\frac{3}{4})\}^2}.$$

(4) $$1-5\left(\frac{1}{2}\right)^5+9\left(\frac{1.3}{2.4}\right)^5-13\left(\frac{1.3.5}{2.4.6}\right)^5+\cdots=\frac{2}{\{\Gamma(\frac{3}{4})\}^4}.$$

(5) $$\int_0^\infty \frac{1+\left(\dfrac{x}{b+1}\right)^2}{1+\left(\dfrac{x}{a}\right)^2}\cdot\frac{1+\left(\dfrac{x}{b+2}\right)^2}{1+\left(\dfrac{x}{a+1}\right)^2}\cdots dx=\tfrac{1}{2}\pi^{1/2}\frac{\Gamma(a+\frac{1}{2})\Gamma(b+1)\Gamma(b-a+\frac{1}{2})}{\Gamma(a)\Gamma(b+\frac{1}{2})\Gamma(b-a+1)}.$$

(6) $$\int_0^\infty\frac{dx}{(1+x^2)(1+r^2x^2)(1+r^4x^2)\cdots}=\frac{\pi}{2(1+r+r^3+r^6+r^{10}+\cdots)}.$$

(7) If $\alpha\beta=\pi^2$, then
$$\alpha^{-1/4}\left(1+4\alpha\int_0^\infty\frac{xe^{-\alpha x^2}}{e^{2\pi x}-1}\,dx\right)=\beta^{-1/4}\left(1+4\beta\int_0^\infty\frac{xe^{-\beta x^2}}{e^{2\pi x}-1}\,dx\right).$$

(8) $$\int_0^a e^{-x^2}dx=\tfrac{1}{2}\pi^{1/2}-\frac{e^{-a^2}}{2a+}\,\frac{1}{a+}\,\frac{2}{2a+}\,\frac{3}{a+}\,\frac{4}{2a+\cdots}.$$

(9) $$4\int_0^\infty\frac{xe^{-x\sqrt{5}}}{\cosh x}\,dx=\frac{1}{1+}\,\frac{1^2}{1+}\,\frac{1^2}{1+}\,\frac{2^2}{1+}\,\frac{2^2}{1+}\,\frac{3^2}{1+}\,\frac{3^2}{1+\cdots}.$$

(10) If $$u=\frac{x}{1+}\,\frac{x^5}{1+}\,\frac{x^{10}}{1+}\,\frac{x^{15}}{1+\cdots},\qquad v=\frac{x^{1/5}}{1+}\,\frac{x}{1+}\,\frac{x^2}{1+}\,\frac{x^3}{1+\cdots},$$
then $$v^5=u\frac{1-2u+4u^2-3u^3+u^4}{1+3u+4u^2+2u^3+u^4}.$$

(11) $$\frac{1}{1+}\,\frac{e^{-2\pi}}{1+}\,\frac{e^{-4\pi}}{1+\cdots}=\left\{\sqrt{\left(\frac{5+\sqrt{5}}{2}\right)}-\frac{\sqrt{5}+1}{2}\right\}e^{2\pi/5}.$$

(12) $$\frac{1}{1+}\,\frac{e^{-2\pi\sqrt{5}}}{1+}\,\frac{e^{-4\pi\sqrt{5}}}{1+\cdots}=\left[\frac{\sqrt{5}}{1+\sqrt[5]{\left\{5^{3/4}\left(\dfrac{\sqrt{5}-1}{2}\right)^{5/2}-1\right\}}}-\frac{\sqrt{5}+1}{2}\right]e^{2\pi/\sqrt{5}}.$$

(13) If $F(k)=1+\left(\frac{1}{2}\right)^2 k+\left(\frac{1.3}{2.4}\right)^2 k^2+\cdots$ and $F(1-k)=\sqrt{(210)}F(k)$, then

$k=(\sqrt{2}-1)^4(2-\sqrt{3})^2(\sqrt{7}-\sqrt{6})^4(8-3\sqrt{7})^2(\sqrt{10}-3)^4(4-\sqrt{15})^4(\sqrt{15}-\sqrt{14})^2(6-\sqrt{35})^2.$

(14) The coefficient of x^n in $(1-2x+2x^4-2x^9+\cdots)^{-1}$ is the integer nearest to

$$\frac{1}{4n}\left(\cosh(\pi\sqrt{n})-\frac{\sinh(\pi\sqrt{n})}{\pi\sqrt{n}}\right).$$

(15) The number of numbers between A and x which are either squares or sums of two squares is

$$K\int_A^x\frac{dt}{\sqrt{(\log t)}}+\theta(x),$$

where $K=0.764\cdots$ and $\theta(x)$is very small compared with the previous integral.

I should like you to begin by trying to reconstruct the immediate reactions of an ordinary professional mathematician who receives a letter like this from an unknown Hindu clerk.

The first question was whether I could recognize anything. I had proved things rather like (7) myself, and seemed vaguely familiar with (8). Actually (8) is classical; it is a formula of Laplace first proved properly by Jacobi; and (9) occurs in a paper published by Rogers in 1907. I thought that, as an expert in definite integrals, I could probably prove (5) and (6), and did so, though with a good deal more trouble than I had expected. On the whole the integral formulas seemed the least impressive.

The series formulas (1)–(4) I found much more intriguing, and it soon became obvious that Ramanujan must possess much more general theorems and was keeping a great deal up his sleeve. The second is a formula of Bauer well known in the theory of Legendre series, but the others are much harder than they look. The theorems required in proving them can all be found now in Bailey's Cambridge Tract on hypergeometric functions.

The formulas (10)–(13) are on a different level and obviously both difficult and deep. An expert in elliptic functions can see at once that (13) is derived somehow from the theory of "complex multiplication", but (10)–(12) defeated me completely; I had never seen anything in the least like them before. A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because, if they were not true, no one would have had the imagination to invent them. Finally (you must remember that I knew nothing whatever about Ramanujan, and had to think of every possibility), the writer must be completely honest, because great mathematicians are commoner than thieves or humbugs of such incredible skill.

The last two formulas stand apart because they are not right and show Ramanujan's limitations, but that does not prevent them from being additional evidence of his extraordinary powers. The function in (14) is a genuine approxi-

mation to the coefficient, though not at all so close as Ramanujan imagined, and Ramanujan's false statement was one of the most fruitful he ever made, since it ended by leading us to all our joint work on partitions. Finally (15), though literally "true", is definitely misleading (and Ramanujan was under a real misapprehension). The integral has no advantage, as an approximation, over the simpler function

$$\frac{Kx}{\sqrt{(\log x)}}, \tag{16}$$

found in 1908 by Landau. Ramanujan was deceived by a false analogy with the problem of the distribution of primes. I must postpone till later what I have to say about Ramanujan's work on this side of the theory of numbers.

It was inevitable that a very large part of Ramanujan's work should prove on examination to have been anticipated. He had been carrying an impossible handicap, a poor and solitary Hindu pitting his brains against the accumulated wisdom of Europe. He had had no real teaching at all; there was no one in India from whom he had anything to learn. He can have seen at the outside three or four books of good quality, all of them English. There had been periods in his life when he had access to the library in Madras, but it was not a very good one; it contained very few French or German books; and in any case Ramanujan did not know a word of either language. I should estimate that about two-thirds of Ramanujan's best Indian work was rediscovery, and comparatively little of it was published in his life-time, though Watson, who has worked systematically through his notebooks, has since disinterred a good deal more.

The great bulk of Ramanujan's published work was done in England.* His mind had hardened to some extent, and he never became at all an "orthodox" mathematician, but he could still learn to do new things, and do them extremely well. It was impossible to teach him systematically, but he gradually absorbed new points of view. In particular he learnt what was meant by proof, and his later papers, while in some ways as odd and individual as ever, read like the works of a well-informed mathematician. His methods and his weapons, however, remained essentially the same. One would have thought that such a formalist as Ramanujan would have revelled in Cauchy's Theorem, but he practically never used it,* and the most astonishing testimony to his formal genius is that he never seemed to feel the want of it in the least.

It is easy to compile an imposing list of theorems which Ramanujan rediscovered. Such a list naturally cannot be quite sharp, since sometimes he found a part only of a theorem, and sometimes, though he found the whole theorem, he was without the proof which is essential if the theorem is to be properly understood. For example, in the analytic theory of numbers he had,

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* Perhaps never. There is a reference to "the theory of residues" on p. 129 of the *Papers*, but I believe that I supplied this myself.

in a sense, discovered a great deal, but he was a very long way from understanding the real difficulties of the subject. And there is some of his work, mostly in the theory of elliptic functions, about which some mystery still remains; it is not possible, after all the work of Watson and Mordell, to draw the line between what he may have picked up somehow and what he must have found for himself. I will take only cases in which the evidence seems to me tolerably clear.

Here I must admit that I am to blame, since there is a good deal which we should like to know now and which I could have discovered quite easily. I saw Ramanujan almost every day, and could have cleared up most of the obscurity by a little cross-examination. Ramanujan was quite able and willing to give a straight answer to a question, and not in the least disposed to make a mystery of his achievements. I hardly asked him a single question of this kind; I never even asked him whether (as I think he must have done) he had seen Cayley's or Greenhill's *Elliptic Functions*.

I am sorry about this now, but it does not really matter very much, and it was entirely natural. In the first place, I did not know that Ramanujan was going to die. He was not particularly interested in his own history or psychology; he was a mathematician anxious to get on with the job. And after all I too was a mathematician, and a mathematician meeting Ramanujan had more interesting things to think about than historical research. It seemed ridiculous to worry him about how he had found this or that known theorem, when he was showing me half a dozen new ones almost every day.

I do not think that Ramanujan discovered much in the classical theory of numbers, or indeed that he ever knew a great deal. He had no knowledge at all, at any time, of the general theory of arithmetical forms. I doubt whether he knew the law of quadratic reciprocity before he came here. Diophantine equations should have suited him, but he did comparatively little with them, and what he did do was not his best. Thus he gave solutions of Euler's equation

$$x^3 + y^3 + z^3 = w^3, \tag{17}$$

such as

$$x = 3a^2 + 5ab - 5b^2,\ y = 4a^2 - 4ab + 6b^2,\ z = 5a^2 - 5ab - 3b^2,\ w = 6a^2 - 4ab + 4b^2; \tag{18}$$

and

$$\begin{gathered} x = m^7 - 3m^4(1+p) + m(2+6p+3p^2),\ y = 2m^6 - 3m^3(1+2p) + 1 + 3p + 3p^2, \\ z = m^6 - 1 - 3p - 3p^2,\ w = m^7 - 3m^4p + m(3p^2 - 1); \end{gathered} \tag{19}$$

but neither of these is the general solution.

He rediscovered the famous theorem of von Staudt about the Bernoullian numbers:

$$(-1)^n B_n = G_n + \frac{1}{2} + \frac{1}{p} + \frac{1}{q} + \cdots \frac{1}{r}, \tag{20}$$

where $p, q, \cdots$ are those odd primes such that $p-1, q-1, \cdots$ are divisors of

$2n$, and G_n is an integer. In what sense he had proved it it is difficult to say, since he found it at a time of his life when he had hardly formed any definite concept of proof. As Littlewood says "the clear-cut idea of what is *meant* by a proof, nowadays so familiar as to be taken for granted, he perhaps did not possess at all; if a significant piece of reasoning occurred somewhere, and the total mixture of evidence and intuition gave him certainty, he looked no further". I shall have something to say later about this question of proof, but I postpone it to another context in which it is much more important. In this case there is nothing in the proof that was not obviously within Ramanujan's powers.

There is a considerable chapter of the theory of numbers, in particular the theory of the representation of integers by sums of squares, which is closely bound up with the theory of elliptic functions. Thus the number of representations of n by two squares is

$$r(n) = 4\{d_1(n) - d_3(n)\}, \tag{21}$$

where $d_1(n)$ is the number of divisors of n of the form $4k+1$ and $d_3(n)$ the number of divisors of the form $4k+3$. Jacobi gave similar formulas for 4, 6 and 8 squares. Ramanujan found all these, and much more of the same kind.

He also found Gauss's theorem that n is the sum of 3 squares except when it is of the form

$$4^a(8k + 7), \tag{22}$$

but I do not attach much importance to this. The theorem is quite easy to guess and difficult to prove. All known proofs depend upon the general theory of ternary forms, of which Ramanujan knew nothing, and I agree with Professor Dickson in thinking it very unlikely that he possessed one. In any case he knew nothing about the number of representations.

Ramanujan, then, before he came to England, had added comparatively little to the theory of numbers; but no one can understand him who does not understand his passion for numbers in themselves. I wrote before

"He could remember the idiosyncrasies of numbers in an almost uncanny way. It was Littlewood who said that every positive integer was one of Ramanujan's personal friends. I remember going to see him once when he was lying ill in Putney. I had ridden in taxi-cab No. 1729, and remarked that the number seemed to me rather a dull one, and that I hoped that it was not an unfavorable omen. 'No,' he replied, 'it is a very interesting number; it is the smallest number expressible as a sum of two cubes in two different ways.'* I asked him, naturally, whether he could tell me the solution of the corresponding problem for fourth powers; and he replied, after a moment's thought, that he knew no obvious example, and supposed that the first such number must be very large."†

* $1729 = 12^3+1^3 = 10^3+9^3$.

† The smallest known is Euler's example

$$635318657 = 158^4 + 59^4 = 134^4 + 133^4.$$

In algebra, Ramanujan's main work was concerned with hypergeometric series and continued fractions (I use the word algebra, of course, in its old-fashioned sense). These subjects suited him exactly, and here he was unquestionably one of the great masters. There are three now famous identities, the "Dougall-Ramanujan identity"

$$(23)\qquad \begin{aligned}&\sum_{n=0}^{\infty}(-1)^n(s+2n)\frac{s^{(n)}}{1^{(n)}}\frac{(x+y+z+u+2s+1)^{(n)}}{(x+y+z+u+s)_{(n)}}\prod_{x,y,z,u}\frac{x_{(n)}}{(x+s+1)^{(n)}}\\ &\qquad=\frac{s}{\Gamma(s+1)\Gamma(x+y+z+u+s+1)}\prod_{x,y,z,u}\frac{\Gamma(x+s+1)\Gamma(y+z+u+s+1)}{\Gamma(z+u+s+1)},\end{aligned}$$

where

$$a^{(n)} = a(a+1)\cdots(a+n-1), \qquad a_{(n)} = a(a-1)\cdots(a-n+1),$$

and the "Rogers-Ramanujan identities"

$$(24)\qquad \begin{aligned}1+\frac{q}{1-q}+\frac{q^4}{(1-q)(1-q^2)}+\frac{q^9}{(1-q)(1-q^2)(1-q^3)}+\cdots\\ =\frac{1}{(1-q)(1-q^6)\cdots(1-q^4)(1-q^9)\cdots},\\ 1+\frac{q^2}{1-q}+\frac{q^6}{(1-q)(1-q^2)}+\frac{q^{12}}{(1-q)(1-q^2)(1-q^3)}+\cdots\\ =\frac{1}{(1-q^2)(1-q^7)\cdots(1-q^3)(1-q^8)\cdots},\end{aligned}$$

in which he had been anticipated by British mathematicians, and about which I shall speak in other lectures. As regards hypergeometric series one may say, roughly, that he rediscovered the formal theory, set out in Bailey's tract, as it was known up to 1920. There is something about it in Carr, and more in Chrystal's *Algebra*, and no doubt he got his start from that. The four formulas (1)–(4) are highly specialized examples of this work.

His masterpiece in continued fractions was his work on

$$(25)\qquad \frac{1}{1+}\,\frac{x}{1+}\,\frac{x^2}{1+\cdots},$$

which includes the theorems (10)–(12). The theory of this fraction depends upon the Rogers-Ramanujan identities, in which he had been anticipated by Rogers, but he had gone beyond Rogers in other ways and the theorems which I have quoted are his own. He had many other very general and very beautiful formulas, of which formulas like Laguerre's

$$(26)\qquad \frac{(x+1)^n-(x-1)^n}{(x+1)^n+(x-1)^n}=\frac{n}{x+}\,\frac{n^2-1}{3x+}\,\frac{n^2-2^2}{5x+\cdots}$$

are extremely special cases. Watson* has recently published a proof of the most imposing of them.

It is perhaps in his work in these fields that Ramanujan shows at his very best. I wrote before

"It was his insight into algebraical formulae, transformation of infinite series, and so forth, that was most amazing. On this side most certainly I have never met his equal, and I can compare him only with Euler or Jacobi. He worked, far more than the majority of modern mathematicians, by induction from numerical examples; all his congruence properties of partitions, for example, were discovered in this way. But with his memory, his patience, and his power of calculation he combined a power of generalization, a feeling for form, and a capacity for rapid modification of his hypotheses, that were often really startling, and made him, in his own peculiar field, without a rival in his day."

I do not think now that this extremely strong language is extravagant. It is possible that the great days of formulas are finished, and that Ramanujan ought to have been born 100 years ago; but he was by far the greatest formalist of his time. There have been a good many more important, and I suppose one must say greater, mathematicians than Ramanujan during the last 50 years, but not one who could stand up to him on his own ground. Playing the game of which he knew the rules, he could give any mathematician in the world fifteen.

In analysis proper Ramanujan's work is inevitably less impressive, since he knew no theory of functions, and you cannot do real analysis without it, and since the formal side of the integral calculus, which was all that he could learn from Carr or any other book, has been worked over so repeatedly and so intensively. Still, Ramanujan rediscovered an astonishing number of the most beautiful analytic identities. Thus the functional equation for the Riemann Zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

namely

$$\zeta(1 - s) = 2(2\pi)^{-s} \cos \tfrac{1}{2}s\pi\ \Gamma(s)\zeta(s), \tag{27}$$

stands (in an almost unrecognizable notation) in the notebooks. So does Poisson's summation formula

$$\alpha^{1/2}\left\{\tfrac{1}{2}\phi(0) + \phi(\alpha) + \phi(2\alpha) + \cdots\right\} = \beta^{1/2}\left\{\tfrac{1}{2}\psi(0) + \psi(\beta) + \psi(2\beta) + \cdots\right\}, \tag{28}$$

where

$$\psi(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\infty} \phi(t) \cos xt dt$$

and $\alpha\beta = 2\pi$; and so also does Abel's† functional equation

* G. N. Watson, *Proceedings of the Cambridge Philosophical Society*, vol. 31, 1935, p. 7.

† The equation was rediscovered by Rogers and is attributed to him in the *Papers* (p. 337); but it is to be found in a posthumous fragment of Abel (*Œuvres*, t.2., p. 193).

$$(29)\qquad L(x) + L(y) + L(xy) + L\left(\frac{x(1-y)}{1-xy}\right) + L\left(\frac{y(1-x)}{1-xy}\right) = 3L(1)$$

for

$$L(x) = \frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \cdots .$$

He had most of the formal ideas which underlie the recent work of Watson and of Titchmarsh and myself on "Fourier kernels" and "reciprocal functions"; and he could of course evaluate any evaluable definite integral. There is one particularly interesting formula, viz.

$$(30)\qquad \int_0^\infty x^{s-1}\{\phi(0) - x\phi(1) + x^2\phi(2) - \cdots\}dx = \frac{\pi\phi(-s)}{\sin s\pi},$$

of which he was especially fond and made continual use. This is really an "interpolation formula", which enables us to say, for example, that, under certain conditions, a function which vanishes for all positive integral values of its argument must vanish identically. I have never seen this formula stated explicitly by any one else, though it is closely connected with the work of Mellin and others.

I have left till last the two most intriguing sides of Ramanujan's early work, his work on elliptic functions and in the analytic theory of numbers. The first is probably too specialized and intricate for anyone but an expert to understand, and I shall say nothing about it now. The second subject is still more difficult (as anyone who has read Landau's book on primes or Ingham's tract will know), but anyone can understand roughly what the problems of the subject are, and any decent mathematician can understand roughly why they defeated Ramanujan. For this was Ramanujan's one real failure; he showed, as always, astonishing imaginative power, but he proved next to nothing, and a great deal even of what he imagined was false.

Here I am obliged to interpolate some remarks on a very difficult subject, *proof* and its importance in mathematics. All physicists, and a good many quite respectable mathematicians, are contemptuous about proof. I have heard Professor Eddington, for example, maintain that proof, as pure mathematicians understand it, is really quite uninteresting and unimportant, and that no one who is really certain that he has found something good should waste his time looking for a proof. It is true that Eddington is inconsistent, and has sometimes even descended to proof himself. It is not enough for him to have direct knowledge that there are exactly

$$136 \cdot 2^{256}$$

protons in the universe; he cannot resist the temptation of proving it; and I cannot help thinking that the proof, whatever it may be worth, gives him a certain amount of intellectual satisfaction. His apology would no doubt be that

"proof" means something quite different for him from what it means for a pure mathematician, and in any case we need not take him too literally. But the opinion which I have attributed to him, and with which I am sure that almost all physicists agree at the bottom of their hearts, is one to which a mathematician ought to have some reply.

I am not going to get entangled in the analysis of a particularly prickly concept, but I think that there are a few points about proof where nearly all mathematicians are agreed. In the first place, even if we do not understand exactly what proof is, we can, in ordinary analysis at any rate, recognise a proof when we see one. Secondly, there are two different motives in any presentation of a proof. The first motive is simply to secure conviction. The second is to exhibit the conclusion as the climax of a conventional pattern of propositions, a sequence of propositions whose truth is admitted and which are arranged in accordance with rules. These are the two ideals, and experience shows that, except in the simplest mathematics, we can hardly ever satisfy the first ideal without also satisfying the second. We may be able to recognise directly that 5, or even 17, is prime, but nobody can convince himself that

$$2^{127} - 1$$

is prime except by studying a proof. No one has ever had an imagination so vivid and comprehensive as that.

A mathematician usually discovers a theorem by an effort of intuition; the conclusion strikes him as plausible, and he sets to work to manufacture a proof. Sometimes this is a matter of routine, and any well-trained professional could supply what is wanted, but more often imagination is a very unreliable guide. In particular this is so in the analytic theory of numbers, where even Ramanujan's imagination led him very seriously astray.

There is a striking example, which I have very often quoted, of a false conjecture which seems to have been endorsed even by Gauss and which took about 100 years to refute. The central problem of the analytic theory of numbers is that of the distribution of the primes. The number $\pi(x)$ of primes less than a large number x is approximately

$$\frac{x}{\log x}; \tag{31}$$

this is the "Prime Number Theorem", which had been conjectured for a very long time, but was never established properly until Hadamard and de la Vallée-Poussin proved it in 1896. The approximation errs by defect, and a much better one is

$$\mathrm{Li}\,x = \int_2^x \frac{dt}{\log t}. \tag{32}$$

In some ways a still better one is

$$\text{(33)} \qquad \mathrm{Li}x - \tfrac{1}{2}\mathrm{Li}x^{1/2} - \tfrac{1}{3}\mathrm{Li}x^{1/3} - \tfrac{1}{5}\mathrm{Li}x^{1/5} + \tfrac{1}{6}\mathrm{Li}x^{1/6} - \tfrac{1}{7}\mathrm{Li}x^{1/7} + \cdots.$$

(we need not trouble now about the law of formation of the series). It is extremely natural to infer that

$$\text{(34)} \qquad \pi(x) < \mathrm{Li}x,$$

at any rate for large x, and Gauss and other mathematicians commented on the high probability of this conjecture. The conjecture is not only plausible but is supported by *all* the evidence of the facts. The primes are known up to 10,000,000, and their number at intervals up to 1,000,000,000, and (34) is true for every value of x for which data exist.

In 1912 Littlewood proved that the conjecture is false, and that there are an infinity of values of x for which the sign of inequality in (34) must be reversed. In particular, there is a number X such that (34) is false for some x less than X. Littlewood proved the existence of X, but his method did not give any particular value, and it is only very recently that an admissible value, viz.

$$X = 10^{10^{10^{34}}},$$

was found by Skewes.* I think that this is the largest number which has ever served any definite purpose in mathematics.

The number of protons in the universe is about

$$10^{80}.$$

The number of possible games of chess is much bigger, perhaps

$$10^{10^{50}}$$

(in any case a second order exponential). If the universe were the chessboard, the protons the chessmen, and any interchange in the position of two protons a move, then the number of possible games would be something like the Skewes number. However much the number may be reduced by refinements on Skewes' argument, it does not seem at all likely that we shall ever know a single instance of the truth of Littlewood's theorem.

This is an example in which the truth has defeated not only all the evidence of the facts and of common sense but even a mathematical imagination so powerful and profound as that of Gauss; but of course it is taken from the most difficult parts of the theory. No part of the theory of primes is really easy, but up to a point simple arguments, although they will prove very little, do not actually mislead us. For example, there are simple arguments which might lead any good mathematician to the conclusion†

$$\text{(35)} \qquad \pi(x) \sim \frac{x}{\log x}$$

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* S. Skewes, *Journal of the London Mathematical Society*, vol. 8, 1933, p. 277.

† $f(x) \sim g(x)$ means that the ratio f/g tends to unity.

of the Prime Number Theorem, or, what is the same thing, to the conclusion that

$$(36) \qquad p_n \sim n \log n,$$

where p_n is the n-th prime number.

In the first place, we may start from Euler's identity

$$(37) \qquad \prod_p \frac{1}{1-p^{-s}} = \frac{1}{(1-2^{-s})(1-3^{-s})(1-5^{-s})\cdots} = \frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\cdots = \sum_n \frac{1}{n^s}.$$

This is true for $s>1$, but both series and product become infinite for $s=1$. It is natural to argue that, when $s=1$, the series and the product should diverge in the same sort of way. Also

$$(38) \qquad \log \prod \frac{1}{1-p^{-s}} = \sum \log \frac{1}{1-p^{-s}} = \sum \frac{1}{p^s} + \sum\left(\frac{1}{2p^{2s}} + \frac{1}{3p^{3s}} + \cdots\right),$$

and the last series remains finite for $s=1$. It is natural to infer that

$$\sum \frac{1}{p}$$

diverges like

$$\log\left(\sum \frac{1}{n}\right),$$

or, more precisely, that

$$(39) \qquad \sum_{p\leqq x} \frac{1}{p} \sim \log\left(\sum_{n\leqq x}\frac{1}{n}\right) \sim \log\log x$$

for large x. Since also

$$\sum_{n\leqq x} \frac{1}{n\log n} \sim \log\log x,$$

formula (39) indicates that p_n is about $n \log n$.

There is a slightly more sophisticated argument which is really simpler. It is easy to see that the highest power of a prime p which divides $x!$ is

$$\left[\frac{x}{p}\right] + \left[\frac{x}{p^2}\right] + \left[\frac{x}{p^3}\right] + \cdots,$$

where $[y]$ denotes the integral part of y. Hence

$$x! = \prod_{p \leqq x} p^{[x/p]+[x/p^2]+\cdots},$$

$$\log x! = \sum_{p \leqq x} \left(\left[\frac{x}{p} \right] + \left[\frac{x}{p^2} \right] + \cdots \right) \log p. \tag{40}$$

The left-hand side of (40) is practically $x \log x$, by Stirling's Theorem. As regards the right-hand, one may argue; squares, cubes, . . . of primes are comparatively rare, and the terms involving them should be unimportant, and it should also make comparatively little difference if we replace $[x/p]$ by x/p. We thus infer that

$$x \sum_{p \leqq x} \frac{\log p}{p} \sim x \log x, \qquad \sum_{p \leqq x} \frac{\log p}{p} \sim \log x,$$

and this again just fits the view that p_n is approximately $n \log n$.

This is broadly the argument used, naturally in a less naïve form, by Tchebychef, who was the first to make substantial progress in the theory of primes, and I imagine that Ramanujan began by arguing in the same sort of way, though there is nothing in the note-books to show. All that is plain is that Ramanujan found the form of the Prime Number Theorem for himself. This was a considerable achievement; for the men who had found the form of the theorem before him, like Legendre, Gauss, and Dirichlet, had all been very great mathematicians; and Ramanujan found other formulas which lie still further below the surface. Perhaps the best instance is (15). The integral is better replaced by the simpler function (16), but what Ramanujan says is correct as it stands and was proved by Landau in 1909; and there is nothing obvious to suggest its truth.

The fact remains that hardly any of Ramanujan's work in this field had any permanent value. The analytic theory of numbers is one of those exceptional branches of mathematics in which proof really is everything and nothing short of absolute rigour counts. The achievement of the mathematicians who found the Prime Number Theorem was quite a small thing compared with that of those who found the proof. It is not merely that in this theory (as Littlewood's theorem shows) you can never be sure of the facts without the proof, though this is important enough. The whole history of the Prime Number Theorem, and the other big theorems of the subject, shows that you cannot reach any real understanding of the structure and meaning of the theory, or have any sound instincts to guide you in further research, until you have mastered the proofs. It is comparatively easy to make clever guesses; indeed there are theorems, like "Goldbach's Theorem",* which have never been proved and which any fool could have guessed.

The theory of primes depends upon the properties of Riemann's function $\zeta(s)$, considered as an analytic function of the complex variable s, and in particu-

* "Any even number greater than 2 is the sum of two primes."

lar on the distribution of its zeros; and Ramanujan knew nothing at all about the theory of analytic functions. I wrote before

"Ramanujan's theory of primes was vitiated by his ignorance of the theory of functions of a complex variable. It was (so to say) what the theory might be if the Zeta-function had no complex zeros. His method depended upon a wholesale use of divergent series. . . . That his proofs should have been invalid was only to be expected. But the mistakes went deeper than that, and many of the actual results were false. He had obtained the dominant terms of the classical formulae, although by invalid methods; but none of them are such close approximations as he supposed.

"This may be said to have been Ramanujan's one great failure . . .",

and if I had stopped there I should have had nothing to add, but I allowed myself again to be led away by sentimentalism. I went on to argue that "his failure was more wonderful than any of his triumphs", and that is an absurd exaggeration. It is no use trying to pretend that failure is something else. This much perhaps we may say, that his failure is one which, on the balance, should increase and not diminish our admiration for his gifts, since it gives us additional, and surprising, evidence of his imagination and versatility.

But the reputation of a mathematician cannot be made by failures or by rediscoveries; it must rest primarily, and rightly, on actual and original achievement. I have to justify Ramanujan on this ground, and that I hope to do in my later lectures.

NOTE ON DEGREE OF APPROXIMATION TO AN INTEGRAL BY RIEMANN SUMS*

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In a number of different investigations it is desirable to approximate a Riemann integral by the corresponding Riemann sums. The object of the present note is to establish the most easily proved results on the degree of such approximation when equidistant ordinates are used, under various conditions on the function integrated.†

1. *Continuous functions.* We prove the following

THEOREM 1. *Let $f(x)$ be continuous in the interval $0 \leqq x \leqq 1$ and possess there the modulus of continuity $\omega(\delta)$ in the sense that for values x and x' in the interval* (0, 1) *the inequality $|x-x'| \leqq \delta$ implies $|f(x)-f(x')| \leqq \omega(\delta)$. Then we have*

$$\left| \int_0^1 f(x)dx - \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \right| \leqq \omega\left(\frac{1}{n}\right). \tag{1}$$

* Presented to the American Mathematical Society, December 1936.

† Of course Theorems 1 and 2 are not to be regarded as novel. They are contained explicitly or implicitly in various treatments of the definite integral. See for instance Veblen and Lennes, Infinitesimal Analysis, New York, 1907, pp. 157–159. But Theorem 5 is believed to be new, and of some interest.

The proof is immediate; throughout the interval $(k-1)/n \leqq x \leqq k/n$ we have $|f(x)-f(k/n)| \leqq \omega(1/n)$, whence by the mean value theorem for integrals

$$\left| \int_{(k-1)/n}^{k/n} \left[f(x) - f\left(\frac{k}{n}\right) \right] dx \right| \leqq \frac{1}{n}\, \omega\left(\frac{1}{n}\right). \tag{2}$$

Addition of these n inequalities yields (1).

2. *Functions of bounded variation.* Let the function $f(x)$ be monotonic non-decreasing in the interval $0 \leqq x \leqq 1$. Then $f(x)$ is bounded and has at most a countable set of discontinuities, hence is Riemann integrable. We have

$$\left| \int_{(k-1)/n}^{k/n} \left[f(x) - f\left(\frac{k}{n}\right) \right] dx \right| \leqq \frac{1}{n} \left[f\left(\frac{k}{n}\right) - f\left(\frac{k-1}{n}\right) \right],$$

$$\left| \int_0^1 f(x)dx - \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \right| \leqq \frac{1}{n}\, [f(1) - f(0)]. \tag{3}$$

An arbitrary function of total variation V in $(0, 1)$ is the difference between two functions monotonic non-decreasing in $(0, 1)$, the sum of whose total variations is V. Both the integral and summation in (3) are additive with respect to the function $f(x)$, so we have established

THEOREM 2. *Let $f(x)$ be of total variation V in the interval $0 \leqq x \leqq 1$. Then we have*

$$\left| \int_0^1 f(x)dx - \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \right| \leqq \frac{V}{n}.$$

3. *Functions with finite jumps.* Our next result is

THEOREM 3. *Let $f(x)$ be continuous in the interval $0 \leqq x \leqq 1$ except in a countable set of points $x_1, x_2, \cdots$. At each point x_ν let the limits of $f(x)$ to the right and left exist and be denoted respectively by $f(x_\nu^+)$ and $f(x_\nu^-)$ (except that $f(0^-)$ and $f(1^+)$ need not exist). Let us suppose the series*

$$\sum_{\nu=1}^{\infty} \left[\left| f(x_\nu^+) - f(x_\nu) \right| + \left| f(x_\nu) - f(x_\nu^-) \right| \right]$$

convergent with sum V and let us set

$$S(x) = \sum_{0 < x_\nu \leqq x} [f(x_\nu) - f(x_\nu^-)] + \sum_{0 \leqq x_\nu < x} [f(x_\nu^+) - f(x_\nu)].$$

The function $f(x) - S(x)$ is continuous in the closed interval $0 \leqq x \leqq 1$; let $\omega(\delta)$ be its modulus of continuity. Then we have

$$\left| \int_0^1 f(x)dx - \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \right| \leqq \omega\left(\frac{1}{n}\right) + \frac{V}{n}. \tag{4}$$

The function $S(x)$ is of total variation V in (0, 1), so by Theorem 2 we may write

$$\left| \int_0^1 S(x)dx - \frac{1}{n} \sum_{k=1}^{n} S\left(\frac{k}{n}\right) \right| \leqq \frac{V}{n}.$$

Inequality (4) is now a consequence of Theorem 1 as applied to the continuous function $f(x) - S(x)$.

We note in particular that the hypothesis on $f(x)$ is satisfied for any function $f(x)$ continuous in (0, 1) except for a finite number of discontinuities of the first kind—that is to say such that the limits of $f(x)$ to the right and to the left exist.

As a corollary to Theorem 3, let $f(x)$ be defined throughout the interval (0, 1) and satisfy a Lipschitz condition $|f(x) - f(x')| \leqq L|x - x'|$, L a constant, in each of a finite number of open subintervals of (0, 1) whose total length is unity. At each end of such an open subinterval the limits of $f(x)$ to the right and left exist (except that at $x = 0$ the limit to the left need not exist, and at $x = 1$ the limit to the right need not exist); the Lipschitz condition is satisfied also in the corresponding closed subinterval (except that in the Lipschitz condition the limits of $f(x)$ at the end points are to be used instead of the values of $f(x)$, if these do not coincide), so $f(x)$ satisfies the requirements of Theorem 3. The function $f(x) - S(x)$ satisfies the given Lipschitz condition in the close dinterval (0, 1) as the reader may readily prove. Hence for suitably chosen K we have

$$\left| \int_0^1 f(x)dx - \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \right| \leqq \frac{K}{n}. \tag{5}$$

Of course Theorems 1, 2, 3, and later theorems are easily extended to include the case that ordinates $f(x)$ are chosen equidistant but not precisely in the points $1/n, 2/n, \cdots, 1$. For instance with the hypothesis and by the method of proof of Theorem 1 we obtain

$$\left| \int_0^1 f(x)dx - \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{2k-1}{2n}\right) \right| \leqq \omega\left(\frac{1}{2n}\right).$$

Those theorems can likewise be extended to apply to an arbitrary interval $(0, a)$, $a > 0$, where the ordinates are chosen $x = ka/n$; here inequality (1) for example is to be replaced by

$$\left| \int_0^a f(x)dx - \frac{a}{n} \sum_{k=1}^{n} f\left(\frac{ka}{n}\right) \right| \leqq a\omega\left(\frac{a}{n}\right).$$

4. *Functions represented by a Fourier series.* The constant term in the Fourier development is, except for a constant factor, precisely the integral of the function developed, so it is to be expected that this development will yield results analogous to the theorems already established.

THEOREM 4. *Let the function $f(x)$ be represented in the interval $0 \leqq x \leqq 2\pi$ by its Fourier development:*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\, dx, \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\, dx.$$

Then we have

$$\int_0^{2\pi} f(x)dx - \frac{2\pi}{N}\sum_{k=1}^{N} f\left(\frac{2k\pi}{N}\right) = -2\pi[a_N + a_{2N} + a_{3N} + \cdots]. \tag{6}$$

For the sake of reference we write the well known equations

$$\cos\theta + \cos 2\theta + \cdots + \cos m\theta = \frac{\sin m\frac{\theta}{2}\cos(m+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}}, \quad \text{if } \sin\frac{\theta}{2} \neq 0,$$
$$\qquad = m, \qquad \text{if } \sin\frac{\theta}{2} = 0. \tag{7}$$

$$\sin\theta + \sin 2\theta + \cdots + \sin m\theta = \frac{\sin m\frac{\theta}{2}\sin(m+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}}, \quad \text{if } \sin\frac{\theta}{2} \neq 0,$$
$$\qquad = 0, \qquad \text{if } \sin\frac{\theta}{2} = 0. \tag{8}$$

The left-hand member of (6) can be written in the form

$$\pi a_0 - \frac{2\pi}{N}\sum_{k=1}^{N}\left[\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2kn\pi}{N} + b_n \sin\frac{2kn\pi}{N}\right)\right]$$
$$= -\frac{2\pi}{N}\sum_{n=1}^{\infty}\sum_{k=1}^{N}\left(a_n \cos\frac{2kn\pi}{N} + b_n \sin\frac{2kn\pi}{N}\right) = -2\pi[a_N + a_{2N} + a_{3N} + \cdots],$$

so Theorem 4 is established.

Theorem 4 is of interest in that it furnishes an exact expression for the difference between the integral and the Riemann sum, and also in that it furnishes more favorable results than Theorems 1, 2, and 3 provided $f(x)$ is sufficiently smooth. Still further results in the latter case are now to be proved.

5. *Functions with given degree of approximation.* We now prove

THEOREM 5. *Let the function $f(x)$ be defined in the interval $0 \leqq x \leqq 2\pi$ and have the property that trigonometric polynomials*

$$S_N(x) = \sum_{n=0}^{N} (a_{Nn} \cos nx + b_{Nn} \sin nx)$$

of respective orders $N=0, 1, 2, \cdots$, *exist such that we have*

$$|f(x) - S_N(x)| \leqq \epsilon_N, \qquad 0 \leqq x \leqq 2\pi. \tag{9}$$

Then we have

$$\left| \int_0^{2\pi} f(x)dx - \frac{2\pi}{N}\sum_{k=1}^{N} f\left(\frac{2k\pi}{N}\right) \right| \leqq 4\pi\epsilon_{N-1}, \qquad N > 0. \tag{10}$$

Equations (7) and (8) yield

$$\int_0^{2\pi} S_{N-1}(x)dx = 2\pi a_{N-1,0} = \frac{2\pi}{N}\sum_{k=1}^{N} S_{N-1}\left(\frac{2k\pi}{N}\right).$$

Consequently we may write

$$\int_0^{2\pi} f(x)dx - \frac{2\pi}{N}\sum_{k=1}^{N} f\left(\frac{2k\pi}{N}\right) = \int_0^{2\pi} [f(x) - S_{N-1}(x)]dx$$
$$- \frac{2\pi}{N}\sum_{k=1}^{N}\left[f\left(\frac{2k\pi}{N}\right) - S_{N-1}\left(\frac{2k\pi}{N}\right)\right].$$

Inequality (10) now follows at once by means of (9).

Various consequences of Theorem 5 are now immediate, thanks to known results* on degree of approximation by trigonometric sums. Thus, if $f(x)$ is a function of period 2π which satisfies a Lipschitz condition of order α: $(|f(x)-f(x')| \leqq L|x-x'|^{\alpha})$, $0<\alpha\leqq 1$, we may set $\epsilon_N = LC/N^{\alpha}$, where C is an absolute constant. If $f(x)$ is a function of period 2π having a p-th derivative which satisfies a Lipschitz condition of order α, $0<\alpha\leqq 1$, we may set $\epsilon_N = LC_p/N^{p+\alpha}$, where C_p is an absolute constant. If $f(x)$ is a function of period 2π having a p-th derivative which has a modulus of continuity $\omega(\delta)$ we may set $\epsilon_N = \Gamma_p\,\omega(2\pi/N)/N^p$, where Γ_p is an absolute constant.

If $f(x)$ is an analytic function of the real variable x of period 2π there exists† a number $R>1$ such that we have $\epsilon_N \leqq M/R^N$, where M is independent of N. If $f(x)$ is an entire function of x of period 2π we have $\lim_{N\to\infty}\epsilon_N^{1/N} = 0$.

6. *Continuous functions; an example.* It is not to be expected that Theorem 1 can be improved so as to obtain a more favorable inequality that will hold simultaneously for *all* continuous functions. To be more explicit, we prove

THEOREM 6. *Let the sequence* $\delta_n > 0$, $n=1, 2, \cdots$, *be given, with* $\lim_{n\to\infty}\delta_n = 0$. *Then there exists a function* $f(x)$ *continuous and periodic with period* 2π *such that we have*

$$\overline{\lim_{n\to\infty}}\, \frac{\Delta_n}{\delta_n} = \infty, \qquad \textit{where} \qquad \Delta_n = \left| \int_0^{2\pi} f(x)dx - \frac{2\pi}{n}\sum_{k=1}^{n} f\left(\frac{2k\pi}{n}\right) \right|.$$

* Dunham Jackson, The Theory of Approximation, New York, 1930, pp. 2–12.

† C. de La Vallée-Poussin, Approximation des Fonctions, Paris, 1919, pp. 123–124.

Let us choose the prime p_1 so that we have $\delta_{p_1}<1/2$, the prime $p_2>p_1$ so that we have $\delta_{p_2}<1/2\cdot 2^2$, and in general the prime $p_k>p_{k-1}$ so that we have $\delta_{p_k}<1/k\cdot 2^k$. We set*

$$f(x) = \sum_{k=1}^{\infty} \frac{\cos p_k x}{2^k},$$

which is continuous and periodic with period 2π. For this function $f(x)$ we may by Theorem 4 set $\Delta_{p_k}=2\pi/2^k$. As a consequence it follows that

$$\frac{\Delta_{p_k}}{\delta_{p_k}} > 2\pi k, \qquad \overline{\lim_{n\to\infty}} \frac{\Delta_n}{\delta_n} = \infty.$$

7. *Trigonometric interpolation.* It is worth noticing that results similar to (but slightly less favorable than) the results of §5 can also be established by considering trigonometric interpolation. Let $f(x)$ be given on the interval $0 \leqq x \leqq 2\pi$, and let

$$T_n(x) = \sum_{k=0}^{n} (a_{nk} \cos kx + b_{nk} \sin kx)$$

be the unique trigonometric polynomial of order n which interpolates to $f(x)$ in the $m=2n+1$ (or $m=2n$ with $b_{nn}=0$) points $2\pi/m, 4\pi/m, \cdots, 2\pi$. By the property of interpolation we have

$$\frac{2\pi}{m}\sum_{k=1}^{m} f\left(\frac{2k\pi}{m}\right) = \frac{2\pi}{m}\sum_{k=1}^{m} T_n\left(\frac{2k\pi}{m}\right),$$

and from equations (7) and (8) we have

$$\frac{2\pi}{m}\sum_{k=1}^{m} T_n\left(\frac{2k\pi}{m}\right) = 2\pi a_{n0} = \int_0^{2\pi} T_n(x)dx.$$

Under various hypotheses on $f(x)$ the degree of convergence of $T_n(x)$ to the function $f(x)$ has been studied.† There are immediate consequences on the degree of convergence to zero of the expression

$$\frac{2\pi}{m}\sum_{k=1}^{m} f\left(\frac{2k\pi}{m}\right) - \int_0^{2\pi} f(x)dx = \int_0^{2\pi} [T_n(x) - f(x)]dx = 2\pi a_{n0} - \int_0^{2\pi} f(x)dx.$$

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* This example is somewhat similar to an example recently given in another connection by Otto Szász, this MONTHLY, vol. 42, 1935, pp. 37–38.

† Jackson, op. cit., Chap. IV.

MAXIMUM AND MINIMUM VALUES OF FUNCTIONS OF SEVERAL VARIABLES

By O. E. BROWN, Case School of Applied Science

1. Sufficient conditions that a function $f(x)$ shall have a relative maximum or minimum value at $x=a$ may be stated in any one of the forms:

A. 1. $f(x)$ is continuous throughout some neighborhood of a, $(x-a)^2<\epsilon$;
 2. $f(x)-f(a)$ has one sign throughout some deleted neighborhood of a, $0<(x-a)^2<\epsilon$; ($f(a)$ is a relative maximum or minimum value of $f(x)$ according as that sign is minus or plus.)

B. 1. $f(x)$ is continuous throughout some neighborhood of a, $(x-a)^2<\epsilon$;
 2. $f'(x)$ is continuous throughout some deleted neighborhood of $x=a$, $0<(x-a)^2<\epsilon$;
 3. $(x-a)f'(x)$ has one sign throughout some deleted neighborhood of a, $0<(x-a)^2<\epsilon$; ($f(a)$ is a relative maximum or minimum value of $f(x)$ according as that sign is minus or plus.)

C. 1. $f(x)$, $f'(x)$, and $f''(x)$ are continuous throughout some neighborhood of a, $(x-a)^2<\epsilon$;
 2. $f'(a)=0$;
 3. $f''(a)\neq 0$; ($f(a)$ is a relative maximum or minimum value of $f(x)$ according as $f''(a)$ is negative or positive.)

Of these three, conditions A form the only set which is both necessary and sufficient, being, in fact, a definition of a relative maximum or minimum value of $f(x)$ at $x=a$. The writer feels that insufficient use is made of conditions A in our elementary courses. After completing the ordinary course in the Differential Calculus a "well trained" student will use derivatives to find maximum and minimum values of such functions as x^2+2x+4, $\sqrt{1+x^4}$, and $6-\sin x \cos x$. If the critical value of the independent variable is known, it is questionable whether any conditions other than A need to be used in order to establish the existence of a maximum or minimum value of any function of one variable. However, if the derived function has been formed, in seeking the critical values, there may be no reason to avoid its use in establishing the existence of a maximum value of the function.

In comparing conditions B and C we note that

(1) B may hold when C does not in case $f'(a)$ is undefined, in case $f''(x)$ is discontinuous in the neighborhood of $x=a$, or in case $f''(a)=0$;
(2) $f''(x)$ may be very complicated;
(3) conditions B are suggested intuitively to the student if he notes that at a point of maximum or minimum ordinate of the curve $y=f(x)$ the slope of the tangent line changes sign.

2. This paper is mainly concerned with the extension of conditions B to functions of several variables. The resulting set of conditions is easily estab-

lished and usually easily applied but seems never to have been published. A clue for the form of the extension may be taken from the informal statement that "if $f(a)$ is a relative maximum value of $f(x)$ then $f(x)$ decreases as x leaves a in either of the two possible directions." Analogously we note that "if $f(a, b)$ is a relative maximum of $f(x, y)$ then $f(x, y)$ decreases as (x, y) leaves (a, b) in any direction."

If the directional derivative,

$$\cos \alpha \cdot f_x(x_1, y_1) + \sin \alpha \cdot f_y(x_1, y_1),$$

of $f(x, y)$ at (x_1, y_1) and in the direction α, is negative, then $f(x, y)$ decreases as (x, y) moves in the direction α through the point (x_1, y_1). Hence we have a maximum value of $f(x, y)$ at (a, b) if this directional derivative is negative for every point (x_1, y_1) in some deleted neighborhood, $0<(x-a)^2+(y-b)^2<\epsilon$, of (a, b), with α chosen as the direction from (a, b) to (x_1, y_1) in every case. Stated formally the conditions take the form B′ below where we give also, for the sake of comparison, A′ and C′, the extensions to a function of two independent variables of conditions A and C above.

A′ 1. $f(x, y)$ is continuous throughout some neighborhood of the point (a, b), $(x-a)^2+(y-b)^2<\epsilon$;

2. $f(x, y)-f(a, b)$ has one sign throughout some deleted neighborhood of (a, b), $0<(x-a)^2+(y-b)^2<\epsilon$; ($f(a, b)$ is a relative maximum or minimum value of $f(x, y)$ according as that sign is minus or plus.)

B′ 1. $f(x, y)$ is continuous throughout some neighborhood of the point (a, b), $(x-a)^2+(y-b)^2<\epsilon$;

2. the partial derivatives $f_x(x, y)$ and $f_y(x, y)$ are continuous throughout some deleted neighborhood of (a, b), $0<(x-a)^2+(y-b)^2<\epsilon$;

3. there exists a positive constant $p_1<\epsilon$ such that the directional derivative
$\cos \alpha \cdot f_x(a+p \cos \alpha, b+p \sin \alpha)+\sin \alpha \cdot f_y(a+p \cos \alpha, b+p \sin \alpha)$
has one sign for every p satisfying the conditions $0<p<p_1$ and every real angle α. ($f(a, b)$ is a relative maximum or minimum value of $f(x, y)$ according as that sign is minus or plus.)

C′ 1. $f(x, y)$ and the partial derivatives $f_x(x, y)$, $f_y(x, y)$, $f_{xx}(x, y)$, $f_{xy}(x, y)$ and $f_{yy}(x, y)$ are continuous throughout some neighborhood of (a, b), $(x-a)^2+(y-b)^2<\epsilon$;

2. $f_x(a, b)=f_y(a, b)=0$;

3. $[f_{xx}(a, b)][f_{yy}(a, b)]-[f_{xy}(a, b)]^2>0$; ($f(a, b)$ is a relative maximum or minimum value of $f(x, y)$ according as $f_{xx}(a, b)$ is negative or positive.)

Conditions A′ form the definition for a relative maximum or minimum value, $f(a, b)$, of $f(x, y)$ at (a, b).

Conditions C′ are classical sufficient conditions for a relative maximum or minimum value of $f(x, y)$ at (a, b) and are found in all books treating the subject.

The proof that when conditions B hold $f(a, b)$ is a relative maximum or minimum value of $f(x, y)$ is easily made. If a function $f(x, y)$ satisfies these conditions the Theorem of the Mean gives

$$f(x, y) - f(a, b) = (x - a)f_x(a + \theta\overline{x - a}, b + \theta\overline{y - b})$$
$$+ (y - b)f_y(a + \theta\overline{x - a}, b + \theta\overline{y - b}), \qquad 0 < \theta < 1.$$

Let x and y satisfy the conditions $0 < (x-a)^2 + (y-b)^2 < p_1^2$ and write

$$x - a = q \cos \alpha, \qquad y - b = q \sin \alpha, \qquad \theta q = p,$$

where $0 < p < q < p_1 < \sqrt{\epsilon}$. Using p, q, and α in the right hand member, the above expression of the Theorem of the Mean becomes

$$f(x, y) - f(a, b) = q[\cos \alpha f_x(a + p \cos \alpha, b + p \sin \alpha)$$
$$+ \sin \alpha f_y(a + p \cos \alpha, b + p \sin \alpha)].$$

Now, q is positive and, by condition B′3, the coefficient of q in the right-hand side has but one sign. The left-hand side, therefore, has that same sign, and conditions A′ are met.

3. Conditions B′ may hold while conditions C′ do not through the occurrence of discontinuities of the first or second partial derivatives of $f(x, y)$ at (a, b), through discontinuities of the second partial derivatives of $f(x, y)$ in the neighborhood of (a, b), or through the vanishing of the expression $[f_{xx}(a, b)][f_{yy}(a, b)] - [f_{xy}(a, b)]^2$. If conditions C′ hold then conditions B′ hold also. Neither of the sets of conditions B′ or C′ is necessary.

An example in which conditions A′ apply but neither B′ nor C′ apply is given by the function

$$f(x, y) \equiv [(a^{2/3} - x^{2/3})^3 - y^2]^{1/2}, \qquad a > 0,$$

which clearly has a maximum at (0, 0). The function $f(x, y)$ is continuous in the neighborhood of (0, 0), but $f_x(x, y)$ is discontinuous on the y-axis.

It is obvious that the function

$$f(x, y) \equiv [a^{2/3} - (x^2 + y^2)^{1/3}]^{3/2}, \qquad a > 0,$$

has a maximum value at (0, 0). For it, both $f_{xx}(x, y)$ and $f_{yy}(x, y)$ are discontinuous at the origin so that conditions C′ do not apply. However conditions B′ do apply, the directional derivative of B′3 taking the form

$$-[a^{2/3} - p^{2/3}]^{-1/2}/p^{-1/3},$$

independent of α and negative for every p satisfying $0 < p < a$.

The function

$$f(x, y) \equiv v^{-1}u$$

where $v(x, y)\equiv 1+x^2+y^2$, and $u(x, y)\equiv 25+(3x+4y)^2-y^4$ satisfies conditions C′1 and C′2, with $a=b=0$, but fails to satisfy condition C′3 because $[f_{xx}(0, 0)][f_{yy}(0, 0)]=[f_{xy}(0, 0)]^2$. It meets conditions B′ with $a=b=0$, the directional derivative involved in B′3 taking the form

$$\frac{-2p\{(2p^2+p^4)\sin^4\alpha+25[1-\sin^2(\alpha+\alpha_1)]\}}{(1+p^2)^2},$$

where $\sin\alpha_1=3/5$ and $\cos\alpha_1=4/5$. Since $1-\sin^2(\alpha+\alpha_1)$ and $\sin^4\alpha$ cannot vanish simultaneously and neither can be negative, this expression is negative for every $p>0$ regardless of α. Hence conditions B′ are met and $f(0, 0)=25$ is the maximum value of $f(x, y)$.

It is to be noticed that in the above examples, the maximum values could have been established immediately by conditions A′. The writer has failed to find examples of functions meeting conditions B′, and failing to satisfy conditions C′, for which the extreme values could not be readily be established by conditions A′. In comparing conditions B′ and C′ he therefore gives little weight to the fact that B′ may hold while C′ does not.

It should be noted that conditions B′ are not equivalent to the statement "$f(x, y)$ has a maximum (or minimum value at (a, b) along every straight line through that point." It is well known* that that statement is not sufficient for a maximum (or minimum) of $f(x, y)$ at (a, b). Consider, for example, the function defined by the formula

$$f(r, \theta)\equiv 2r^6-15r^4\theta^2+24r^2\theta^4, \qquad -\frac{\pi}{2}<\theta\leqq\frac{\pi}{2},$$

where $x=r\cos\theta$, and $y=r\sin\theta$. The equation $z=f(r, \theta)$ defines a surface, any section of which by a plane $\theta=$constant has a minimum value of z at the point $r=0$, $z=0$; yet in every neighborhood of the point $r=0$ are points (r, θ) at which $f(r, \theta)-f(0, 0)$ is negative.

For this function, the directional derivative of conditions B′3 reduces to

$$12p(p-\alpha)(p+\alpha)(p-2\alpha)(p+2\alpha).$$

This quantity is positive for $0<\alpha<\frac{1}{2}p$ and negative for $\frac{1}{2}p<\alpha<p$. It therefore does not have a unique sign for all values of α for any $p>0$ and conditions B′ are not met.

4. Conditions B′ are readily extended to a function f of n variables, the directional derivative involved taking the form

$$\sum_{i=1}^{n}\cos\alpha_i\cdot f_{x_i}(a_1+p\cos\alpha_1, a_2+p\cos\alpha_2, \cdots, a_n+p\cos\alpha_n).$$

* See Harris Hancock, Theory of Maxima and Minima, p. 35; or James Pierpont, The Theory of Functions of Real Variables, vol. I, p. 329.

If, for a sufficiently small positive number p_1, this sum has only one sign for every p satisfying the conditions $0<p<p_1$ and every set of α's satisfying the conditions $\sum_{i=1}^{n}\cos^2\alpha_i = 1$, then $f(a_1, a_2, \cdots, a_n)$ is a relative maximum or minimum value of $f(x_1, x_2, \cdots, x_n)$ according as that sign is minus or plus.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions, and Notes in the Monthly is open to all forms of activity in collegiate mathematics including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A THEOREM CONCERNING CIRCLES

By W. M. STEWART, University of Wyoming

The object of this paper is to prove a theorem believed to be new concerning circles related to an arbitrary triangle. The theorem may be stated as follows: If three circles whose centers are the vertices of any triangle and whose radii are the medians to the opposite side be drawn, then the radical center of these circles is a point which bisects the segment of the Euler line joining the orthocenter and the center of the nine point circle of the triangle.

The reader may recall (1) that the Euler line of a triangle is the line upon which lie the orthocenter, the circumcenter, the center of the nine point circle, and the median point of the triangle; (2) that the nine point circle passes through nine notable points, namely the feet of the altitudes, the mid-points of the sides, and the mid-points of the line segments joining the orthocenter and the vertices; and (3) that the radical axis of two intersecting circles is their common chord, and the radical center of three circles is the point of concurrency of their radical axes.

The theorem is most simply proved by assuming a rectangular coordinate system such that the vertices A, B, and C of the triangle have the coordinates $(2a, 0)$, $(2b, 0)$, and $(0, 2c)$ respectively. Then the equations of the three circles are

$$(x-2a)^2+y^2=b^2+c^2+4a(a-b),$$
$$(x-2b)^2+y^2=a^2+c^2+4b(b-a),$$
$$x^2+(y-2c)^2=a^2+b^2+4c^2+2ab.$$

The radical center of these three circles is the point

$$\left(\frac{a+b}{4}, \frac{c^2-5ab}{4c}\right).$$

Now, the coordinates of the orthocenter of triangle ABC are found to be $(0, -2ab/c)$ and the coordinates of the center of the nine point circle are

$$\left(\frac{a+b}{2}, \frac{c^2-ab}{2c}\right).$$

The point bisecting the line segment jointing these two points is obviously the radical center of the three circles, and so the theorem is established.

Note by the Editor. Professor F. E. Wood of Northwestern University makes the following interesting observation: By the analytic method of the present paper or otherwise it is easy to prove the lemma—*If three circles are drawn having as centers the vertices of a triangle and radii ρk, ρl, ρm, respectively, where k, l, m are fixed, then as ρ varies, the locus of the radical center of the three circles is a straightline (or part of a line if only real circles are used).* This lemma can be applied to the triangle ABC with medians AA', BB', CC' to obtain the theorem: *If circles are described with centers at A, B, C and radii $\rho AA'$, $\rho BB'$, $\rho CC'$ respectively, then as ρ varies the locus of the radical center is the Euler line of the triangle ABC.*

For since the locus is a line and since for $\rho=0$ the radical center is the circumcenter, and for $\rho=\frac{2}{3}$ it is the intersection of the medians, then the locus is the Euler line.

Mr. Stewart's theorem is, in part, a special case ($\rho=1$) of this more general theorem. R. E. G.

A Note on $m^2=n!+1$

By D. H. Lehmer, Lehigh University

In the January, 1936 number of this Monthly, pages 32–34, H. Gupta discussed the equation $m^2=n!+1$ and tabulated the actual square roots of $n!+1$ with remainders in order to show that this equation has no solutions for $7<n\leqq 50$. The author was apparently unaware of the investigation of M. Kraitchik: Recherches sur la Théorie des Nombres, Paris, 1924, pp. 38–41, who by a more efficient method showed that the equation in question has no solutions for $n<1020$. His scheme may be described briefly as follows.

Consider the equation $n!+1=m^2$ as a congruence modulo p, where p is a prime $>n$, and let $p-n=k$. Multiplying both sides of the congruence by

$$\begin{aligned}(n+1)(n+2)\cdots(p-1) &= (p-k+1)(p-k+2)\cdots(p-1)\\ &\equiv (-1)^{k-1}(k-1)! \pmod{p},\end{aligned}$$

we get

$$(p-1)!+(-1)^{k-1}(k-1)!\equiv m^2(-1)^{k-1}(k-1)! \pmod{p}.$$

Since by Wilson's theorem $(p-1)!\equiv -1 \pmod{p}$, we have

$$m^2\equiv 1+(-1)^k/(k-1)!=R \pmod{p}.$$

Hence a necessary condition for the solution of the equation is that R should be a quadratic residue of p. To apply the method we need only examine the first few values of k until a p is found such that the corresponding R is a quadratic non-residue. Thus for $k=2$ and $k=3$ we have $m^2\equiv 2$, and $m^2\equiv 1/2 \pmod{p}$. Since 2 is a quadratic non-residue of primes of the form $8x\pm 3$, the equation has

no solutions if $n+2$ or $n+3$ is a prime of the form $8x \pm 3$, thus giving an infinite number of cases for which the equation has no solutions. Although this is an excellent method for proving the non-existence of solutions, it would be impractical to use it or any other known method to exhibit a solution for $n \geqq 1020$, since 1020! has 2707 digits.

Gupta's statement that the congruence $(p-1)!+1 \equiv 0 \pmod{p^2}$ has no solutions for $7 < p < 50$ is not true since $12!+1$ is divisible by 13^2.

Note by the Editor. This reference to Kraitchik was also supplied by Dr. N. G. W. H. Beeger of Amsterdam.

The Quadratrix and the Associated Cochleoid

By L. S. Johnston, University of Detroit

In Figure 1 let $OP_0 = a$, angle $P_0OP_1 = \pi/4$, angle $P_1OP_2 = \pi/8, \cdots$, angle $P_{n-1}OP_n = \pi/2^{n+1}$; let P_0P_1 be normal to OP_0, P_1P_2 be normal to $OP_1, \cdots$, $P_{n-1}P_n$ be normal to OP_{n-1}; let P_0P_1' be normal to OP_1, $P_1'P_2'$ be normal to $OP_2, \cdots, P_{n-1}'P_n'$ be normal to OP_n. There are thus established two sequences of points $[P] \equiv P_0, P_1, P_2, \cdots$ and $[P'] \equiv P_0, P_1', P_2', \cdots$. We wish to find

(1) the positions of the limit point P_∞ of $[P]$ and the limit point P_∞' of $[P']$, and

(2) polar equations of two smooth curves, one through each of these sequences including the respective limit points. Hereafter $[P]$ and $[P']$ will be understood to include their respective limit points, and the two curves sought will be designated by C and C' respectively.

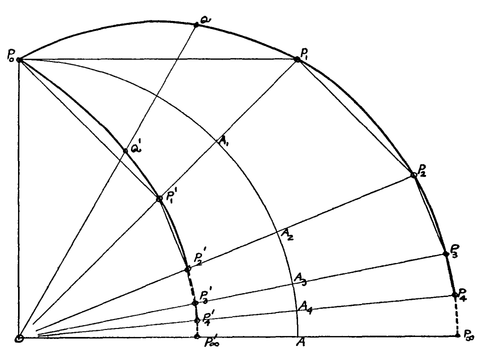

Fig. 1

The problem has already been solved for the sequence $[P]$ and its curve C (see the writer's note "An Unusual Spiral," this Monthly, vol. 40, 1933, p. 596). The point P_∞ is the point on OA such that $OP_\infty = \pi a/2$, that is, OP_∞ is the length of the quadrant of the circumference of radius a. The polar equation of C is

$$\rho = \frac{\pi a}{2}\,\frac{\sin\theta}{\theta},\ \theta \neq 0; \qquad \rho = \frac{\pi a}{2},\ \theta = 0.$$

The curve is known as the cochleoid, and is the "unusual spiral" discussed in the note cited.

By a process exactly similar to that used in the note already mentioned, a polar equation of one curve C' is found to be

$$\rho' = \frac{2a}{\pi}\,\frac{\theta}{\sin\theta},\quad \theta \neq 0; \qquad \rho' = \frac{2a}{\pi},\quad \theta = 0.$$

We have used ρ and ρ' to indicate radii vectores to C and C' respectively. It is seen at once that C' is the quite familiar and historic quadratrix.

Since $\rho\,\rho' = a^2$, we see that C and C' are inverse curves with respect to the point O, the constant of inversion being a^2, that is, the radius of the circle is the mean proportional between the radii vectors ρ and ρ'. Here, then, is a very interesting relation between the familiar quadratrix and the much less familiar cochleoid.

Several other interesting relations between the two curves may be found by the reader. One of them is that any radius vector intersects the two curves at the same angle, if the two angles of intersection be measured in opposite directions from the radius vector. This is of course a familiar property of any pair of inverse curves. It follows that the circle bisects the angle between the two curves at P_0.

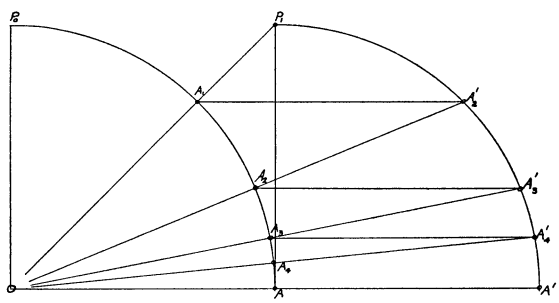

FIG. 2

Figure 2 shows an extremely simple and rapid method of constructing the successive bisectors OP_1, OP_2, etc. The quadrant $P_1A_2'A_3' \cdots A'$ is drawn with the radius a, and the various steps in the construction are self evident from the figure.

Referring again to Figure 1, it is worth noting that the construction by which the points P and P' were located give excellent approximations to the quadrature of the circle. The radius vector OP_4 differs from OP_∞ by less than one half of one percent, and the vector OP_4' is equally close to OP_∞'. For $n>4$ the approximations are of course much more nearly accurate.

RECENT PUBLICATIONS

EDITED BY W. R. LONGLEY, Yale University

All books for review should be sent directly to the editor of this department, at the American Mathematical Monthly, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Differential and Integral Calculus. By R. Courant. Volume 2. Translated by E. J. McShane. London and Glasgow, Blackie and Son, 1936; x+677 pages; 30 *s.*

Elements of Probability. By H. Levy and L. Roth. Oxford, Clarendon Press, 1936. x+196 pages. 15 *s.*

Mathematical Analysis. By M. Philip. New York, Longmans, Green, and Company, 1936. xi+275 pages.

Elementary Analytical Conics. By J. H. S. Bailey. London, Oxford University Press, 1936. 378 pages. $2.75.

Analytical and Applied Mechanics. By G. R. Clements and L. T. Wilson. New York, McGraw-Hill Book Company, 1935. ix+420 pages. $3.75.

Plane and Spherical Trigonometry with Applications. By J. Shibli. Second Edition. Boston, Ginn and Company, 1936. xii+242+94 pages. $1.96.

Special Topics in Theoretical Arithmetic. By J. Bowden. New York, Bowden, 1936. xi+217 pages. $2.50.

Mathematics of Finance. By T. M. Simpson, Z. M. Pirenian, and B. H. Crenshaw. Second Edition. New York, Prentice-Hall, Inc., 1936. xiii+330+126 pages. $3.75.

REVIEWS

Vorlesungen über elementare Mechanik. By J. Nielsen. Übersetzt und bearbeitet von Werner Fenchel. Berlin, Springer, 1935. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen. Band XLIV. x+500 pages, 164 figures. RM 38; gebunden, RM 39.60.

The title of the book under review suggests the question: How elementary should an elementary text be? The answer is given by this book, which begins by laying a very broad foundation starting with the ideas which group around the concepts "scalars, vectors, matrices, and tensors." With this beginning the author treats in a very remarkably clear and complete manner the entire field of elementary mechanics. There is no need of giving a list of contents, and to get the flavor of the book one must read it one's self.

Unfortunately, there are only a very few engineering schools where this text could be taught in the regular mechanics courses. As a matter of fact, a great deal of attention is given by Professor Nielsen to the applications, particularly to truss analysis and stress diagrams. The problems at the end of each section are of the stimulating kind rather than of numerical application; in fact, there are none whatever of the latter sort, but each set of problems is a gem in developing the power of the student to think "mechanically."

The only omission which the reviewer has noted is that of the use of the minimum principle. Whether this principle should be included in an elementary text is very debatable, but there is so much already present that a brief section on the use of the minimum principle in mechanics might have been included. One might also raise the question as to whether it would have been worth while to introduce the skew-symmetric matrix as a substitute for the cross or vector product of two vectors early in the book instead of making a reference to this possibility on page 381. By so doing one avoids completely the necessity of more than one kind of multiplication of vectors. Special attention and commendation should be given to the 164 figures in the book, especially to the space figures which are well drawn in accordance with the most exact principles of descriptive geometry.

H. J. ETTLINGER

Differential and Integral Calculus. By G. L. Parsons. Cambridge, University Press, 1936. I. Elementary Differential Calculus, xxiii+220 pages; II. Elementary Integral Calculus, viii+127 pages. $2.75.

The usual introductory discussion of limits and continuity is followed by the definition of the derivative and the development of the standard formulas for differentiation. This is followed by the applications. The treatment is similar to that of the average American text with the several exceptions noted below.

The expansion of a function into a power series is considered with practically no discussion of the convergence question. The rather vague warning that "Care must, of course, be taken in regard to the convergency of the series employed, and also of the results obtained," constitutes almost the only reference to this subject.

The applications to geometry are much more extensive than in the average American text, there being five chapters devoted to the subject. One chapter is on envelopes and associated loci, and one is on rectilinear asymptotes.

Partial differentiation is introduced quite informally in Chapter III in connection with the differentiation of implicit functions. The subject is further developed in Chapter XII. There is no formal definition of the partial derivative as a limit, and there is no mention of its geometrical significance—in fact, Chapter XII does not contain a single figure.

The so-called "indeterminate forms" which probably receive more attention than their importance warrants in most American texts, are omitted entirely from this book except for a casual mention of L'Hôpital's rule.

There are practically no errors of any consequence. On page 98 is the careless statement that "a constant has no derivative." On page 169 the word "differential" is used to mean "derivative," or "differential coefficient." The reviewer believes that the use of the term "trigonometric series" to mean the power series expansion of a trigonometric function should be avoided since this term is used quite generally in another sense.

Of the ten chapters in the second part, the first five are devoted to formal integration and the next three to the usual applications to area, length, center of gravity, moment of inertia, etc. Chapter IX is on applications to mechanics, and the final chapter is devoted to elementary differential equations. Thus the content is similar to that of the average American text with the exception that multiple integration is omitted.

It is evident that the author has deliberately sacrificed rigor in an attempt to produce a book from which a student can grasp the main ideas without being confused by a mass of detail. For example, the equation $\overline{\Delta s^2}=\overline{\Delta x^2}+\overline{\Delta y^2}$ is justified by the statement that "if P and P' are two close points on a curve we may in the limit regard the chord PP' as indistinguishable from the arc PP'." As another example, it may be mentioned that improper integrals are treated without the formality of defining their values as limits.

In view of the author's avowed purpose, the reviewer believes that such things as the above are not open to serious criticism. On the other hand, there is serious objection to the attempt to sum up the whole idea of the Fundamental Theorem by the simple statement that "$\int_a^b\phi(x)dx$ is the sum of all expressions like $\phi(x)dx$ between $x=b$ and $x=a$." Here the process of simplification of an idea has been carried to the point where the main part of the idea has been lost. It is, to say the least, a pedagogical error to give such a statement in bold-face type at the close of a discussion of the Fundamental Theorem. Students invariably memorize it—and in so doing they lose the proper mental picture of this important limiting process.

The problems are adequate in number and are well-chosen, those on the applications to mechanics being particularly good. The book could be considerbly improved by the addition of more figures and an improvement in the drawing of the space pictures.

R. R. MIDDLEMISS

An Introduction to Mathematical Analysis (Revised Edition). By F. L. Griffin. Boston, Houghton Mifflin Company, 1936. x+546 pages. $2.75.

This book is a unified mathematics text for freshmen. Rejecting the traditional tripartite course (algebra, trigonometry, analytic geometry), Professor Griffin develops, on the basis of the function concept, not only the substance of freshman mathematics but also the elements of calculus. An enumeration of the chapter headings gives some idea of the scope though hardly of the detailed organization which, while for the most part excellent, necessarily admits of considerable shuffling. The chapter titles are: I. Functions and Graphs; II. As to Exact Relations; III. Differentiation; IV. Integration; V. Trigonometric Functions; VI. Logarithms; VII. Exponential and Logarithmic Functions; VIII. Rectangular Coodinates; IX. Solution of Equations; X. Polar Coordinates and Trigonometric Functions; XI. Trigonometric Analysis; XII. Definite Integrals; XIII. Progressions and Series; XIV. Combinations, Probability, and Statistical Method; XV. Complex Numbers.

Professor Griffin has the knack of making mathematics interesting. And he does not sacrifice exactness; his treatment of limits and instantaneous rates (pp. 57–68), for example, is a gem of textbook-writing alike from the standpoint of content and of style.

The difficulty under which he labors inheres in the nature of the undertaking. The enormous number of facts and relations he seeks to weave into a logical pattern forces him to extremes both of compactness and of discursiveness. Thus into one brief chapter of twenty pages he crowds definite integrals and applications, multiple integration, partial derivatives, maxima and minima of functions of two variables, mean values, and approximate integration. But the opposite tendency is even more pronounced. For instance, while the slope of a curve is discussed in Chapter II as an instantaneous rate, the inclination of a line or curve must be withheld until Chapter V where the tangent function is defined; the expression for the slope of a line in terms of coordinates is derived in Chapter VIII, followed soon by the slope-intercept formula, thirty-three pages later by the point-slope formula, then by tangents and normals, and finally, at the end of Chapter XI, by the angle between two lines or curves. The closely related topics just listed are, in Griffin's *Mathematical Analysis*, scattered over nearly three hundred and fifty pages, as contrasted with a spread of only eight pages in a well-known elementary text on calculus.

The changes made in the revised edition under review are, in general, improvements. Topics missing in the original edition, such as determinants, the discussion of curves with reference to symmetry, intercepts, extent, and asymptotes, and the equations of tangents and normals, are supplied in the revision. In the long intervals devoted to algebra, trigonometry, and analytic geometry, numerous exercises specifically designated as involving calculus have been inserted. Perhaps as an admission that the book is too comprehensive for the average freshman, articles which can be "omitted without creating later difficulties" are labeled accordingly. These articles include the logarithmic solution of triangles, a subject which, in the opinion of the reviewer, is usually overemphasized, but with less justification they also include functions of double and half angles. The chapter on probability has been largely rewritten; a faulty article on errors of artillery fire has been deleted, and the theory of insurance and the elements of statistics have been added. It may finally be noted that in the revised edition the author's predilection for mechanical and astronomical applications has been checked to the extent of relegating these to the supplementary paragraphs.

For a class of superior freshmen who are really interested in mathematics and require but little drill in algebraic operations, Griffin's *Mathematical Analysis* would be a boon, but the ordinary run of students will benefit more by the old plan which postpones calculus until a knowledge of prerequisites has been gained.

MEYER SALKOVER

MATHEMATICS CLUBS

EDITED BY F. W. OWENS AND HELEN B. OWENS, State College, Pa.

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to F. W. Owens, 462 East Foster Ave., State College, Pa.

CONTEST PAPERS

Reviews of four of the 1935–36 contest papers have appeared in recent issues of the MONTHLY. In this number we present one more, the last for the contest of 1935–36.

Clubs should watch their programs closely for the papers to be sent to this Department in June for the contest of 1936–37. It is hoped that many will be entered. Remember the classifications: (1) historical, (2) expository, (3) original

V. A LOCUS PROBLEM

By RICHARD FOWLER, Albion College

The Mathematics Club of Albion College sent this interesting paper, presented by Richard Fowler.—Starting with the problem, to find the locus of the midpoint of the chord connecting the points of contact of the tangents drawn from any point on a given straight line to a circle, the writer generalized his problem to a consideration of the locus of the midpoint of the chords of contact of the tangents drawn from any point on a given curve to a given conic. Using properties of polars and parametric equations he obtains, for $y=f(x)$ as the locus of the pole, and the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

as base curve,

$$y=\frac{a^2y^2+b^2x^2}{a^2b^2}\cdot f\left(\frac{a^2b^2x}{a^2y^2+b^2x^2}\right)$$

as the locus of the midpoint of the chord of contact and similar equations for other conics.

LOAN LIBRARY OF STUNTS

This department has been so frequently requested to furnish ideas for the lighter side of club meetings that in December a questionnaire on the subject was sent to one hundred thirty-five clubs concerning the advisability of establishing a Loan Library of Stunts.

The response was encouraging. Many of the mathematics clubs confine themselves strictly to programs of mathematical topics and mathematical discussions but about two-thirds of the clubs include social activities. For the use of these clubs there are now being collected manuscripts of mathematical plays,

mathematical songs, games, tricks, competitions, and stunts of various types. Contributions to this library will be welcomed by this department. As soon as sufficient material has been collected, announcements of its availability will be made.

Clubs which have used mimeographed copies of puzzles, games and so forth will help greatly if they contribute spare copies of such mimeographed sheets together with the pertinent information as to the method of conducting the stunt. Suggestions and materials will be most welcome.

Club Reports
1935–1936

Naperian Club, DePauw University

President, J. Gavin; Vice President, D. Nicodemus; Secretary, Dorothy Taggart; Treasurer, L. Levin. Opening with a social meeting and closing with the annual steak roast, the Club held monthly meetings with papers by various members. Two talks, illustrated by stereopticon slides, on "Famous mathematicians" added variety to the program while an evening devoted to "Early mathematicians in Indiana and their work" brought to the club greater appreciation of their state's share in the development of the science.

Pi Mu Epsilon, Iowa State College

Director, E. Timm; Vice President, Margaret Ralph; Secretary, Madelyn Kerr; Treasurer, G. Higgins; Librarian, J. S. Johnson; Faculty Adviser, Dr. D. L. Holl. This chapter sponsors a mathematics club and is successful in getting a large number of Junior College students to appear on the programs; one program included ten short talks on such subjects as "Wire model coordinate system," "Wood model of intersecting cylinders" and "Minimum area bounded by a parabola and a focal chord." At another meeting the subject "Education experience in Argentina" suggested a wide range of interests. Monthly program meetings were held but the chapter held several business meetings and an initiation banquet distinct from these.

The Pi Mu Epsilon prize for "highest scholastic average in first two years of college work" was won by Alice A. Churchill.

Pi Mu Epsilon and Newtonian Society, State College of Washington

The two organizations in addition to their separate business meetings held sixteen joint program meetings including discussion of such up-to-date problems as "Price determination under conditions of partial monopoly," and "Symbolic logic" but not forgetting the classical "Trisection of an angle." At the annual joint banquet the guest speaker, Dr. J. A. Colley, professor of mathematics, University of Idaho, discussed "An interesting transformation."

The Newtonian Society in the late spring entertained the visiting Yakima Junior College Mathematics Club.

Kappa Mu Epsilon, State Teachers College of Nebraska

President, J. Larson; Vice President, Donabelle Stuart; Secretary, Betty Grosvenor; Treasurer, F. Ostopoff; Corresponding Secretary, E. Marie Hove. The social program of this group included two initiations, a tea, a picnic and a banquet. Yet they found time for ten program meetings with programs ranging from reports of the National State Teachers Association mathematics meetings and the St. Louis meetings of the American Mathematical Society and the Mathematical Association of America to "Numbers from integers to quaternions" and "The measurement of time," "Mathematical recreations" and "The future of mathematics."

SOLUTIONS

E 225 [1936, 431]. *Proposed by A. Gloden, l'Athênê de Luxembourg.*

Determine a pentagonal number in the decimal system with digit pattern *aabb*, and show that the solution is unique. [Note: a pentagonal number is of the form $n(3n-1)/2$, with n a positive integer.]

Solution by E. P. Starke, Rutgers University.

We have at once $aabb=11(100a+b)=n(3n-1)/2$, so that n is of one of the forms, $11k$ or $11k+4$, where k is some positive integer. Further, since $100\leqq 100a+b\leqq 908$, we must have $2200\leqq n(3n-1)\leqq 19976$. Hence n must lie between 27 and 82. Consequently, n must be one of the numbers, 33, 37, 44, 48, 55, 59, 66, 70, 77 and 81. These are easily tested against the requirement that a and b shall be digits, by direct computation of $100a+b=n(3n-1)/22$. Even this arithmetic may be simplified by noting that an increment of 11 in n produces an increment of $3n+16$ in the function $n(3n-1)/22$. Since only $n=77$ satisfies this test, 8855 is the unique solution.

Also solved by J. A. Benner, W. E. Buker, Mary L. Constable, Fred Discepoli, D. M. Dribin, C. T. Oergel, B. C. Schwanda, W. R. Talbot, J. E. Trevor, C. W. Trigg, Simon Vatriquant, and Z. W. Wilchinsky.

E 226 [1936, 431]. *Proposed by Cezar Coşniţă, Roumanian Mathematical Institute.*

If AD and BC are the diagonals of the square $ABCD$, and M is any point on its circumscribed circle, show that the points in which MA and MD cut BC are concyclic with the points in which MB and MC cut AD.

Solution by C. E. Springer, University of Oklahoma.

Let the vertices of the square be $A(0, -a)$, $B(-a, 0)$, $C(a, 0)$, and $D(0, a)$. If M has the coordinates $(a \cos\theta, a\sin\theta)$, MA cuts BC at $P(a[1-\sin\theta]/\cos\theta, 0)$, MB cuts AD in $Q(0, a[1-\cos\theta]/\sin\theta)$, MC cuts AD in $R(0, a[1+\cos\theta]/\sin\theta)$, and MD cuts BC in $S(a[1+\sin\theta]/\cos\theta, 0)$. The points P, Q, R and S lie on the circle $x^2+y^2-2ax\sec\theta-2ay\csc\theta+a^2=0$.

As M moves around the circumcircle of the square, the center of this other circle describes the curve $x^{-2}+y^{-2}=a^{-2}$.

Also solved by W. B. Clarke, Fred Discepoli, L. M. Kelly, D. L. MacKay, E. R. Ott, D. K. Pease, A. V. Richardson, and Simon Vatriquant.

E 227 [1936, 431]. *Proposed by V. Thébault, Le Mans, France.*

Obtain the general solutions in integers for the pair of simultaneous equations:

$$(2n+1)x+1=y^2, \tag{1}$$

$$2nx+1=z^2, \tag{2}$$

and show that each of the numbers, $(n+1/2)x+1$ and $nx+1$, is equal to the sum of the squares of two consecutive integers.

Solution by B. W. Jones, Cornell University.

By addition and subtraction of the given equations, we obtain

$$(4n+1)x+2=y^2+z^2 \quad \text{and} \tag{3}$$

$$x=y^2-z^2. \tag{4}$$

Any integers which satisfy (1) and (2) satisfy (3) and (4), and conversely. We proceed to find one equation whose solutions are known, and which yields all the solutions of (3) and (4), and hence of (1) and (2).

Let

$$y=z+k; \tag{5}$$

then (4) becomes

$$x=2kz+k^2. \tag{6}$$

Using (6) and substituting from (5) in (3) we get, after a little manipulation, $2nk^2=z^2-4nkz-1$. Adding $4n^2k^2$ to both sides and quadrupling gives

$$k^2(16n^2+8n+1)-k^2=4(z-2nk)^2-4,$$

which may be readily reduced to

$$a^2-k^2(b^2-1)/4=1, \tag{7}$$

where

$$a=z-2nk, \qquad \text{and} \qquad b=4n+1. \tag{8}$$

If $b-1$ is a multiple of 4 and if a and k are integers, n and z defined by equations (8) will be integers.

Now, if x, y, z, and n are integers satisfying (1) and (2), they define by means of (5) and (8) integers k, a, and b which satisfy (7) and make $b-1$ a multiple of 4. Conversely, any integer solutions of (7) for which $b-1$ is a multiple of 4, will define by means of (8), (6), and (5) integer values of n, z, x, and y, which satisfy equations (1) and (2). In other words, the finding of all integer solutions of (1) and (2) is equivalent to finding all integer solutions of (7) which make $b-1$ a multiple of 4.

But (7) is the noted Pell equation since the values of b under consideration make $(b^2-1)/4$ an integer. It is known from the theory of the Pell equation [Chrystal, Algebra, vol. 2, pp. 452–453] that (7) has an unlimited number of solutions in integers whenever $(b^2-1)/4$ is an integer not a perfect square. If $(b^2-1)/4=r^2$, then $b^2-4r^2=1$, or $(b+2r)(b-2r)=1$. In this case, we have $b+2r=b-2r=\pm 1$; hence $r=0$, $b=1$, $n=0$, $a=\pm 1=z$, $x=\pm 2k+k^2$, $y=k\pm 1$, which, for all values of k, satisfy (1) and (2). It remains to consider the solutions of (7) when $b\neq 1$.

First, if $k=\pm 1$, $3=4a^2-b^2$, and either $3=2a+b$ and $1=2a-b$ or else $-1=2a+b$ and $-3=2a-b$. In either of these cases $b^2=1$, which has already been considered.

Secondly, if $k=\pm 2$, (7) reduces to $0=a^2-b^2$, and therefore, $a=bd$, where $d=\pm 1$, $n=(b-1)/4$, $z=b(d\pm 1)\mp 1$, $x=4(1\pm z)$, $y=z\pm 2$, which are solutions of (1) and (2) for all values of b for which n is an integer.

Since for each b under consideration, $k=2$, $a=|b|$ is the first, that is, least, positive solution, then all the positive solutions, a, k, of (7) can be found by equating the rational and irrational parts of the equation

$$a + k\sqrt{C} = (|b| + 2\sqrt{C})^m \tag{9}$$

where $C=(b^2-1)/4$, and m is allowed to range over all positive integers. All the solutions of (7), positive and negative, may then be found by changing the signs of one or both of a and k. From this, all the solutions of (1) and (2) may be found.

If a, b, and k are positive solutions of (7), with $(b-1)/4$ a positive integer, then obviously x, y, z, and n are positive integer solutions of (1) and (2). Conversely, if x, y, z, and n are positive, (4) implies that k is positive, (8) makes b exceed 1, and $a+2nk$ is positive. From (7) we have $(4n^2+2n)k^2+1=a^2$, and hence $|a|$ exceeds $2nk$, so that if $a+2nk$ is positive, a is. Thus all the positive solutions of (1) and (2) can be found from the positive solutions of (7) which make $(b-1)/4$ a positive integer.

It is interesting to note that the smallest possible value of k greater than 2, for b positive, is $k=20$, in which case $b=5$, $a=49$, $x=3960$, $y=109$, $z=89$, and $n=1$.

To complete the solution of the problem as stated, note that any solution z of (2) must be odd. Thus $z=2t+1$ defines an integer t. Substituting this in (2), we have $nx+1=t^2+(t+1)^2$, as specified. Equation (9) shows that k must always be even and hence (5) implies that y must always be odd. Substitution of $y=2u+1$ in (1) yields $(n+1/2)x+1=u^2+(u+1)^2$.

Also solved by W. B. Carver, Daniel Finkel, E. P. Starke, Simon Vatriquant, and the proposer.

E 228 [1936, 432]. *Proposed by Virgil Claudian, Roumanian Mathematical Institute.*

Solve the equation:

$$x^7 + 7px^5 + 14p^2x^3 + 7p^3x + q = 0.$$

Solution by C. W. Trigg, Cumnock College, Los Angeles.

Upon making the substitution, $x=y-p/y$, the given equation reduces to $y^7-(p/y)^7+q=0$, or to $y^{14}+qy^7-p^7=0$. Solving this as a quadratic equation for y^7, we secure $y^7=-q/2\pm r$, where $r^2=q^2/4+p^7$. Now if we choose one value of y and call it A, and let $B=-p/A$, then

$$A = (-q/2 + r)^{1/7} \qquad \text{and} \qquad B = (-q/2 - r)^{1/7}.$$

Now let j be a primitive seventh root of unity. Then the fourteen different values of y are Aj^k and Bj^k $[k=0, 1, 2, \cdots 6]$. If these be paired so that we combine Aj^k with Bj^{7-k}, the product of each pair will be $-p$. Hence the sum of the two

members of such a pair is a value of x, and the seven roots of the given equation are

$$x = (-q/2 + r)^{1/7}j^k + (-q/2 - r)^{1/7}j^{7-k}, \qquad k = 0, 1, 2, \cdots, 6,$$

with $r=(q^2/4+p^7)^{1/2}$.

(This is essentially problem 4, page 140, L. E. Dickson, First Course in the Theory of Equations, 1922.)

Also solved by R. W. Cowan, A. V. Richardson, J. Rosenbaum, E. P. Starke, and Simon Vatriquant.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3820. *Proposed by Paul Erdös, The University, Manchester, England.*

Let $a_1<a_2< \cdots <a_n<2n$ be positive integers such that no one of them is divisible by any other member of the sequence; then $a_1\geqq 2^k$, where k is defined by the inequalities $3^k\leqq 2^n<3^{k+1}$. This estimate for a_1 is the best possible.

3821. *Proposed by V. Thébault, Le Mans, France.*

Given an orthocentric tetrahedron $ABCD$ and two variable points M and M' diametrically opposite on the inscribed sphere, the parallels to AM, BM, CM, DM and to AM', BM', CM', DM', drawn from the orthocenter H cut, respectively, the corresponding faces in four points on planes π and π'. Show that: (a) The planes π and π' envelope a quadric surface of revolution Q inscribed in the tetrahedron and concentric with the second sphere of twelve points. (b) The planes π and π' intersect in a line Δ in the directrix plane of Q relative to its focus H. (c) The chord of contact turns about H remaining perpendicular to the plane (H, Δ).

3822. *Proposed by R. Goormaghtigh, Bruges, Belgium.*

In a triangle, the inscribed conics passing through the intersections of the circum- and conjugate circles are tangent to the nine-point circle.

3823. *Proposed by J. R. Musselman, Western Reserve University.*

Given any triangle $T_1T_2T_3$, let us denote by T_1', T_2', T_3' the reflections of T_1, T_2, and T_3 in any diameter of the circumcircle. The lines through T_i perpendicular to $T_j'T_k'$ meet at a point R_t, similarly the lines through T_i' perpendicular to T_jT_k meet at a point M_t. It is known that the points R_t and M_t also

lie on the circumcircle and are symmetric with respect to the above mentioned diameter. Given three sets of three points T_i, V_i, W_i on the same circle, locate with reference to some diameter the points R_t, R_v, R_w. Similarly for the triangles $T_1V_1W_1$, $T_2V_2W_2$ and $T_3V_3W_3$ find R_1, R_2, and R_3. Show that the triangles $R_1R_2R_3$ and $R_tR_vR_w$ have the same R point. A like theorem can be stated for the M point.

SOLUTIONS

3735 [1935, 324]. *Proposed by B. P. Gill, College of the City of New York.*

Solve in real, non-zero integers: $(x^2+y^2+z^2)(x+y+z)+xyz=0$. An equivalent statement is: Can the equation $t^5+at+b=0$, $b\neq 0$, have three real integers t for roots?

Remarks by A. A. Bennett, Brown University.

It one is to credit at face value certain statements by J. J. Sylvester the only conclusion is that (1) $(x^2+y^2+z^2)(x+y+z)+xyz=0$ has no solution of the form desired. These are from his collected *Mathematical Papers*.

I. From Vol. 1, p. 118 (stated without proof). "The equation in integers,

$$a(x^3+y^3+z^3)+c(x^2y+y^2z+z^2x+xy^2+yz^2+zx^2)+exyz=0,$$

may always be transformed so as to depend upon the equation,

$$fu^3+gv^3+hw^3=(6a-e)uvw,$$

wherein

$$fgh=ae^2-(c^2+3a^2)e+9a^3-3ac^2-2c^3,\text{"} \qquad \text{(misprinted, } 9a^2).$$

II. Sylvester shows explicitly, Vol. I, p. 111, how to transform

$$fu^3+gv^3+hw^3=Muvw,$$

into

$$x^3+y^3+Az^5=Mxyz, \qquad \text{where} \qquad A=fgh.$$

III. From Vol II, pp. 63–4 (translated) (stated without proof). "The equation,

$$x^3+y^3+Az^3=Mxyz,$$

is not solvable [in integers all different from zero] in the following circumstances. Let us write $M^3-27A=\Delta^3\Delta'$, where Δ' contains no cubic factor. Then if Δ' is even and contains [as proper factor] no number of the form f^2+3g^2 and if A is a prime, the equation is not solvable save in the cases when $\sqrt{(-M/A)}$ is an integer." (Further cases also considered.)

In the present instance (1) is of the form considered in I, with $a=c=e=1$. It reduced indeed to

$$u^3+v^3+5w^3=5uvw, \tag{2}$$

which fulfills all the conditions required for III above, with $M=A=5$, $\Delta=-1$, $\Delta'=10$.

The details of the reduction from (1) to (2) in the present instance may be effected as follows without recourse to a general theory. Write $T=x+y+z$, $U=2T-x$, $V=2T-y$, $W=2T-z$; The equation (1) reduces by use of these and the observation that $U+V+W=5T$, to $UVW=5T^3$. Thus the original problem is equivalent to that of finding rational numbers U/T, V/T, W/T, whose sum and whose product is 5. If any such triad exists, there is seen to exist a reducing triad for which U, V, W, are relatively prime by pairs. Hence each given prime factor of T occurs to the third power in just one among U, V, W. Since one of these must also contain the factor 5, we may write $U=u^3$, $V=v^3$, $W=5w^3$, $T=uvw$, for the reduced case, thus obtaining (2) as desired.

I have found no published proof to the effect that (2) has no solutions in integers all different from zero. The most complete results seem to be those given by A. Hurwitz, *Über ternäre diophantische Gleichungen dritten Grades*, Viertelsjahrschrift Naturf. Ges. Zürich, v. 62, (1917) 207–229. As applied to (2) they lead only to the conclusion that if integral solutions (u, v, w) exist other than those proportional to the trivial solutions $(1, -1, 0)$ infinitely many solutions must exist where the ratios among (u, v, w) represent points lying everywhere dense upon the single real branch of the curve of genus one represented in the projective (u, v, w) plane by (2).

3743 [1935, 397]. *Proposed by Norman Anning, University of Michigan.*

Two congruent coplanar parabolas have the same line as axis and open in the same direction. Tangents are drawn to the inner from any point of the outer. Prove that the area bounded by the tangents and the arc joining their points of contact is invariant.

I. *Solution by L. Richardson, The University of British Columbia.*

Let R, S be the points of contact of tangents drawn from any point T on the outer parabola to the inner congruent coaxial parabola. Let TQP, parallel to the axis, cut the latter parabola in Q and the chord RS in P. Let θ be the inclination of RS to the axis. Then we have the following well known results: (1) $TQ=QP$, (2) $SP=PR$, (3) $PR^2 \sin^2 \theta=4a.QP$, (4) *Area parabolic sector* $QPR=2/3$ *parallelogram* QR.

With the given conditions we see from (1) that TQ, QP and TP are invariant. From (2) we see that the required area is double the sectorial area TQR. From (1) and (3) we see that the altitude and the area of the triangle TPR are constant. It follows from (4) that the sectorial area TQR is constant.

II. *Solution by Harry Levy, University of Illinois.*

We can determine by direct computation that the unimodular affine transformations which leave invariant the parabola $y^2=2px+a$ are given by $x'=x+ky+\frac{1}{2}pk^2$, $y'=y+kp$. We observe that these transformations form a one parameter group, and that they leave invariant every parabola of the family $y^2=2px+a$

(with parameter a). It follows readily that there exists a transformation of the group which will carry a point of a given parabola of the family into any other point of that parabola.

Let A and A' be two points of the outer parabola, B, C, and B', C' be the points of contact of the tangents to the inner parabola from A and A' respectively. The transformation of the group which carries A into A' must carry B into B' and C into C'. Since the determinant of the transformation is unity, area is preserved.

Solved also by M. G. Boyce, B. LeF. Brown, C. E. Buell, W. B. Campbell, A. Pelletier, Leon Recht, E. P. Starke, C. W. Trigg, F. Underwood, Maud Willey, Mrs. D. E. Woodbridge, and the proposer.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items of interest to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

Professor R. A. Fisher, Galton Professor of Eugenics at University College, London, is a visiting professor at the University of California for the academic year 1936–37.

Dr. Israel Halperin has been appointed a Sterling Fellow in mathematics at Yale University.

Dr. Frances Harshbarger of Kent State University has been promoted to an assistant professorship.

One of the Townsend Harris Prizes of the College of the City of New York has been awarded to Professor Edward Kasner of Columbia University for his achievements in the field of mathematics.

Professor Tullio Levi-Civita of the University of Rome gave a series of three mathematical lectures at the Rice Institute in November 1936.

Professor C. E. Love of the University of Michigan has leave of absence for the second semester 1936–37.

The retirement of Professor M. A. MacKenzie of the University of Toronto is announced.

Dr. R. S. Martin, last year at the Institute for Advanced Study, has been appointed an associate at the University of Illinois for the academic year 1936–37.

Dr. Gertrude I. McCain has been appointed professor of mathematics at Brenau College.

Professor R. L. Menuet served as acting president of Tulane University prior to the recent appointment of a new president.

Dr. L. W. Nordheim, formerly lecturer in theoretical physics at the University of Göttingen, is visiting professor at Purdue University for this academic year.

Assistant Professor G. E. Raynor of Lehigh University has been promoted to an associate professorship.

Dr. Robin Robinson of Dartmouth College has been promoted to an assistant professorship.

Professor C. F. Roos, professor of economics at Colorado College and research director of the Cowles Commission for Research in Economics, has resigned to accept a position as director of research with the Mercer Allied Corporation.

Professor T. R. Running of the University of Michigan retired from active teaching service, February 1937.

Assistant Professor N. E. Rutt of Northwestern University has been appointed to an associate professorship at Louisiana State University.

Assistant Professor C. A. Shook of Lehigh University has been promoted to an associate professorship.

Dr. A. J. Smith of the University of Pennsylvania has been appointed professor of mathematics at Susquehanna College.

Professor P. F. Smith of Yale University retired on July 1, 1936.

Assistant Professor I. S. Sokolnikoff of the University of Wisconsin has been promoted to an associate professorship.

Frances Stribic of the University of Colorado has been promoted to an assistant professorship.

Dr. Otto Szasz was appointed research fellow in mathematics at the University of Cincinnati for the year 1936–37.

Assistant Professor W. J. Trjitzinsky of the University of Illinois has been promoted to an associate professorship.

Dr. G. R. Trott of Johns Hopkins University has been appointed to a professorship at Blue Mountain College, Mississippi.

W. A. Vezeau has been appointed to a professorship at St. Joseph's College, Philadelphia.

Associate Professor Aurel Wintner of Johns Hopkins University has been granted leave of absence for the year 1937–38 to attend the Institute for Advanced Study.

Associate Professor Oscar Zariski of Johns Hopkins University has been promoted to a professorship.

The following appointments to instructorships have been announced:

Brown University, Robert Rawhouser
Butler University, Dr. B. C. Getchell
Columbia University, Dr. F. J. Murray
Georgia School of Technology, W. B. Coleman
Iowa State College, Dr. R. A. Higdon
Michigan State College, J. D. Hill
New York University, Dr. Leo Zippin
Notre Dame University, Rev. J. H. Kenna
Pennsylvania State College, Dr. Aline H. Frank
Coleman Herpel
Princeton University, Harwood Rosser
South Dakota State College, Dr. L. E. Mehlenbacher
Teachers College of Connecticut, Margaret C. Weeber
Tulane University, Dr. E. G. H. Comfort
Dr. Charles Hopkins
University of Kansas, P. O. Bell
University of Minnesota, Dr. Joel Brenner
Whitman College, Dr. I. E. Highberg

Dana P. Bartlett, professor emeritus of mathematics at the Massachusetts Institute of Technology, died September 9, 1936.

Professor N. A. Pattillo of Randolph-Macon Woman's College died September 8, 1936.

Dr. J. N. V. Vedder, professor of thermodynamics at Union College, died on December 26th, 1936.

The University of Notre Dame announces a symposium on the Calculus of Variations to be held on April 7–8. Professors L. M. Graves, A. A. Haas, Solomon Lefschetz, W. Mayer, Karl Menger, Marston Morse, L. W. Nordheim, Tibor Radó, and Dr. W. T. Reid have accepted invitations to present papers.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

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BIRKHOFF PRESIDENT OF A.A.A.S.

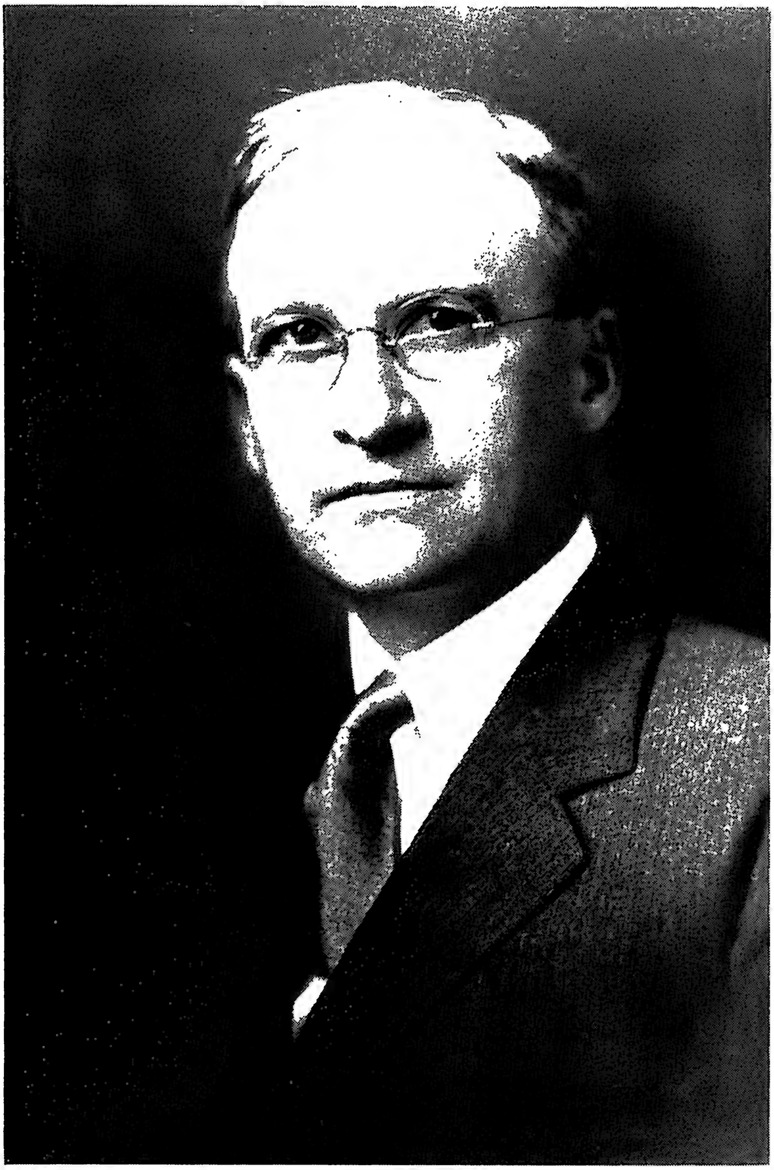
For the second time in half a century a mathematician has been honored by election to the presidency of the American Association for the Advancement of Science. This honor has come to Dean G. D. Birkhoff of Harvard University. Former presidents of the Association who did work in mathematical fields include William Chauvenet (1870), Joseph Lovering (1873), Simon Newcomb (1877), H. A. Newton (1885), and E. H. Moore (1921).

This honor to Birkhoff is only one of many that have come to him for his brilliant mathematical work. At the age of thirty-four he was elected to membership in the National Academy of Sciences; only a few men such as Hale, Michelson, Newcomb, and Theodore Richards have won this distinction at such an early age. At forty he was elected president of the American Mathematical Society; and the presidency of the A.A.A.S. comes at fifty-two.

He has been elected to honorary membership in the French Academy of Sciences, the Göttingen Scientific Society, the Royal Danish Academy of Sciences and Letters, the National Academy of the Lincei, and the Academy of Sciences at Bologna. The new Pontifical Academy of Sciences not only included him in its limited list of foreign fellows, but awarded to him in 1933 its prize for his investigations of systems of differential equations. He has also received from Venice the Quirini-Stampalia prize. Recently he has been made an officer of the French Legion of Honor, and honorary doctorates have been conferred on him by a number of universities including Brown, Harvard, Wisconsin, Paris, and Poitiers. In 1923 the American Mathematical Society awarded to Birkhoff the Bôcher prize for his paper on *Dynamical Systems with Two Degrees of Freedom;* and in December 1926 he won the A.A.A.S. $1000 prize for his paper entitled *A Mathematical Critique of Some Physical Theories.*

Books published by Birkhoff include *Relativity and Modern Physics* (1923), *The Origin, Nature, and Influence of Relativity* (1925), *Dynamical Systems* (1927), and *Aesthetic Measure* (1933). His published papers include eighty of a mathematical nature, and would fill at least three large volumes. While differential equations, difference equations, dynamics, relativity and foundations of mathematical physics have held a central place in his work, the catholicity of his interests is indicated by his book on aesthetic measure and one on elementary geometry (written in collaboration with a colleague).

Birkhoff's early papers on boundary value problems in differential equations, published when he was about twenty-five years of age, were indicative of his genius. He probably first attracted attention abroad by a short paper in 1913 in which he proved a geometrical theorem which Poincaré, Europe's leading mathematician, had announced but had not succeeded in proving. Birkhoff considers the ergodic theorem, which he proved in 1931, an exceptionally important contribution. His paper on the mathematical theory of electricity and gravitation, presented at the December 1936 meeting of the A.A.A.S., shows the most recent direction of his researches.



GEORGE DAVID BIRKHOFF

Of Dutch ancestry, Birkhoff was born at Overisel, Michigan, on March 21, 1884. His education included study at Lewis Institute, the University of Chicago, and Harvard University. He received the A.B. in 1905 and the A.M. in 1906 at Harvard, and the Ph.D. in 1907 at Chicago where he worked under the stimulating guidance of E. H. Moore. After teaching two years at the University of Wisconsin, and three years at Princeton where he was a full professor at the age of twenty-seven, he accepted in 1912 an assistant professorship at Harvard, where he has continued his activities, now holding the titles of Perkins Research Professor of mathematics, and Dean of the Faculty of Arts and Sciences.

E. J. MOULTON

THE OCTOBER MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The fifth regular meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at Geneva College, Beaver Falls, Pennsylvania, on Saturday, October 26, 1935. Professor C. S. Atchison, Chairman of the Section, presided at both the morning and afternoon sessions.

The total attendance was sixty-two, including the following thirty members of the Association: C. S. Atchison, O. F. H. Bert, H. L. Black, Helen Calkins, J. F. Calvert, W. E. Cleland, L. L. Dines, H. L. Dorwart, F. A. Foraker, N. C. Grimes, E. E. Hess, H. C. Hicks, B. P. Hoover, R. P. Johnson, Sister Marie Gertrude McNeil, W. I. Miller, David Moskovitz, L. T. Moston, J. H. Neelley, E. G. Olds, J. B. Rosenbach, H. C. Shaub, C. S. Shively, R. E. Smith, J. C. Stayer, J. S. Taylor, R. W. Thomas, W. J. Wagner, E. D. Wells, E. A. Whitman.

At the afternoon session the following officers of the Section were elected: Chairman, L. L. Dines, Carnegie Institute of Technology; Secretary-Treasurer, J. S. Taylor, University of Pittsburgh; Member of Executive Committee, H. L. Black, Westminster College. The Chairman and Secretary-Treasurer were elected for one year with the term of the newly elected member of the executive committee two years, Professor W. E. Cleland, Geneva College, continuing for the second year of his term as the additional member of the executive committee. The spring meeting was set for Saturday, May 2, at the Pennsylvania College for Women, Pittsburgh, Pennsylvania.

Following a welcoming address by President M. M. Pearce of Geneva College the following six papers were read:

1. "Reciprocals with respect to a space cubic" (preliminary report) by Dr. Mary M. Speer, University of Pittsburgh, introduced by Professor F. A. Foraker.

2. "A recent solution of the biquadratic with geometric interpretations" by Professor J. H. Neelley, Carnegie Institute of Technology.

3. "Teaching differential equations" by Professor C. S. Shively, Juniata College.

4. "Minimum voids and maximum mobility in clay products" by Professor C. S. Atchison, Washington and Jefferson College.

5. "The method of finite differences applied to various problems in elasticity" by Professor J. J. Stoker, Carnegie Institute of Technology, introduced by Professor L. L. Dines.

6. "Some problems in electrical machine design" by Dr. J. F. Calvert, Westinghouse Electric and Manufacturing Company.

Abstracts of these papers follow, with the numbers corresponding to the numbers in the list of titles:

1. Dr. Speer extended the investigation of the process of reciprocation with respect to a space cubic to include a study of the reciprocals of the axes and planes of the moving trihedral, of the surfaces generated or enveloped by them, of the edges of regression of some of these, and of conics. Properties of reciprocal loci and relations of reciprocal to given loci were obtained.

2. In the method presented by Professor Neelley the biquadratic is changed into a pencil of quadratics by a transformation in which the parameter is chosen so as to factor the quadratic. Then the solution depends merely upon the solution of two quadratics. The pencil interpreted as conics locates the roots as the ordinates of the intersections of the base conics. A projective transformation makes these base conics two congruent parabolas with perpendicular axes. Hence graphical solutions are readily obtained by superimposing these parabolas with proper orientation.

3. Professor Shively argued that a course in differential equations should train students in the statement of problems in the language of calculus and cultivate skill in the use of certain processes peculiar to the solution of equations thus formed. The student may lose sight of these aims in what may seem to him a confusion of difficult integrations and unusual algebraic manipulations. Methods of avoiding this danger were discussed.

4. Professor Atchison discussed the problem of retaining minimum voids in the aggregate in concrete or clay products and at the same time obtaining maximum mobility of the aggregate. Solutions were obtained by means of mathematical calculations concerning the fitting together of spheres.

5. Recalling that the method of finite differences for the approximate solution of boundary value problems has long been known, Professor Stoker called attention to the fact that it can be used with great success in solving certain classes of problems in elasticity, the solution of which by other methods is difficult if not impossible, as has been shown by H. Marcus.

6. Dr. Calvert presented a solution for the penetration of a lightning surge into the first coil of an armature winding. This involved the determination of the voltages and currents on n parallel lines when the line on which the wave enters is discontinuous in one direction from the point of entrance.

J. S. Taylor, *Secretary*

THE MAY MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The sixth regular meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at the Pennsylvania College for Women, Pittsburgh, Pennsylvania, on Saturday, May 2, 1936. Professor L. L. Dines, Chairman of the Section, presided at both the morning and afternoon sessions. Following the afternoon session those in attendance were entertained by the Pennsylvania College for Women at a very delightful tea held in Berry Hall.

The number of those in attendance was sixty-six, including the following twenty-five members of the Association: C. S. Atchison, O. F. H. Bert, H. L. Black, Helen Calkins, L. L. Dines, H. L. Dorwart, L. T. Dunlap, W. O. Gordon, E. E. Hess, H. C. Hicks, W. I. Miller, David Moskovitz, L. T. Moston, J. H. Neelley, E. G. Olds, F. W. Owens, Helen B. Owens, Walter Penney, J. B. Rosenbach, C. A. Rupp, C. S. Shively, J. C. Stayer, J. S. Taylor, E. D. Wells, E. A. Whitman.

Following a welcoming address by Dean M. Helen Marks of the Pennsylvania College for Women the following five papers were read:

1. "Orthogonal transformations" by Professor A. E. Staniland, University of Pittsburgh, introduced by the Secretary.

2. "Problems of geophysics" by Dr. L. L. Nettleton, Gulf Research and Development Corporation, introduced by the Secretary.

3. "On a theorem of projective geometry and its converse" by Professor C. A. Rupp, Pennsylvania State College.

4. "Some remarks on the length of an ellipse" by Professor H. C. Hicks, Carnegie Institute of Technology.

5. "On the validity of the mathematical theory of elasticity and photoelasticity" by Professor Max Frocht, Department of Mechanics, Carnegie Institute of Technology, introduced by the Secretary.

Abstracts of these papers follow, with the numbers corresponding to the numbers in the list of titles:

1. Professor Staniland discussed the general linear, homogeneous, nonsingular transformation of matrix A such that $AA'=A'A=I$, or such that $A=(A')^{-1}=(A^{-1})'$, pointing out that an orthogonal transformation is thus one for which the usual distinction between cogredience and contragredience fails to exist. The general n-ary orthogonal transformation depends upon $n(n-1)/2$ independent real parameters. Cayley succeeded in expressing the $n^2\, a_{ij}$'s of A for a transformation of positive determinant as rational functions of these parameters. His procedure was equivalent to taking $A=(1+S)/(1-S)$ where S is the general skew symmetric matrix, the symbolic fractional form indicating the commutativity of $1+S$ and $(1-S)^{-1}$. The Cayley matrix does not yield all orthogonal transformations of positive type. If, however, we take $E=[e_{ij}]$, where $e_{ij}=1$ for $i=j$ and $e_{ij}=0$ for $i\neq j$, the matrix of *any* orthogonal

transformation may be obtained by appropriate specialization of the elements of $E[(1+S)/(1-S)]$.

2. Dr. Nettleton explained that geophysical prospecting for oil involves the measurement at the surface of certain physical quantities and the interpretation of these measurements in terms of the probable nature and attitude of rocks below the surface. In the gravitational and magnetic methods, certain components of potential fields are measured. These measured quantities must be separated into the parts or anomalies which probably arise from density or magnetic contrast within depths which are of interest from parts which arise from depths much too great to be of interest. Anomaly pictures for comparison with an interpretation of the measured anomalies are built up, with the use of various graphical and mechanical aids to simplify and speed up the calculations. In the seismic methods, the times of travel of elastic waves from shot point to detector are measured. Time-distance curves can be used to determine the wave velocity in and the depths of beds traversed by refracted waves. With certain arrangements of the detectors, reflected waves can be recognized, the depths of reflecting layers can be calculated, and accurate maps of deep geologic horizons can be constructed.

3. Professor Rupp called attention to the fact that the assertion that three triangles are pairwise perspective was sometimes ambiguous. There is no ambiguity in the well known theorem that three triangles which are pairwise perspective from the same center have perspective axes which concur. The usual converse states that three pairwise perspective triangles whose perspective axes concur have a common perspective center. It is a question of the orders of corresponding elements in the three pairs of perspective triangles. For one of the six possible orders the converse stated above is true, for three of the orders the conclusion is that the three perspective centers lie on a line, and for two of the orders the three centers have no particular relation.

4. The paper of Professor Hicks suggested that various approximation formulae for the perimeter of an ellipse could be replaced by approximate graphical or numerical computation based on the following theorem: the perimeter of an ellipse equals the circumference of a corresponding circle whose radius is the mean value of the radii vectors of the ellipse (from its center) averaged with respect to their corresponding eccentric anomaly. This theorem is proved and it is noted that there is a similar theorem concerning any elliptic arc.

5. Professor Frocht dealt with the loci of maximum shear stresses in a number of two dimensional cases as determined from known stress functions in the infinite and semi-infinite plane. Among the cases considered were: the concentrated load on a semi-infinite plane normal to the edge, parallel to the edge, and inclined to the edge; wedge type beams with symmetrical and unsymmetrical loading; uniform and non-uniform pressure on a finite interval acting on the straight edge of the semi-infinite plane; and a circular disk subjected to concentrated diametral loads.

J. S. TAYLOR, *Secretary*

THE FALL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The fall meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the National Bureau of Standards, Washington, D. C., on Saturday, December 5, 1936. The chairman, Dr. John Williamson of Johns Hopkins University, presided over both sessions, morning and afternoon. The members of the staff of the Bureau of Standards prepared a number of exhibits and arranged a tour of the laboratories before the regular meeting. Five papers were read at the morning session while in the afternoon, at the invitation of the Section, Professor A. E. Landry of the Catholic University of America delivered a lecture on "Cremona transformations and their applications to algebraic function theory."

The attendance was fifty, including the following thirty-five members of the Association: O. S. Adams, N. H. Ball, G. A. Bingley, Archie Blake, W. E. Byrne, C. C. Bramble, Abraham Cohen, L. S. Dederick, Alexander Dillingham, J. A. Duerksen, P. J. Federico, E. J. Finan, Michael Goldberg, Harry Gwinner, L. M. Kells, W. D. Lambert, A. E. Landry, J. A. Larrivee, C. M. Lennahan, J. J. Luck, Florence M. Mears, T. W. Moore, K. S. Purdie, O. J. Ramler, C. H. Rawlins, J. N. Rice, A. W. Richeson, J. B. Scarborough, M. A. Scheier, J. L. Stearn, J. H. Taylor, H. W. Tyler, John Tyler, John Williamson, E. W. Woolard.

The spring meeting will be held on May 8, 1937 at Randolph-Macon Woman's College, Lynchburg, Virginia.

The following six papers were read:

1. "Hyperconformal transformations" by G. F. Alrich, University of Maryland, introduced by the Secretary.

2. "An elementary transformation of rectangular axes" by Professor J. W. Blincoe, University of Virginia.

3. "On the role of a basis in vector analysis" by Professor J. H. Taylor, George Washington University.

4. "Certain factorial sums" by Dr. Solomon Kullback, Office of the Chief Signal Officer, Washington, D. C., introduced by the Secretary.

5. "An episode in the life of Sylvester" by Professor R. C. Yates, University of Maryland, introduced by the Secretary.

6. "Cremona transformations and their applications to algebraic function theory" by Professor A. E. Landry, Catholic University of America.

In the absence of Professor J. W. Blincoe his paper was read by title only. Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Replacing the classical element of arc by an n-ic differential form which may be resolved into n linear factors in the differentials of the coordinates and imposing the requirement that each of these factors be a relative invariant under transformation, we obtain transformations in space of n dimensions which

have properties analogous to those of plane conformal transformations. Mr. Alrich confined his attention to the form:

$$\begin{vmatrix} dx & dz & dy \\ dy & dx & dz \\ dz & dy & dx \end{vmatrix}.$$

The transformations considered were shown to admit as absolute invariants the cross ratios $R=(JQ, JQ'; JI, JK)$ and $S=(KQ, KQ'; KI, KJ)$ where Q, Q' are the traces on the plane at infinity of two generic lines through any point of space and I, J, K are the vertices of the absolute. The vector (R, S) which plays the role of angle was designated as hyperangle. A set of normal functions was introduced permitting the expression of the components of the hyperangle by the equations

$$3\theta = w^2 \log R + w \log S, \qquad 3\phi = w \log R + w^2 \log S, \qquad w = e^{2\pi i/3}.$$

A simple method of generating the transformations was shown.

2. Dr. Blincoe's paper was concerned with a method of referring the equation of a plane algebraic curve, $f(x)=0$, to two arbitrary perpendicular lines, $h_1(x)=0$ and $h_2(x)=0$, as coordinate axes, by expressing the coefficients of the transformed equation as simultaneous euclidean invariants of the three forms, $f(x)$, $h_1(x)$, and $h_2(x)$.

3. Professor Taylor indicated how the use of systems of base vectors may be expected to clarify such notions as linear vector spaces, linear dependence, and application of linear operators like scalar and vector products, differentiation.

4. Dr. Kullback considered sums defined by

$$S_{rst\cdots}(n, N) = \sum \frac{N!}{x_1!x_2! \cdots x_n!}$$

where the summation is for those integral solutions of $x_1+x_2+ \cdots +x_n=N$ which do not involve $r, s, t, \cdots$. These sums arise in connection with the following problems in probability. Suppose there exist n mutually exclusive events, each with probability of occurrence $p=1/n$. In N independent observations, what is the probability that k of the events occur r times each? What is the probability for the simultaneous occurrence of k events r times each, l events s times each, etc.?

5. The paper by Professor Yates appears in full in this MONTHLY, pages 194–201.

6. The first part of Professor Landry's paper was an exposition of the principal properties of plane Cremona transformations including: (a) general formulas for the number of fundamental points of each order of multiplicity, (b) fundamental curves as maps of the fundamental points and as forming the Jacobian of the net of transforms of lines, (c) possibility of obtaining any

Cremona transformation as the product of quadratic transformations, (d) by way of illustration, the transformation set up by the bisecants of a space cubic. In the second part the principal theorems concerning the reduction of singularities involving contact of branches were given and illustrated by the case of the curve

$$y^2 = \prod_{i=1}^{6} (x - e_i),$$

which by two quadratic transformations was reduced to a quartic with one double-point at which the two tangents are distinct.

MICHAEL GOLDBERG, *Secretary*

THE ELEVENTH ANNUAL MEETING OF THE PHILADELPHIA SECTION

The eleventh annual meeting of the Philadelphia Section of the Mathematical Association of America was held at the University of Pennsylvania, Philadelphia, Pa., on Saturday, November 28, 1936, Professor Clawson presiding.

The attendance was forty-one, including the following twenty-eight members of the Association: Laura M. Ashbaugh, J. A. Benner, H. W. Brinkmann, S. S. Cairns, P. A. Caris, J. W. Clawson, Mary L. Constable, J. E. Davis, Arnold Dresden, Tomlinson Fort, W. L. Graves, V. V. Latshaw, D. H. Lehmer, D. L. McDonough, J. S. Mikesh, H. H. Mitchell, M. A. Nordgaard, C. O. Oakley, T. S. Peterson, G. E. Raynor, C. J. Rees, J. B. Reynolds, George Rosengarten, J. A. Shohat, C. A. Shook, L. L. Smail, W. M. Smith, A. H. Wilson.

At the business meeting the following officers were elected for next year: Chairman, H. H. Mitchell, University of Pennsylvania; Secretary, P. A. Caris, University of Pennsylvania; Program Committee, A. H. Wilson and Richard Morris. It was agreed to hold the next meeting at Haverford College, Haverford, Pa., on Saturday, November 27, 1937.

The following papers were presented:

1. "Remarks on abstract spaces" by Dr. J. A. Clarkson, University of Pennsylvania, introduced by Professor Oakley.

2. "Triangulations and related problems" by Professor S. S. Cairns, Lehigh University.

3. "Inverse probability and fiducial inference" by Professor S. S. Wilks, Princeton University, introduced by Professor Oakley.

4. "The undergraduate comprehensive examination" by W. R. Murray, Franklin and Marshall College, introduced by Professor Long.

Abstracts of the papers follow:

1. Dr. Clarkson discussed convexity properties of Banach spaces and their subsets, with special reference to the notion of uniform convexity of such spaces. A simple proof was offered of the Radon-Riesz theorem on weak convergence

in spaces L_p for p exceeding unity, exhibiting the connection between this theorem and the uniform convexity of these spaces.

2. In triangulation problems, one investigates the possibility of dividing a point set, subject to various restrictions, into the cells of a simplicial complex. Professor Cairns said that such problems are important in connection with the relationship between point set and combinatorial topology. While the problems most fundamental from this viewpoint remain unsolved, the author has given solutions for regular manifolds and various other loci subject to regularity restrictions. These solutions have applications outside topology. As examples, Professor Cairns gave (1) a generalization of the usual arc length definition to an area definition for regular r-manifolds in euclidean n-space and (2) a method for proving the generalized theorem of Stokes under hypotheses weaker than those previously used.

3. Professor Wilks explained some of the difficulties which arise in attempting to apply Bayes's Theorem on inverse probability to the problem of making inferences about unknown population parameters from observations, and conditions under which such difficulties can be avoided by fiducial argument. In particular, the principle of "insufficient reason" or the assumption of equal *a priori* probabilities was shown to lead to inconsistencies. In the case of a continuous observable variate x and an unknown parameter θ, conditions were given for the existence of a fiducial probability function of θ for a given value of x. From this fiducial probability function it was shown that confidence intervals can be set up for θ for a given value of x, and definite probability statements can be made about θ independently of the *a priori* probability function of θ. Extensions of the fiducial notion to several parameters, to problems of making inferences about samples from sub-samples, and to problems involving discrete variates were pointed out.

4. Mr. Murray maintained that the system of comprehensive examinations offers many decided advantages in coordinating the student's course of study in his specialized field. The effectiveness of the plan is largely determined by the amount of systematic guidance that can be given. The field of mathematics, while gaining less than other studies in the degree of coordination that may be obtained, is still adaptable to the comprehensive plan. The experience at Franklin and Marshall College has indicated several desirable gains, but the experiment is still too new and the operation too imperfect to make any enthusiastic claims for the plan as it operates there.

P. A. CARIS, *Secretary*

"Mathematics in its widest signification is the development of all types of formal, necessary, deductive reasoning." A. N. Whitehead, Universal Algebra, Cambridge, 1898, Preface, p. vi.

"The whole of mathematics consists in the organization of a series of aids to the imagination in the process of reasoning." A. N. Whitehead, loc. cit., p. 12.

SYLVESTER AT THE UNIVERSITY OF VIRGINIA*

By R. C. YATES, University of Maryland

I would like to take you back approximately one hundred years and ask you to think of yourself as a citizen of perhaps the leading state in the Union, Virginia. You were undoubtedly a slave owner and violently opposed to a group of "ignorant Yankee fanatics" called Abolitionists who were gradually eating their way into an otherwise peaceful South. Your loyalty to Southern traditions and to the almost God-given right to possess slaves, mingled with an inherited hatred of foreigners, lent you principles which you were ready to defend at all costs. An attack on these principles was an attack on your honor and deserved to be dealt with in an appropriate manner.

Although turbulent and restless on affairs of the moment, the national government in the hands of Andrew Jackson was financially sound and stood without a single debt that it could not liquidate. This fortunate condition was without precedent and that fact would have made you doubly proud had you been able to peer into the future and see it not duplicated, at least during the next one hundred years.

Illinois was an almost impenetrable frontier, with Lincoln a young man of thirty struggling for a political foothold in Sangamon County. A tale of being transported at thirty miles per hour on a railroad was scoffed at on the ground that no human could sustain life at such an excessive rate of speed. Shipping on the Eastern seaboard was sought after by new-fangled contraptions that supplemented sail power with steam—although the clipper ship was by no means replaced since wood for fuel had to be carried at the expense of cargo space. Steam gauges and good lubrication were unknown with the result that parts moved in shrieking discord and wore out in short order or boilers burst their hand-riveted seams to the destruction of life and limb. The head of the Patent Office in Washington submitted his resignation in view of the fact that with the railroad engine and the steamboat the possibilities of invention were about exhausted and little more need be expected. Before the forties came, steam had been applied to machinery on land, factories had sprung up, and our booming commercial trade with Great Britain had brought us into more intimate contact with foreigners.

Into the very heart of this scene came a young man from England, James Joseph Sylvester, at the age of 27, to take the chair of mathematics at the University of Virginia. Although a Jew and a foreigner, he was accepted with little hesitation on the part of the Board of Visitors—due no doubt to the excellence of a score or more of letters and testimonials from eminent people across the water. The Richmond newspapers of the day were in violent opposition to this appointment (together with that of Dr. Kraitsir, a Catholic) and spared no pains to make their feelings widespread. Although the post was tendered temporarily for the first year, as was the custom, and carried an assured salary of only $1000,

* Read before the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America, Washington, D. C., December 5, 1936.

Sylvester accepted, believing the social life in the new country for an educated Jew more desirable than that in England where he thought discrimination rampant. It is to be remembered here that because of his religion—and he was staunchly loyal to his faith—his well-earned degree at Cambridge was withheld from him. St. John's College was controlled by the Church of England and although they allowed unorthodox applicants to attend, the laws required a test of faith before the formal degree could be conferred. Sylvester must have felt this discrimination deeply since he was Second Wrangler and probably desired to return in some sort of official capacity. There was no particular intolerance of the Jew in England at this time because they were respected for their financial acumen if for nothing else. Disraeli himself was the leader of the home government and a guiding spirit in the national life. We can only conclude, therefore, that Sylvester's grievances against his lot were somewhat fanciful and distorted into a feeling of personal wrong. There was a chip on his shoulder in this dusty town of Charlottesville—a chip, unfortunately, transported safely all the way from England.

This salary of $1000 from the University was to be supplemented with fees of $25 from each student registering in mathematics—making to all events, an annual sum of well over $2500—substantial indeed for the time.

Sylvester arrived in November, late for the session of 1841–42, and was warmly greeted by the student body. The arches on the campus were lighted with flares and there was general good feeling on both sides. He responded to this welcome with an address in which either nervousness or self-consciousness made him scarcely fluent—to all appearances he was extremely modest and retiring.* This seems remarkably strange in the light of his later and more mature nature. He entered into his new duties with enthusiasm, directing the work of about four dozen students in arithmetic, geometry, and the calculus. Soon he was appointed to a faculty committee to arrange hours for special lectures—presumably for advanced work, but whether or not these were actually started is not known. The entire enrollment of the University was short of two hundred—their academic needs cared for by ten professors.

It seems evident that the students—or at least some of them—must have resented somewhat the presence of these two foreigners, the gala reception notwithstanding. Each professor was annoyed from time to time by unseemly action on the part of one or several students in the classroom, as evidenced by reports to the faculty. Inattention was presumably a serious offense—of the same magnitude as drunkenness—since it was generally the custom for the chairman of the faculty to write to the parents or guardian and at the same time reprimand the student. At a meeting of the faculty on February 1, Professor Sylvester presented a much more serious offense—that of disrespect on the part of Mr. William H. Ballard, a first year student from New Orleans. It was at this same meeting, incidentally, that the Proctor's report of assessments for the month of January was submitted in which Sylvester was charged $2.50 for five

* Life and Letters of William Barton Rogers, Houghton Mifflin & Co., 1896.

window glasses. To this charge he objected (for reasons unknown and thereby perhaps hangs a tale of student destruction) and the matter was referred back to the Proctor. Again there came trouble with Mr. Ballard. Not three weeks had passed when a distressing scene occurred in the lecture room just at the close of the hour—a scene best described by the minutes of the faculty meeting of February 23, 1842. Present at this meeting were Gessner Harrison, chairman, Professors Rogers, Kraitsir, Howard, Sylvester, and Judge Tucker. We read:

> Case of W. H. Ballard. Professor Sylvester complained of Mr. Ballard's action in class on February 21, reading the following notes, commenting upon and explaining them:
>
> "He engaged (to all appearance) in reading some book, held beneath the desk during the lecture, at all events paying no attention to the examination then going on.
>
> "Answered to his own examination (but this I passed over) in a very unbecoming way, contradicting and interrupting as if he were disputing with me.
>
> "A general call to attention had the effect of causing him for a few minutes to redress his position; but presently after, he gave his exclusive attention to the book (or whatever else it might have been) that he continued to hold under the desk.
>
> "On being called up after the lecture and privately recommended to pay more attention in future—he answered (still with increased violence of demeanor and tone) that he understood the subject—could follow the lecture without looking at me and had not his spectacles.
>
> "On this as he was leaving it occurred to me to recommend him to bring them with him in future. Hereupon he answered in a very violent tone, manner and language, but the exact words I do not remember. It is proper to observe that Mr. Ballard's mode of addressing me has been almost uniformly marked with insolence and defiance.
>
> "I then felt it right to state that such conduct must not be persisted in and that if Mr. Ballard could not alter his conduct he must cease to attend my lecture room.
>
> "On this he answered with increased insolence and violence—but I cannot recall the terms employed. I answered that I had been in different parts of the world, but had never witnessed similar conduct in persons brought up amongst gentlemen.
>
> "To this he replied that 'I was not to prate to him, to hold my jaw, that I might go to hell'—and other abusive terms which I cannot recall.
>
> "I declined altercation with him and ordered him to leave the room, which he declared he would not do.
>
> "I here turned to such of the class as remained, made some remarks on the disgracefulness and discreditable character of such conduct and language, and left the room. Before my leaving Mr. Ballard was guilty of additional abusive language.

"The above is a faithful sketch of what passed to the best of my belief, and the extent of my impressions.

"Mr. Ballard had twice rendered himself in the preceeding month obnoxious to special censure, and has been reprimanded by the chairman, by order of the Faculty, for rudeness and insubordination.

"Since then, he has left the lecture room towards the close, and answered with great rudeness when informed that it was against order.

"On the following day he answered to his name on being called in such a way as to excite laughter in the whole class, and drew down from me the reprimand for persisting to adopt this peremptory tone in addressing me—he forgot what was due to his own position and not mine.

"The manner was such as to make me apprehend the probability of personal violence."

Mr. Ballard was summoned before the faculty and asked to make a statement. Before he concluded, Professor Sylvester withdrew from the meeting to return after Ballard had gone. His testimony is quoted in part: "Among other things, Mr. Sylvester said I was no gentleman—or something equivalent—the exact words I do not recollect. I considered that I was imposed upon, and spoken to in an authoritative manner—as an overseer speaks to a negro slave." When challenged with Professor Sylvester's accusation that he was disrespectful, he said: "I always answer Professor Sylvester as I do other professors—that I mean no disrespect to him or to them." He added later: "Till I was ordered out of the classroom my language was not violent—or I did not intend it should be so." Ballard asked when he had finished that some of the bystanders be examined. This was agreed upon by the faculty and Mr. W. F. Weeks was sent for—another first year student and a probable intimate of Ballard's since he also came from Louisiana. Weeks reported vividly and at some length on the whole affair from beginning to end. He told that when ordered out Ballard said Mr. Sylvester had no authority after the finish of his lecture and that he would see him in hell first. Sylvester, turning to the few members of the class that remained, said this could not be considered brave or gentlemanly and that Mr. Ballard had never associated with gentlemen or been in genteel society. Ballard then told Mr. Sylvester that the manner in which he had treated him (Ballard) indicated that he (Sylvester) had himself never associated with gentlemen. The faculty minutes state:

"After a full consideration of all the circumstances of the case as above recited, the following resolutions were adopted:

First: The reprimand bestowed by the Professor of Mathematics upon Mr. Ballard being considered by him and the faculty as adequate to the offense committed during the lecture and the professor not having reported Mr. Ballard for this offense—Resolved that in this particular no subject is presented for the action of the Faculty.

Second: In view of the transactions posterior to the termination of the lecture, and the reprimand of the Professor; while the Faculty cannot but

reprehend the violent language indulged in by Mr. Ballard towards Professor Sylvester, being unable to determine in how far the remarks of the Professor as understood by Mr. Ballard might extenuate his conduct in reply—Resolved, that it is expedient that this much of the transaction in question be submitted to the Visitors of the University at their next meeting for their decision.

Professor Kraitsir asked to be excused from voting on these resolutions."

Thus the matter ended for the day. Before the night had passed, Professor Sylvester had an opportunity to examine the written minutes and found much with which to be dissatisfied. Taking considerable pains to make his stand clear he asked that the following be spread upon the minutes of the faculty meeting on the next day, February 24:

"Mr. Sylvester protests against the principle of introducing students of a class to give evidence against their professor. He believes that besides other grave inconveniences this practice has the effect of bringing the professor into a position of direct controversy and altercation with those over whom he is placed in a position of authority.

"The evidence on the occasion in question was obtained from students present after the lecture at the time of Mr. Ballard being called on to be privately admonished. That such presence is in itself an act of insubordination, and very generally made the means, and occasioned by the desire, of encouraging the student subject to reprimand in resistance to the lawful authority of his professor.

"That Mr. Weeks in particular is habitually ill prepared and has been himself admonished and reported by the Professor of Mathematics.

"That he has some reason to believe that previous communication had passed between Mr. Ballard and the witnesses, at all events the question as to this previous communication was not put to the witnesses.

"Finally he calls attention to the fact that Mr. Ballard was unable, and did not offer during the first part of his examination when called on by Mr. Sylvester to do so, to point out any inaccuracy or exaggeration in the statement of the Professor of Mathematics.

"Mr. Sylvester denies absolutely having ever said to Mr. Ballard 'that he had never associated with gentlemen or been in genteel society'—or having used a word to this effect.

"He denies further that Mr. Ballard ever said that this was the first time he had contradicted Mr. Sylvester, or one word to this effect. He affirms on the contrary, that from the beginning, Mr. Ballard's attitude was that of open defiance."

At this meeting Sylvester made the remark that there were good students in the class as well as bad ones and that *they* might have been examined by the faculty. He named three such students and said that they could be expected to make correct statements of the affair. Sylvester withdrew from the meeting, and the faculty, thinking they were consulting his wishes and sense of justice, sent

for these men to give testimony. Their version was not greatly different from that of Ballard and Weeks and served only to increase the hopeless tangle of misunderstanding between professor and student. The faculty, of course, was caught between loyalty to university laws and a colleague on the one hand and evident sympathy for Ballard on the other. It cannot be assumed that they themselves were immune to the feeling against foreigners which was still alive in the community—did not Dr. Kraitsir, who was treated in a similar fashion by the students, silently censure their action when he withheld his vote on their resolutions?

Sometime later Sylvester, after reading the record of this second meeting, sent the following protest to the faculty, asking that it be placed in the minutes. It appears under the date of March 19:

> "Mr. Sylvester is not conscious of having expressed and certainly never entertained a desire that further testimony should be obtained from students after he had withdrawn from the faculty meeting of February 24th. On being interrogated by the Chairman as to the character of certain members of his class, he spoke favorably of them, and on the question being put to him by the Chairman, may have gone so far as to say that he would have objected less to testimony being sought for from them than from Mr. Weeks. But he left the room without having the faintest idea that when the case had already been judged by the Faculty and their decision published, fresh depositions were to have been taken; otherwise he would certainly, under the altered circumstances of his own position, have remained to address questions to the witnesses."

At this point our record presents a suspicious blank. The only inference that I can possibly draw is that Sylvester, finding himself in complete discord with his fellow faculty members, hurt and dissatisfied with their decision (or rather lack of decision), "viciously lied upon" by his own students, at odds with every one—submitted his unconditional resignation. The Board of Visitors, comprised of Chapman Johnson, John H. Cocke, Thomas J. Randoph, Samuel Taylor, met on March 22, 1842 to consider this—the only record of their investigation and discussion being the following:

> "Resolved that the resignation of Mr. Sylvester be accepted, to take effect from and after the 29th day of the present month or at any earlier period that he may elect; that a copy of this resolution be forthwith communicated to him by the secretary, and that he be informed that in accepting his resignation the Board has not deemed it necessary to investigate the merits of the matter in difference between himself and the student Ballard, and does not mean to impute to Mr. Sylvester any blame in that matter."

Thus to a swift and tragic end came a career that might have been a brilliant one—that might have added glory to the University of Virginia and to the Union. What happened to this unfortunate and friendless man in the distressing year or two that followed is not known. Two of Sylvester's brothers were in this country and it is entirely possible that he stayed with them during this interven-

ing period, perhaps revolting at the thought of returning to England. He did apply to Columbia College for a position and upon his request in the Spring of 1843 his former colleagues brought forth the following unanimously adopted resolution:

> "The faculty of the University of Virginia having been informed that Mr. Sylvester is a candidate for the chair of Mathematics in Columbia College, and having learned that his prospects of success are likely to be injuriously affected by erroneous impressions as to the circumstances of his separation from this Institution, desire, in justice to him, to correct any misconceptions on this subject which may now be operating to his disadvantage. They, therefore, beg leave to state that his separation from the University was entirely his own voluntary act occasioned as they conceive by his dissatisfaction at the course which his colleagues thought it proper to adopt towards a student whom Mr. Sylvester had reprimanded for inattention in the lecture room and whom in their view of the circumstances they were unwilling to punish to the extent Mr. Sylvester required."

He did not receive the appointment at Columbia—for what reason I am not prepared to state. What our mathematical history would have been had Sylvester found things to his liking over here no one can say. For thirty-five years—until he returned to inaugurate the program at Johns Hopkins—we were deprived of his contagious enthusiasm and mental genius. Our first opportunity to entertain one of the greatest intellectual giants of the century was lost to posterity. On the other hand, had he remained in America he would have missed forming that life-long and intimate friendship with Cayley—that meek, kind, and retiring little man who complemented Sylvester so completely that together they breathed life once again into mathematics in England.

I cannot close without presenting the following account quoted from *Ten British Mathematicians* by Alexander Macfarlane, John Wiley & Sons, 1916:

> "In 1841 he became professor of mathematics at the University of Virginia. In almost all notices of his life nothing is said about his career there; the truth is that after the short space of four years it came to a sudden and rather tragic termination. Among his students were two brothers, fully imbued with the Southern ideas about honor. One day Sylvester criticised the recitation of the younger brother in a wealth of diction which offended the young man's sense of honor; he sent word to the professor that he must apologize or be chastised. Sylvester did not apologize, but provided himself with a swordcane; the young man provided himself with a heavy walking-stick. The brothers lay in wait for the professor; and when he came along the younger brother demanded an apology, almost immediately knocked off Sylvester's hat, and struck him a blow on the bare head with his heavy stick. Sylvester drew his swordcane, and pierced the young man just over his heart; who fell back into his brother's arms, calling out 'I am killed.' A spectator, coming up, urged Sylvester away from the spot. Without waiting to pack his books the professor left for New York, and took the earliest possible passage for England. The student was not seriously hurt; fortunately

the point of the sword had struck fair against a rib."

This account cannot be accepted as reliable in its entirety since there are several errors to be noted. First, Sylvester did not stay in Charlottesville four years; his resignation was received within four months of his arrival. Second, Ballard did not have a brother at the University since he was the only student of that name registered. The principal theme of the story I cannot establish and affirm even though I have searched diligently through the newspapers and letters in the University collection and those in the Library of Congress. The possibility of a truthful foundation must be admitted, however, since in Sylvester's first report to the faculty he stated that Ballard's "manner was such as to make me apprehend the probability of personal violence." Furthermore, the student Weeks who testified for Ballard and against Sylvester did have a brother who was a member of the same class in mathematics. Incidentally, this brother must have been rather troublesome too, since he was denied the privilege of attending the class after March 2.

Thus the episode ends. Its effect is depressing. For fear of leaving you with unpleasant thoughts, let me tell you a little story connected with his successor. Edward H. Courtenay was appointed the next fall to the chair vacated by Sylvester and served in complete harmony with and to the admiration of both his colleagues and his students. In presenting his lectures, Courtenay found it more convenient to stencil the text of his analytics and calculus on long rolls of cotton cloth than to make use of the blackboards. Upon his death, a Dr. Fleming of Hanover County bought some of his personal effects—among them these cotton rolls. "When the spectre of impoverishment stalked through the state towards the end of the Civil War, Dr. Fleming was at a great loss how to prevent his youthful negroes from reverting joyfully to the complete nudity of the African jungles. Apparently, there was no stuff except hickory and oak leaves with which to clothe them, for all the sheep and oxen had been consumed by the armies, and the old garments were now too tattered to be patched. The presence of the rolls in the garret luckily leaped to his mind; the mammies were set busily to work; and soon the pickininnies were tumbling about the yard with half the problems of geometry and calculus conspicuously imprinted upon their backs. It was said at the time that an observant traveller passing that way could have found no difficulty in furbishing up his whole mathematical education by studying the wonderful display of figures on the persons of these little grinning and animated blackboards."*

Postscript: Since delivering this paper, I have had the opportunity to inspect the David Eugene Smith Presentation Volume of the *Osiris* in which R. C. Archibald presents thirty hitherto unpublished letters of Sylvester. These letters indicate clearly that he spent the year 1842–43 with his brother in New York City—all the while trying desperately to obtain work either as a teacher or as an actuary. One of the letters to Peirce of Harvard gives, as his most urgent reason for obtaining a position at almost any salary—a love affair!

* History of the University of Virginia, by P. A. Bruce.

ON THE SHAPE OF LEVEL CURVES OF GREEN'S FUNCTION*

By J. L. WALSH, Harvard University

The level curves of Green's function—that is to say, the images of concentric circles in a smooth conformal map—have been widely studied, particularly with reference to convexity [**1**]†, star-shapedness [**2**, p. 82], and centers of curvature [**3**, **4**]. It is the object of the present note to obtain some new results on the behavior of a level curve in the large, notably its approximation to the shape of a circle.

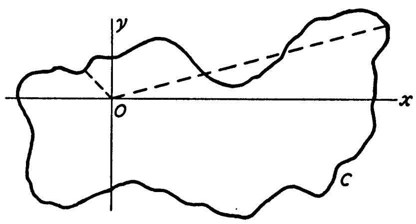


FIG. 1

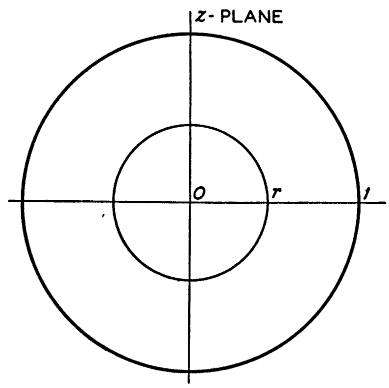


FIG. 2

1. *The circularity of a curve.* Let C be a closed curve with or without multiple points (Figure 1), or more generally an arbitrary closed limited point set which separates the point O from the point at infinity. *By the circularity $K(C)$ of C (with respect to O) we understand the ratio*

$$K(C) = \frac{\text{Distance from } O \text{ to nearest point of } C}{\text{Distance from } O \text{ to farthest point of } C}. \tag{1}$$

The quantity $K(C)$ obviously depends on O, is always positive, and is never greater than unity. The quantity $K(C)$ is a rough measure of the closeness with which C approximates to the shape of a circle with center O. Obviously the circularity $K(C)$ equals unity when and only when C is a circle whose center is O.

For convenience we formulate here a well known result [**2**, pp. 2, 103; or **5**, p. 39] to which we shall make frequent reference:

SCHWARZ'S LEMMA. *Let $\Phi(z)$ be analytic and bounded: $|\Phi(z)| \leqq M$ for $|z| < 1$, with $\Phi(0) = 0$. Then we have*

$$|\Phi(z)| \leqq rM \quad \text{for} \quad |z| \leqq r < 1, r > 0; \tag{2}$$

the strong inequality is valid in (2) unless $\Phi(z)/z$ is a constant of modulus M.

* Presented to the American Mathematical Society, December, 1936.

† Bold faced numbers in square brackets refer to references listed at the end of the paper.

2. *Monotonic character of* $K(C_r)$. Our first result on the shape of level curves is

THEOREM 1. *Let the function* $f(z)$ *be analytic and bounded for* $|z|<1$, *with* $f(0)=0$, $f'(0)\neq 0$, *and* $f(z)$ *bounded from zero except in the neighborhood of the origin. Let* $C_r(0<r\leqq 1)$ *denote the boundary in the w-plane of the image of the region* $|z|<r$ *under the map* $w=f(z)$. *Then* $K(C_r)$, *the circularity of* C_r *with respect to* $w=0$, *increases monotonically as r decreases*:

$$K(C_{r_1}) \geqq K(C_{r_2}) \quad \text{if} \quad r_1 < r_2, \tag{3}$$

and $K(C_r)$ *approaches unity as r approaches zero. The strong inequality holds in* (3) *unless* $f(z)/z$ *is a constant—that is to say, unless* $K(C_r)$ *is identically unity.*

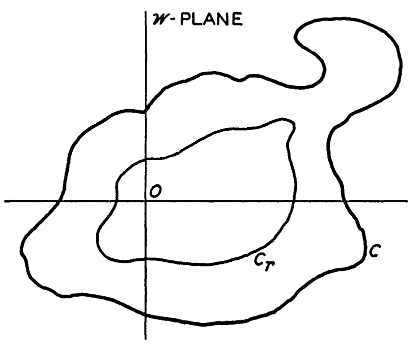


FIG. 3

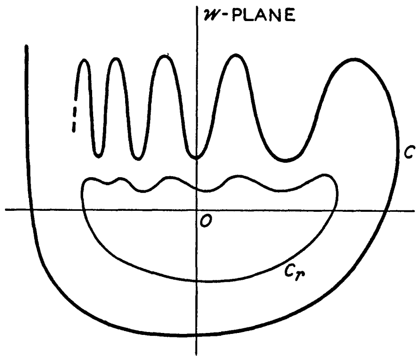


FIG. 4

Figure 2 represents the situation in the z-plane, with Figures 3 and 4 as suggestions of the numerous possibilities in the w-plane.

Here and in the sequel we denote by M_r and m_r the greatest and least values of $|f(z)|$ on the circle $|z|=r<1$; by M and m we denote the superior and inferior limits of $|f(z)|$ as z approaches the circumference $|z|=1$. By hypothesis m is greater than zero. Thus we have

$$K(C_r) = \frac{m_r}{M_r}, \qquad K(C_1) = \frac{m}{M}; \tag{4}$$

it is to be noticed that the neighborhood of $z=0$ is mapped by the transformation $w=f(z)$ onto the neighborhood of $w=0$, so the boundary C_r must separate $w=0$ from the point at infinity. By Schwarz's Lemma we have

$$\frac{M_r}{r} \leqq M. \tag{5}$$

The function $z/f(z)$, when suitably defined at the origin, is analytic and different from zero for $|z|<1$. When $|z|$ approaches unity, the function $|z/f(z)|$ can ap-

proach no limit greater than $1/m$, so by the principle of maximum modulus for an analytic function [6, p. 146; or 7, p. 136; or 8, p. 4], we have for $|z|<1$

$$\left|\frac{z}{f(z)}\right| \leqq \frac{1}{m},$$

$$\frac{r}{m_r} \leqq \frac{1}{m}. \tag{6}$$

By equations (4) and inequalities (5) and (6) we now have

$$\frac{m_r}{M_r} \geqq \frac{m}{M}, \quad \text{or} \quad K(C_r) \geqq K(C_1). \tag{7}$$

We notice that C_{r_1}, the image of the circle $|z|=r_1<r_2$ under the transformation $w=f(z)$, is the image of the circle $|z'|=r_1/r_2<1$ under the transformation $w=f(r_2z')$, which transforms $|z'|<1$ into a region bounded by C_{r_2}. Inequality (3) now follows from the second of inequalities (7). If $f(z)/z$ is not a constant, the strong inequality holds in (5), hence also in (7) and (3).

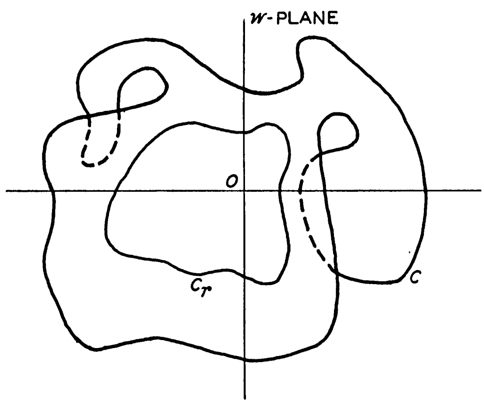


FIG. 5

For all values of z, $|z|<1$, we can write

$$\begin{aligned} f(z) &= a_1z + a_2z^2 + a_3z^3 + \cdots, \qquad a_1 \neq 0, \\ &= a_1z + z^2\chi(z), \end{aligned}$$

where $\chi(z)$ is analytic. Since $f(z)$ is bounded for $|z|<1$, so also is $\chi(z)$. If we suppose $|\chi(z)| \leqq h$, we have for $r<1$

$$M_r \leqq |a_1| r + hr^2, \qquad m_r \geqq |a_1| r - hr^2.$$

Thus we have

$$K(C_r) = \frac{m_r}{M_r} \geqq \frac{|a_1| r - hr^2}{|a_1| r + hr^2} = \frac{|a_1| - hr}{|a_1| + hr},$$

which approaches unity as r approaches zero. Then $K(C_r)$ also approaches unity as r approaches zero, so Theorem 1 is established.

In Theorem 1 the image in the w-plane of the region $|z|<1$ may overlap itself as in Figure 5. That is to say, the function $f(z)$ of Theorem 1 may take on the same value w in several distinct points z, $|z|<1$; the function $f(z)$ is not necessarily *univalent* (*schlicht*) for $|z|<1$.

3. *Direct study of green's function.* Theorem 1 can be proved by methods of conformal mapping, as above, or by methods of pure potential theory, as we proceed to indicate. We now direct our attention to the plane of $w=x+iy$ itself, without necessarily referring to the mapping onto the z-plane.

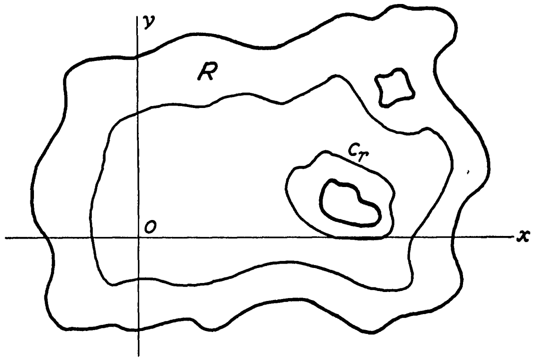


FIG. 6

THEOREM 2. *Let R (Figure 6) be a bounded region of the (x, y)-plane whose Green's function $G(x, y)$ with pole in the interior point O exists. Let C_r $(0<r\leqq 1)$ denote the locus $G(x, y)=\log r$ in the closed region R, so that C_r for $r<1$ consists of a finite number of Jordan curves. Then the circularity of C_r increases monotonically as r decreases, and approaches unity as r approaches zero.*

By Green's function $G(x, y)$ for R with pole at O: $(0, 0)$, we mean the unique function which is harmonic interior to R except at O, which approaches zero whenever (x, y) in R approaches a point of the boundary of R, and which in the neighborhood of O can be expressed

$$G(x, y) = G_0(x, y) + \log \rho, \qquad \rho = (x^2 + y^2)^{1/2}, \tag{8}$$

where $G_0(x, y)$ is harmonic throughout the neighborhood of O. The locus C_r obviously separates O from the boundary of R, hence separates O from the point at infinity.

If the region R of the plane for $w=x+iy$ is simply connected, and is mapped conformally in a one-to-one manner onto the region $|z|<1$ of the z-plane by the function $w=f(z)$ with $f(0)=0$, then [**9**, p. 717; or **2**, p. 57; or **10**, p. 365] this transformation can also be written $z=e^{G(x,y)+iH(x,y)}$, where the function $H(x, y)$ is conjugate harmonic to $G(x, y)$ interior to R. That is to say, we have under this transformation $|z|=e^{G(x,y)}$, so the locus C_r: $G(x, y)=\log r$ defined in Theorem 2 is precisely the locus C_r (the transform of $|z|=r$) defined in Theorem 1.

Under the hypothesis of Theorem 2, equation (8) defines a function $G_0(x, y)$ harmonic throughout the interior of R, continuous in the corresponding closed region. Let D_r denote the greatest distance and d_r the least distance from O to a point of C_r. By the definition of C_r: $G(x, y) = \log r$ we have

$$\begin{aligned} \max\,[G_0(x, y), \text{ on } C_r] &= \log r - \log d_r, \\ \min\,[G_0(x, y), \text{ on } C_r] &= \log r - \log D_r. \end{aligned} \tag{9}$$

The function $G_0(x, y)$ is harmonic in the region bounded by C_{r_2} ($r_2 \leqq 1$) and containing O, continuous in the corresponding closed region; that region contains in its interior the locus C_{r_1}, $r_1 < r_2$, so we have

$$\begin{aligned} \max\,[G_0(x, y), \text{ on } C_{r_1}] &\leqq \max\,[G_0(x, y), \text{ on } C_{r_2}], \\ \min\,[G_0(x, y), \text{ on } C_{r_1}] &\geqq \min\,[G_0(x, y), \text{ on } C_{r_2}]. \end{aligned} \tag{10}$$

These inequalities, by virtue of equations (9), can be written

$$\begin{aligned} \log r_1 - \log d_{r_1} &\leqq \log r_2 - \log d_{r_2}, \\ \log r_1 - \log D_{r_1} &\geqq \log r_2 - \log D_{r_2}, \end{aligned}$$

from which we derive

$$\begin{aligned} \log d_{r_1} - \log D_{r_1} &\geqq \log d_{r_2} - \log D_{r_2}, \\ \log K(C_{r_1}) &\geqq \log K(C_{r_2}), \\ K(C_{r_1}) &\geqq K(C_{r_2}), \end{aligned} \tag{11}$$

as we were to prove. The strong inequality holds in (10) and therefore in (11) unless $G_0(x, y)$ is identically constant, that is to say, unless R is the interior of a circle.

For r sufficiently small, the locus C_r is a small Jordan curve containing O in its interior, and which approaches O monotonically when r approaches zero. If we set $G_0(0, 0) = g$, equations (9) can be written respectively

$$\begin{aligned} \log d_r &= \log r - (g + \epsilon_1), \\ \log D_r &= \log r - (g - \epsilon_2), \end{aligned}$$

where ϵ_1 and ϵ_2 depend on r and approach zero with r. Then we have

$$\log K(C_r) = \log \frac{d_r}{D_r} = -\epsilon_1 - \epsilon_2,$$

which approaches zero with r, so $K(C_r)$ approaches unity as r approaches zero. Theorem 2 is established.

The proof and conclusion of Theorem 2 apply even if the region R overlaps itself, as in Figure 5 or Figure 7, provided each point of the neighborhood of O is covered but once. That is to say, the region R need not be smooth (schlicht),

if each point of the neighborhood of O is covered only once. Under these circumstances, the distances ρ, d_r, D_r from O to a point of R are as before merely the rectilinear distances, measured as if R were entirely in one plane.

The proof and conclusion of Theorem 2 remain valid even if the region R overlaps itself a finite number of times throughout the neighborhood of O in such a way that several sheets come together at O (that is, if the Riemann surface on which R lies has an algebraic branch point at O), provided *all* the sheets of the surface covering O are connected at O, as in Figure 8. But the proof of Theorem 2 breaks down if the neighborhood of O is covered more than once by sheets that are not connected at O, for in that case the function $G_0(x, y)$ defined by (8) is no longer harmonic in R except at a single point of R; the reader may notice that the conclusion also fails in this case.

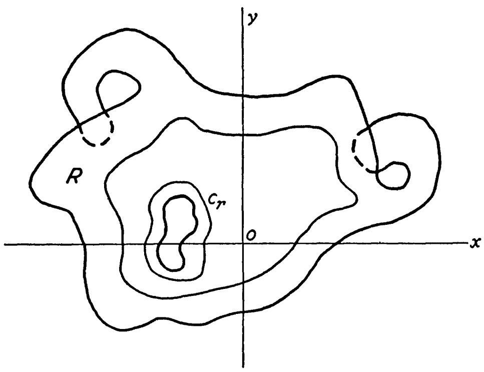


FIG. 7

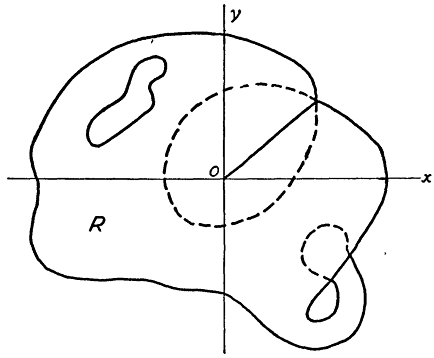


FIG. 8

Theorem 2 as formulated has the advantage over Theorem 1 of applying in the case of an arbitrary multiply connected region, provided merely that $G(x, y)$ exists. But Theorem 1 can also be extended to apply more generally to the mapping of a multiply connected region; let us assume $f(z)$ not necessarily single-valued but bounded and analytic except perhaps for algebraic branch points for $|z| < 1$, with $f(0) = 0$, $f'(0) \neq 0$, $f(z)$ bounded from zero except in the neighborhood of the origin, and let us assume $|f(z)|$ single valued for $|z| < 1$. The proof and conclusion hold without change.

Under the hypothesis of Theorem 1 fairly precise bounds can be obtained for the dependence of $K(C_r)$ on r. But here as in Theorem 1 itself the methods are again those of analytic functions. As a preliminary to the obtaining of these bounds, we need to make a detailed study of a certain conformal map.

4. *Mapping of an annulus on a circle.* Let A denote the annulus in the W-plane: $0 < m < |W| < M$, $m < 1 < M$; we shall map A conformally onto the interior of the unit circle in the Z-plane so that $W = 1$ corresponds to $Z = 0$. For the map to be possible and one-to-one, A must be interpreted as a simply-connected region; we therefore consider A as an infinite strip of width $M - m$

which winds in both senses from the line segment $m<W<M$ infinitely many times around the outside of the circle $|W|=m$.

The function

$$W_1 = \log \frac{W}{M}$$

maps the region A as just interpreted onto the strip*

$$\log \frac{m}{M} < \Re(W_1) < 0, \tag{12}$$

and the point $W=1$ corresponds to $W_1 = -\log M$.

The function

$$W_2 = \exp\left[\pi i\left(1 - \frac{W_1}{\log m - \log M}\right)\right] = \exp\left[\pi i \frac{\log m - \log W}{\log m - \log M}\right]$$

maps the strip (12) onto the upper half plane $\Im(W_2)>0$. The point $W_1 = -\log M$ corresponds to the point

$$W_2 = \exp\left[\pi i \frac{\log m}{\log m - \log M}\right] = e^{i\theta} = \alpha, \qquad 0 \leqq \theta \leqq \pi. \tag{13}$$

The line segment $\Im(W)=0$, $m<W<M$, corresponds to the segment $\Im(W_1)=0$, $\log(m/M)<W_1<0$, which corresponds to the semicircle S satisfying the conditions $|W_2|=1$, $0<-i \log W_2<\pi$.

The function

$$Z = \frac{W_2 - \alpha}{W_2 - \bar{\alpha}}$$

transforms the upper half plane $\Im(W_2)>0$ onto the interior of the circle $|Z|<1$ in such a way that $W_2=\alpha$ corresponds to $Z=0$. The semicircle S corresponds to the line segment L joining the two points

$$\beta = \frac{1-\alpha}{1-\bar{\alpha}} = -\alpha \quad \text{and} \quad -\beta.$$

The point $Z=\beta r$ $(0<r<1)$ on L corresponds to the point on S

$$W_2 = \frac{\bar{\alpha}Z - \alpha}{Z - 1} = \frac{1}{\alpha}\,\frac{r+\alpha}{r+\bar{\alpha}}. \tag{14}$$

* The notations $\Re[\Gamma]$ and $\Im[\Gamma]$ indicate respectively the real part of Γ and the quotient by i of the pure imaginary part of Γ.

If ω denotes the angle (argument, amplitude) of the quantity $r+\alpha$, we compute at once by (13)

$$\omega = \tan^{-1} \frac{\sin \theta}{r + \cos \theta},$$

whence equation (14) can be replaced by

$$\begin{aligned} W_2 &= \exp \left\{ i \left[2 \tan^{-1} \frac{\sin \theta}{r + \cos \theta} - \theta \right] \right\} \\ &= \exp \left\{ 2i \tan^{-1} \frac{(1 - r) \sin \theta}{(1 + r)(1 + \cos \theta)} \right\} = e^{i\delta}, \end{aligned} \tag{15}$$

where we set

$$\delta = 2 \tan^{-1} \frac{(1 - r) \sin \theta}{(1 + r)(1 + \cos \theta)}, \qquad 0 < \delta < \pi. \tag{16}$$

In a similar way, we find that the point $Z = -\beta r$ on L corresponds to the point on S

$$W_2 = e^{i\Delta}, \tag{17}$$

with the notation

$$\Delta = 2 \tan^{-1} \frac{(1 + r) \sin \theta}{(1 - r)(1 + \cos \theta)}, \qquad 0 < \Delta < \pi. \tag{18}$$

The circle $|Z| = r < 1$ cuts L orthogonally in the points βr and $-\beta r$, hence corresponds to a circle in the W_2-plane which cuts S orthogonally in the two points $e^{i\delta}$ and $e^{i\Delta}$. This circle in the W_2-plane therefore lies in the (closed) sector whose vertex is the origin and whose bounding rays make respective angles of δ and Δ with the positive direction of the axis of reals. This sector $\delta \leqq \Im(\log W_2) \leqq \Delta$ corresponds to the vertical strip

$$\left(1 - \frac{\delta}{\pi}\right) \log \frac{m}{M} \leqq \Re(W_1) \leqq \left(1 - \frac{\Delta}{\pi}\right) \log \frac{m}{M},$$

which corresponds to the annulus

$$m\left(\frac{M}{m}\right)^{\delta/\pi} \leqq |W| \leqq m\left(\frac{M}{m}\right)^{\Delta/\pi}. \tag{19}$$

Of course the entire closed region $|Z| \leqq r$ corresponds to a closed region in the W-plane which lies in the closed region (19).

5. *Application to the modulus of an analytic function.* The properties of the conformal map just analyzed are to be used in the proof of

THEOREM 3. *Let the function* $F(z)$ *be analytic for* $|z|<1$, *with* $F(0)=1$. *If we have*

$$0 < m \leqq |F(z)| \leqq M \quad for \quad |z| < 1,$$

then we have also $(r>0)$

$$m\left(\frac{M}{m}\right)^{\delta/\pi} \leqq |F(z)| \leqq m\left(\frac{M}{m}\right)^{\Delta/\pi} for \quad |z| \leqq r < 1, \tag{20}$$

where δ *and* Δ *are given by* (13), (16), (18).

Let the functions $W(Z)$ and $Z(W)$ map the Z-plane onto the W-plane in the manner already described (§4). Then the function $Z[F(z)]$ is analytic and of modulus less than unity for $|z|<1$, with $Z[F(0)]=0$. By Schwarz's Lemma, stated in §1, we have

$$|Z[F(z)]| \leqq r \quad \text{for} \quad |z| \leqq r < 1. \tag{21}$$

But every point of the region $|Z| \leqq r$ corresponds to a point of the region (19), so inequality (21) implies (19) for the values

$$W = W\{Z[F(z)]\} \equiv F(z),$$

and (20) is established.

The bounds that occur in (20) are actually taken on by an admissible function, so inequality (20) cannot be improved without restricting the class of admissible functions.

6. *Application to conformal mapping*. Theorem 3 is to be used in the proof of

THEOREM 4. *Let the function*

$$w = f(z) \equiv z + a_2z^2 + a_3z^3 + \cdots \tag{22}$$

be analytic for $|z|<1$, *be different from zero for* $0<|z|<1$, *and map* $|z|<1$ *onto a region* R *of the* w-*plane whose boundary is* C. *Let* $M<\infty$ *and* $m>0$ *be respectively the greatest and least distances from* O *to a point of* C.

Let the function (22) *map* $|z|<r<1$ *onto a region* R_r *whose boundary is* C_r, *and let* M_r *and* m_r *be respectively the greatest and least distances from* O *to a point of* C_r. *Then we have*

$$mr\left(\frac{M}{m}\right)^{\delta/\pi} \leqq m_r \leqq M_r \leqq mr\left(\frac{M}{m}\right)^{\Delta/\pi}, \tag{23}$$

where δ *and* Δ *are given by* (13), (16), (18).

The function $f(z)$ is not assumed univalent (schlicht) in the region $|z|<1$, but points of the neighborhood of the origin $w=0$ cannot be covered more than once in the map.

The function $F(z) \equiv f(z)/z$, when suitably defined for $z=0$, satisfies the hypothesis of Theorem 3. Inequalities (23) follow from (20).

Theorem 4 was established by a similar but somewhat different method by Koebe [11]. The present method is likewise similar to one that had been previously employed by Carathéodory [12]. The details are provided here for the purpose of deriving the inequalities (25) and (26) below.

7. *Inequalities for* $K(C_r)$. Under the hypothesis of Theorem 4 we find from (23) one of our main results:

$$K(C_r) = \frac{m_r}{M_r} \geqq \left(\frac{m}{M}\right)^{(\Delta-\delta)/\pi} \geqq \frac{m}{M}; \tag{24}$$

this first inequality is more precise than (7), and can be used to give a complete proof of Theorem 1. The first inequality in (24) can also be written

$$K(C_r) \geqq \left(\frac{m}{M}\right)^{\eta/\pi}, \qquad \eta = 2\tan^{-1}\frac{2r\sin\theta}{1-r^2}. \tag{25}$$

The strong inequalities are valid in (20) unless $Z[F(z)] \equiv z$ or $F(z) \equiv W(z)$; the strong inequalities are valid in (23), (24), and (25) unless $f(z) \equiv zW(z)$.

Inequality (25) is exact, in the sense that it cannot be improved for the totality of functions of the kind admitted in Theorem 4. But a simpler inequality is also of interest. We note the relations

$$\eta = \Delta - \delta = 2\tan^{-1}\frac{2r\sin\theta}{1-r^2} \leqq 2\tan^{-1}\frac{2r}{1-r^2} = 4\tan^{-1} r.$$

Consequently we may write from (25) our fundamental inequality

$$K(C_r) = \frac{m_r}{M_r} \geqq \left(\frac{m}{M}\right)^{(4/\pi)\tan^{-1} r} = [K(C_1)]^{(4/\pi)\tan^{-1} r}, \tag{26}$$

an inequality of relatively simple form, which does not involve explicitly $f'(0)$, and involves m and M only through their ratio $K(C_1)$. The quantities $K(C_r)$ are all invariant under a transformation of the form $w' = kw$, where $k \neq 0$ is constant. The situation of Theorem 1 $[f'(0) \neq 0]$ can be transformed by such a transformation into the situation of Theorem 4 $[f'(0) = 1]$, and therefore *inequality* (26) *is valid under the conditions of Theorem* 1. By (13), the relation $\sin\theta = 1$ is satisfied whenever we have $m = 1/M$, so (26) is exact in the sense that it cannot be improved for the class of functions which satisfy the hypothesis of Theorem 1.

Under the hypothesis of Theorem 4, it follows from a general inequality [11, p. 73] on the modulus of an analytic function that we have

$$\frac{M_r}{r} \leqq M^{2r/(1+r)}, \qquad \frac{r}{m_r} \leqq \frac{1}{m^{2r/(1+r)}}.$$

Thus we may write also

$$K(C_r) = \frac{m_r}{M_r} \geqq \left(\frac{m}{M}\right)^{2r/(1+r)} = [K(C_1)]^{2r/(1+r)},$$

an inequality only slightly less favorable than (26).

Let now an arbitrary function $w=f(z)$ analytic for $|z|<1$ with $f(0)=0$, $f'(0)=1$, map $|z|<1$ smoothly (schlicht) onto a limited or unlimited region of the w-plane; we have the well known inequalities [**2**, p. 78] for $|z| \leqq r<1$:

$$\frac{r}{(1+r)^2} \leqq |f(z)| \leqq \frac{r}{(1-r)^2},$$

so we may write

$$\frac{r}{(1+r)^2} \leqq m_r, \qquad M_r \leqq \frac{r}{(1-r)^2}, \qquad K(C_r) \geqq \left(\frac{1-r}{1+r}\right)^2,$$

an inequality which has the advantage of not involving $K(C_1)$. Indeed, under the present circumstances the point set C_1 need not separate the point $w=0$ from the point at infinity, so $K(C_1)$ need not be defined. This last inequality for $K(C_r)$ obviously remains valid if the assumption $f'(0)=1$ is omitted.

8. *Mapping of infinite regions.* If the region $|z|<1$ is mapped conformally onto an *infinite* region of the w-plane so that $z=0$ corresponds to $w=\infty$, results can be obtained similar to those already given. Indeed, *the circularity $K(C)$ of a point set C in the z-plane with respect to $z=0$ is invariant under the transformation $z'=1/z$.* For let C' denote the transformed set. The distance from O to the nearest point of C is the reciprocal of the distance from O to the farthest point of C', and the distance from O to the farthest point of C is the reciprocal of the distance from O to the nearest point of C'. Thus from (1) we have

$$K(C) = K(C').$$

For simplicity we formulate the result analogous to Theorem 1 and to (26) rather than (25):

THEOREM 5. *Let the function*

$$f(z) \equiv \frac{a}{z} + a_0 + a_1 z + a_2 z^2 + \cdots, \qquad a \neq 0,$$

be analytic for $0<|z|<1$ and bounded except in the neighborhood of the origin. Let C_r $(0<r\leqq 1)$ denote the boundary in the w-plane of the image of the region $|z|<r$ under the map $w=f(z)$. Suppose a neighborhood of the point $w=\gamma$ is left uncovered by the image of $|z|<1$. Then $K(C_r)$, the circularity of C_r with respect to $w=\gamma$, increases monotonically as r decreases:

$$K(C_{r_1}) \geqq K(C_{r_2}) \quad \text{if} \quad r_1 < r_2, \tag{3}$$

and $K(C_r)$ approaches unity as r approaches zero. The strong inequality holds in (3) *unless $z[f(z)-\gamma]$ is a constant. Inequality* (26) *is valid.*

We emphasize the fact that Theorem 5 involves circularity with respect to an *arbitrary* point $w=\gamma$, provided a neighborhood of γ is left uncovered by the image of $|z|<1$.

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EXPANSION OF CERTAIN LOGICAL FUNCTIONS

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In the theories of probability and choice we are confronted with questions of the following type. What is the probability that a card drawn from an ordinary deck will *not* be a heart? In how many ways can one draw a card from a deck of 52 *and* then draw another from the remaining 51? When a die has been thrown, what is the probability that its upper face is an ace *if* it is already known that this face is an ace *or* a deuce? Thus we form complex events from simple ones by means of the words "not," "and," "or," "if." These complex events are called logical functions of the simple events. The words "not," "and," "or," "if" may be regarded as logical operators by means of which the functions are formed. By defining logical operations in terms of the operations of ordinary algebra, we can obtain certain series expansions for logical functions. This method enables us to give simple derivations of some of the well known formulas in probability and choice and to derive new formulas.

1. *Logical functions.* The arguments of the logical functions will be infinite sequences

$$x_i = x_i^{(1)}, x_i^{(2)}, \cdots, x_i^{(k)}, \cdots$$

where $x_i^{(k)}$ is a number and $i=1, 2, 3, \cdots$. We shall define both algebraic and logical operations on these sequences and develop various relations* between

* These elementary relations have already appeared in the literature in the following articles: P. Delans, Comptes Rendus, vol. 195, 1932, p. 686; Hassler Whitney, A logical expansion in mathematics, Bulletin of the American Mathematical Society, 1932, pp. 572–579; and Characteristic functions in the algebra of logic, Annals of Mathematics, 1933, pp. 405–414; A. H. Copeland, Admissible numbers in the theory of probability, American Journal of Mathematics, 1928, pp. 536–552; Admissible numbers in the theory of geometrical probability, American Journal of Mathematics, 1931, pp. 153–162; and The theory of probability from the point of view of admissible numbers, Annals of Mathematical Statistics, 1932, pp. 143–156.

the operations. Let $f(u_1, u_2, \cdots u_n)$ be any function of n variables and let

$$f(x_1, x_2, \cdots, x_n) = f(x_1^{(1)}, x_2^{(1)}, \cdots, x_n^{(1)}), f(x_1^{(2)}, \cdots, x_n^{(2)}), \cdots, f(x_1^{(k)}, \cdots, x_n^{(k)}), \cdots.$$

Thus a function of sequences is itself a sequence. If x_i is such that its kth term $x_i^{(k)}$ is equal to 1 or 0 for every k, then x_i can be interpreted as a sequence of trials of an event. The event succeeds on its kth trial if $x_i^{(k)}=1$ and fails if $x_i^{(k)}=0$. If $x_1, x_2, \cdots x_n$ are sequences of this type, the product $x_1 \cdot x_2 \cdots x_n$ can be interpreted as the conjunction of these events. That is, the kth term of this product is $x_1^{(k)} \cdot x_2^{(k)} \cdots x_n^{(k)}$ which is equal to 1 provided all n of the events succeed on their kth trials and is equal to 0 if any one of the events fails.

A sequence is said to be a constant provided all of its terms are the same. Thus 1 may be regarded as the sequence $1, 1, 1, \cdots$. The kth term of the sequence $1-x_i$ is $1-x_i^{(k)}$. This term is equal to 1 when $x_i^{(k)}=0$ and equal to 0 when $x_i^{(k)}=1$. That is, the event $1-x_i$ succeeds when x_i fails and fails when x_i succeeds. We shall use the symbol $\sim x_i$ to represent the event "not x_i." Then

$$\sim x_i = 1 - x_i.$$

We shall let $x_i \vee x_j$ symbolize the event "x_i or x_j." Then by De Morgan's theorem

$$x_i \vee x_j = \sim (\sim x_i \cdot \sim x_j) = x_i + x_j - x_i \cdot x_j.$$

The expression $\sim(x_1 \vee x_2 \vee \cdots x_n) = \sim x_1 \cdot \sim x_2 \cdots \sim x_n$ means "none of the events $x_1, x_2, \cdots x_n$." Then

$$\sim (x_1 \vee x_2 \vee \cdots x_n) = (1 - x_1)(1 - x_2) \cdots (1 - x_n).$$

Let

$$\begin{aligned} f(z) &= (1 - zx_1)(1 - zx_2) \cdots (1 - zx_n), \\ S^{(\nu)} &= \sum_{k_1, k_2, \cdots, k_\nu = 1}^{n} x_{k_1} \cdot x_{k_2} \cdots x_{k_\nu}, \qquad \nu \leqq n, \\ &= 0, \qquad \nu > n, \end{aligned}$$

where each product $x_{k_1} \cdot x_{k_2} \cdots x_{k_\nu}$, for which the subscripts are all distinct, appears once and only once in the summation. We have the following symbolic expression for $f(z)$:

$$\begin{aligned} f(z) &= 1 - zS^{(1)} + z^2S^{(2)} - z^3S^{(3)} + \cdots \\ &= (1 + zS)^{-1}, \end{aligned} \tag{1}$$

where the νth power of S in the binomial expansion of $(1+zS)^{-1}$ symbolizes the summation $S^{(\nu)}$. Thus we may write

$$\sim (x_1 \vee x_2 \vee \cdots x_n) = f(1) = (1 + S)^{-1}.$$

The expression

$$x_1 \vee x_2 \vee \cdots \vee x_n = 1 - (1+S)^{-1} = S(1+S)^{-1}$$
$$= S^{(1)} - S^{(2)} + S^{(3)} \cdots$$

means "at least one of the events $x_1, x_2, \cdots x_n$." If these events are mutually exclusive, $x_i \cdot x_j = 0, 0, 0, \cdots$ whenever $i \neq j$. Then $S^{(2)} = S^{(3)} = \cdots = 0$ and $x_1 \vee x_2 \vee \cdots x_n = S^{(1)} = x_1 + x_2 + \cdots + x_n$. That is, for mutually exclusive events the signs $+$ and $\vee$ can be interchanged.

The negative derivative of $f(z)$ at $z=1$ can be interpreted as another logical function. Thus

$$\begin{aligned} -f'(1) &= S(1+S)^{-2} \\ &= x_1 \cdot (1-x_2) \cdot (1-x_3) \cdots (1-x_n) \\ &\quad + x_2 \cdot (1-x_1) \cdot (1-x_3) \cdots (1-x_n) \\ &\quad \cdots\cdots\cdots\cdots\cdots \\ &\quad + x_n \cdot (1-x_1) \cdot (1-x_2) \cdots (1-x_{n-1}). \end{aligned}$$

The expressions $x_1 \cdot (1-x_2) \cdot (1-x_3) \cdots (1-x_n)$, $x_2 \cdot (1-x_1) \cdot (1-x_3) \cdots (1-x_n)$, $\cdots$ are respectively the conjunctive events "x_1 and none of the events $x_2, x_3, \cdots x_n$," "x_2 and none of the events $x_1, x_3, \cdots x_n$," $\cdots$. Since these conjunctive events are mutually exclusive, we can replace the $+$ signs by $\vee$ signs. Hence $S(1+S)^{-2}$ means "exactly one of the events $x_1, x_2, \cdots x_n$." Similarly $f''(1)/2! = S^2(1+S)^{-3}$ means "exactly two of the events $x_1, x_2, \cdots x_n$" and in general $(-1)^r f^r(1)/r! = S^r(1+S)^{-r-1}$ means "exactly r of the events $x_1, x_2, \cdots x_n$." The expression

$$S^r(1+S)^{-r-1} + S^{r+1}(1+S)^{-r-2} + \cdots = S^r(1+S)^{-r}$$

means "r or $r+1$ or $r+2$ or $\cdots$ of the events $x_1, x_2, \cdots, x_n$," that is, "at least r of these events." Finally

$$1 - S^{r+1}(1+S)^{-r-1}$$

means "at most r of the events $x_1, x_2, \cdots x_n$."

2. *Probability*. We shall obtain formulas for the probabilities associated with these logical functions. We shall let $p_n(x)$ denote the average of the first n terms of the sequence $x = x^{(1)}, x^{(2)}, \cdots$. Thus

$$(2) \qquad p_n(x) = \sum_{k=1}^{n} x^{(k)}/n \qquad \text{or} \qquad np_n(x) = \sum_{k=1}^{n} x^{(k)}.$$

When x is an event sequence, the expression $np_n(x)$ is equal to the total number of 1's (or successes) in the first n trials of this event. Hence $p_n(x)$ is the success ratio for the first n trials of x. We shall let

$$(3) \qquad p(x) = \lim_{n \to \infty} p_n(x)$$

whenever this limit exists. When x is an event sequence, $p(x)$ is the probability associated with this sequence. It is easily seen that

$$p(x_i + x_j) = p(x_i) + p(x_j)$$

for any two sequences x_i and x_j for which $p(x_i)$ and $p(x_j)$ exist. Thus for example

$$\begin{aligned} p[(1+S)^{-1}] &= p(1) - p(S^1) + p(S^2) - p(S^3) + \cdots \\ &= 1 \quad - p(S^1) + p(S^2) - p(S^3) + \cdots , \end{aligned} \tag{4}$$

where

$$p(S^\nu) = \sum_{k_1, k_2, \cdots, k_\nu = 1}^{n} p(x_{k_1} \cdot x_{k_2} \cdots x_{k_\nu}).$$

In a similar manner we obtain formulas for the probabilities associated with the functions $S^r(1+S)^{-r-1}$, $S^r(1+S)^{-r}$, and $1-S^{r+1}(1+S)^{-r-1}$. The expressions thus obtained reduce to King's formulas for the case in which the events $x_1, x_2, \cdots x_n$ are independent. When these events are independent, we have the relation

$$p(x_{k_1} \cdot x_{k_2} \cdots x_{k_\nu}) = p(x_{k_1}) \cdot p(x_{k_2}) \cdots p(x_{k_\nu}).$$

We shall derive a formula for the conjunction $x_{k_1} \cdot x_{k_2} \cdots x_{k_\nu}$ which is applicable when these events are dependent. We shall let $x_i \subset x_j$ denote the event "x_i if x_j." We shall also interpret $x_i \subset x_j$ as a sequence. A given term of x_i shall belong to this sequence *if* the corresponding term of x_j is a 1 (i.e., a success). Let us suppose that m_n of the first n terms of x_i belong to the sequence $x_i \subset x_j$. Then the corresponding terms of x_j are all 1's and hence $m_n = np_n(x_j)$. A term of $x_i \subset x_j$ is equal to 1 if and only if the corresponding terms of x_i and x_j are both equal to 1. Hence the number of 1's in the first m_n terms of $x_i \subset x_j$ is equal to $np_n(x_i \cdot x_j)$. Therefore

$$p_{m_n}(x_i \subset x_j) = np_n(x_i \cdot x_j)/np_n(x_j)$$

or

$$p_n(x_j) \cdot p_{m_n}(x_i \subset x_j) = p_n(x_i \cdot x_j).$$

It follows that

$$p(x_j) \cdot p(x_i \subset x_j) = p(x_i \cdot x_j)$$

whenever $p(x_i)$ and $p(x_i \subset x_j)$ exist and $\lim_{n\to\infty} m_n = \infty$. By mathematical induction we obtain the relation

$$p(x_{k_1} \cdot x_{k_2} \cdots x_{k_\nu}) = p(x_{k_1}) p(x_{k_2} \subset x_{k_1}) p(x_{k_3} \subset x_{k_1} \cdot x_{k_2}) \cdots p(x_{k_\nu} \subset x_{k_1} \cdots x_{k_{\nu-1}}) \tag{5}$$

For most cases of conjunctive events this formula is easily applied.

3. *Choice.* We shall let $N(x)$ be the number of ways in which a given event x can occur. If an event x_j can occur in $N(x_j)$ ways and after this event has occurred, an event x_i can occur in $N(x_i \subset x_j)$ ways, then the number of ways in which both events can occur is given by the equation

$$N(x_j) \cdot N(x_i \subset x_j) = N(x_i \cdot x_j).$$

We have the following generalization of this formula

$$\begin{aligned}N(x_{k_1} \cdot x_{k_2} \cdots x_{k_\nu}) = {} & N(x_{k_1}) N(x_{k_2} \subset x_{k_1}) \\ & N(x_{k_3} \subset x_{k_1} \cdot x_{k_2}) \cdots N(x_{k_\nu} \subset x_{k_1} \cdots x_{k_{\nu-1}}).\end{aligned} \tag{6}$$

The theorem implied in this equation is fundamental to the theory of permutations and combinations. We have given the theorem a concise symbolic form instead of the usual verbal statement.

We can also obtain formulas in which the operator $N(x)$ is applied to the logical functions developed above. For example*

$$N[(1+S)^{-1}] = N(1) - N(S^1) + N(S^2) - N(S^3) + \cdots, \tag{7}$$

where

$$N(S^\nu) = \sum_{k_1, k_2, \cdots, k_\nu = 1}^{n} N(x_{k_1} \cdot x_{k_2} \cdots x_{k_\nu}).$$

In this equation $N(1)$ is the number of ways in which the events considered in a given problem can occur. These $N(1)$ possibilities must at least include all of the ways in which all the conjunctive events $x_{k_1} \cdot x_{k_2} \cdots x_{k_\nu}$ can occur. They may also include ways of occurrence of events which have no relation to the events $x_1, x_2, \cdots x_n$. Of the $N(1)$ ways considered, the number of ways in which none of the events $x_1, x_2, \cdots x_n$ occur is $N[(1+S)^{-1}]$. In order to derive this equation, we shall construct a probability situation in which all of the $N(1)$ possibilities have the same probability. Then $p(x_{k_1} \cdot x_{k_2} \cdots x_{k_\nu}) = N(x_{k_1} \cdot x_{k_2} \cdots x_{k_\nu})/N(1)$ and hence $p(S^\nu) = N(S^\nu)/N(1)$. Moreover $p[(1+S)^{-1}] = N[(1+S)^{-1}]/N(1)$. It follows from equation (4) that

$$N[(1+S)^{-1}]/N(1) = 1 - N(S^1)/N(1) + N(S^2)/N(1) - \cdots.$$

Multiplying both sides of the equation by $N(1)$ we obtain equation (7). In a similar manner formulas for $N[S^r(1+S)^{-r-1}]$, $N[S^r(1+S)^{-r}]$, and $N[1-S^{r+1}(1+S)^{-r-1}]$ can easily be obtained. In each case the operator $N(x)$ is distributive.

The example of the game of rencontre will perhaps clarify the symbolism. Suppose we have n cards numbered $1, 2, 3, \cdots n$. They are shuffled, then taken from the deck one at a time and counted. If the kth card counted is the card with the number k (i.e., the kth card is in its proper place), this constitutes a rencontre and will be called the event x_k. We shall consider both arrange-

* A special case of this formula has been obtained by Whitworth; see proposition 14 in his book, Choice and Chance. Also compare the two articles by Whitney previously referred to.

ments of cards and probabilities connected with this game. The number of arrangements for which the cards k_1, k_2, $\cdots$ k_ν are all in their proper places is equal to the number of ways in which the remaining $(n-\nu)$ cards can be arranged, i.e., $N(x_{k_1} \cdot x_{k_2} \cdots x_{k_\nu}) = (n-\nu)!$. Since the number of terms in the summation $N(S^\nu)$ is equal to the number of ways in which ν things can be selected from n, it follows that

$$N(S^\nu) = {}_nC_\nu \cdot (n - \nu)! = n!/\nu!.$$

The total possibilities considered will be the number of ways in which the deck can be arranged, i.e., $N(1) = n!$. If all arrangements of the deck are equally likely, $p(S^\nu) = 1/\nu!$. It follows that the probability that there will be no rencontre is

$$p[(1 + S)^{-1}] = 1 - 1/1! + 1/2! - \cdots + (-1)^n/n!.$$

This probability differs from e^{-1} by less than $1/(n+1)!$. The number of ways in which the deck can be so arranged that there will be no rencontre is obtained by multiplying the probability by $N(1)$, and hence $N[(1+S)^{-1}]$ is the nearest integer to $n!/e$.

The formulas which we have developed form a basis for practically all of the computational problems in the theories of probability and choice.

ON A DETERMINANT FUNCTION INVOLVING THE PARAMETER OF A PLANE CURVE*

By CLIFFORD BELL, University of California at Los Angeles

In a recent paper† the author made a study of the determinant function $F_i(t)$, where t is the parameter of a plane curve. The first, second and third columns of the determinant are, respectively, the functions representing the homogeneous coordinates of a point having the parameter t, their first derivatives and their ith derivatives with respect to the parameter. In this paper a similar function of the parameter is defined where non-homogeneous coördinates are used in the parametric representation of the curve. The properties of this function are developed and certain applications are discussed.

Let the equations of the plane curve be

$$(1) \qquad x = f(t), \qquad y = g(t),$$

and let $f_i(t)$ and $g_i(t)$ represent derivatives of the ith order with respect to t. A function $H_{ij}(t)$ is defined by the determinant $|f_i(t) \; g_i(t)|$. As an immediate consequence of the definition, it follows that $H_{ij}(t) \equiv 0$ for $i = j$, $H_{ij}(t) \equiv -H_{ji}(t)$, and $dH_{ij}(t)/dt \equiv H_{ij+1}(t) + H_{i+1j}(t)$.

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* Presented to the American Mathematical Society, November 30, 1935.

† Clifford Bell, On the properties of a determinant function, National Mathematics Magazine, vol. 10, 1936, pp. 171–174.

THEOREM 1. *The function, $H_{ij}(t)$, is invariant under the non-singular affine transformation, $x'=a_1x+b_1y+c_1$, $y'=a_2x+b_2y+c_2$.*

Proof. Under the transformation in question, the equations (1) become $x'=a_1f(t)+b_1g(t)+c_1$, $y'=a_2f(t)+b_2g(t)+c_2$, and it readily follows that $H'_{ij}(t)=\left|a_1f_i(t)+b_1g_i(t),\ a_2f_j(t)+b_2g_j(t)\right|$, where $H'_{ij}(t)$ is the transformed $H_{ij}(t)$. Making use of the multiplication rule for determinants, this reduces to $\left|a_1\ b_2\right|\cdot\left|f_i(t)\ g_j(t)\right|$. Since the transformation is non-singular the determinant $\left|a_1\ b_2\right|\neq 0$, and hence, as $H'_{ij}(t)=\left|a_1\ b_2\right|\cdot H_{ij}(t)$, the theorem is verified.

COROLLARY. *Under the rotation and translation transformations, $H_{ij}(t)$ is an absolute invariant.*

The question of the invariancy of $H_{ij}(t)$ under the general non-singular projective transformation naturally arises at this point. It may readily be shown, by the use of a particular case of such a transformation, that $H_{ij}(t)$ is then not invariant. However, projective theorems may still arise from this function as will be verified by the theorems which follow. A somewhat analogous condition exists in connection with d^2y/dx^2, where $y=f(x)$ is a plane curve. This derivative is not invariant under the general non-singular projective transformation, but it exhibits a projective property in that at an inflection point of the original curve d^2y/dx^2 vanishes, and at the corresponding point the second derivative of the transformed curve vanishes.

It will be assumed in the following work that to each ordinary point on the curve there corresponds one and only one value of the parameter, and that derivatives of $f(t)$ and $g(t)$ exist to the order indicated in each of the theorems.

THEOREM 2. *At every point (t_1) for which the tangent line has $(k+1)$-point contact with the curve* (1), $H_{ij}(t_1)=0$, $(i, j=1, 2, \cdots, k)$, and $H_{ik+1}(t_1)\neq 0$ *when i takes on at least one of the values* $1, 2, \cdots, k$.

Proof. The parameters of the points of intersection of a tangent line, $ax+by+c=0$, with the curve (1) are given by the roots of $af(t)+bg(t)+c=0$. This tangent line has at least $(k+1)$-point contact at (t_1) if*

$$af_i(t_1)+bg_i(t_1)=0, \qquad i=1, 2, \cdots, k. \tag{2}$$

To insure contact of no higher order the following inequality must hold,

$$af_{k+1}(t_1)+bg_{k+1}(t_1)\neq 0. \tag{3}$$

A condition that equations (2) hold simultaneously is that all second order determinants that can be formed from the matrix of the coefficients should vanish. These determinants are the quantities $H_{ij}(t_1)$, $(i, j=1, 2, \cdots, k)$. Furthermore the inequality (3) gives the information that at least one of the quantities $H_{ik+1}(t_1)\neq 0$, $(i=1, 2, \cdots, k)$, for if they were all zero the tangent line would have at least $(k+2)$-point contact with the curve.

* See Goursat-Hedrick, Mathematical Analysis, vol. 1, p. 447.

THEOREM 3. (*Converse of Theorem* 2.) *If* $H_{ij}(t_1)=0$, $(i, j=1, 2, \cdots, k)$ *and* $H_{ik+1}(t_1)\neq 0$ *when* i *takes on at least one of the values* $1, 2, \cdots, k$, *the tangent line has* $(k+1)$-*point contact with the curve at the point* (t_1).

THEOREM 4. *If the equations* (1) *represent a straight line,* $H_{ij}(t)\equiv 0$ *in* i, j, *and* t.

Proof. If equations (1) represent a line, let its non-parametric form be $Ax+By+C=0$. It follows that $Af(t)+Bg(t)+C\equiv 0$, and hence $Af_i(t)+Bg_i(t)\equiv 0$ in i and t. Therefore, as the derivatives of $f(t)$ and $g(t)$ of any given order are proportional to those of any other given order, $H_{ij}(t)\equiv 0$ in i, j, and t. A proof may also be derived from Theorem 2.

THEOREM 5. *If* $H_{12}(t)\equiv 0$, *equations* (1) *represent a straight line.*

Proof. If we take the derivative of $dy/dx\equiv g_1(t)/f_1(t)$, it becomes evident that $H_{12}(t)\equiv f_1^3(t)d^2y/dx^2$. If $H_{12}(t)\equiv 0$ two possibilities arise—either $f_1(t)\equiv 0$ or $d^2y/dx^2\equiv 0$. In either case equations (1) represent a straight line, and furthermore if the first identity holds the line is parallel to the Y-axis.

COROLLARY. *If* $H_{12}(t)\equiv 0$ *in* t, $H_{ij}(t)\equiv 0$ *in* i, j, and t.

At a multiple point of order r, the parameter takes on r values, some of which may be equal. For convenience in stating the following theorems, a multiple point at which two or more values of the parameter are equal is designated as being of the cuspidal type. A double point, for which the two values of the parameter are equal, is a special multiple point of this type in that it is the ordinary cusp.

THEOREM 6. *If the curve* (1) *has a multiple point of the cuspidal type at* (t_1), $H_{1j}(t_1)\equiv 0$ *in* j.

Proof. The derivatives $f_1(t)$, $g_1(t)$ vanish for $t=t_1$, if (t_1) is a multiple point of the cuspidal type. The determinant, $H_{1j}(t_1)$, therefore has a column of zeros and hence vanishes irrespective of the value of j.

THEOREM 7. *If* $f(t)$ *and* $g(t)$ *are analytic, the curve* (1) *has a multiple point of the cuspidal type at* (t_1), *provided that* $H_{1j}(t_1)\equiv 0$ *in* j *and* $H_{12}(t)\not\equiv 0$ *in* t.

Proof. Four cases arise:

(a) $f_1(t_1)\neq 0,\ g_1(t_1)\neq 0;$ (b) $f_1(t_1)=0,\ g_1(t_1)\neq 0;$

(c) $f_1(t_1)\neq 0,\ g_1(t_1)=0;$ (d) $f_1(t_1)=0,\ g_1(t_1)=0.$

In view of the condition that $f(t)$ and $g(t)$ are analytic, it follows that

$$x = f(t_1)+f_1(t_1)(t-t_1)+f_2(t_1)(t-t_1)^2/2!+\cdots, \tag{4}$$

$$y = g(t_1)+g_1(t_1)(t-t_1)+g_2(t_1)(t-t_1)^2/2!+\cdots. \tag{5}$$

In case (a), the equation $f_1(t_1)g_j(t_1)-g_1(t_1)f_j(t_1)=0$ gives $f_j(t_1)=g_j(t_1)f_1(t_1)/g_1(t_1)$, which substituted in (4) gives, after simplification,

$$[x-f(t_1)]g_1(t_1)/f_1(t_1) = g_1(t_1)(t-t_1)+g_2(t_1)(t-t_1)^2/2!+\cdots.$$

Substituting this in equation (5), we obtain the straight line

$$y - g(t_1) = [x - f(t_1)]g_1(t_1)/f_1(t_1).$$

In like manner it may be shown that cases (b) and (c) give straight lines. Hence cases (a), (b) and (c) are impossible under the condition that $H_{12}(t)$ does not vanish identically in t. Therefore the conditions of case (d) hold and curve (1) has a multiple point of the cuspidal type at (t_1).

It should be noted that the order of contact of the tangent line at a multiple point, such as mentioned in Theorem 7, may be determined by the use of Theorem 3. It may happen that $H_{ij}(t_1) \equiv 0$ in i and j, from which one might conclude that the tangent line has contact of infinite order at (t_1). The derivatives of $H_{12}(t)$ are linear combinations of the functions $H_{ij}(t)$ and hence if $H_{ij}(t_1) \equiv 0$ in i and j, $H_{12}(t)$ and all its derivatives vanish at (t_1). Therefore, if $f(t)$ and $g(t)$ are analytic, $H_{12}(t) \equiv 0$ and equations (1) represent a straight line; if non-analytic, the curve is either a straight line or a non-analytic curve for which the order of contact of the tangent line at (t_1) is infinite.

Theorem 3 may also be applied to the problem of determining the inflection and certain types of turning points of curve (1). Thus if k is even, the parameters of the inflection points are found among those common roots of $H_{ij}(t) = 0$, $(i, j = 1, 2, \cdots, k)$, which satisfy the condition, $H_{ik+1}(t) \neq 0$, when i takes on at least one of the values $1, 2, \cdots, k$. The curve has, of course, $(k+1)$-point contact with its tangent line at such points. If k is odd, those roots which satisfy the above condition and also make $g_1(t)$ vanish give turning points at which the tangent line has $(k+1)$-point contact.

In conclusion a word should be said about a previous supposition—namely that projective theorems may arise from the $H_{ij}(t)$ function. This has happened, as is readily seen from the above theorems, even though $H_{ij}(t)$ is not invariant under the general non-singular projective transformation. But it should also be noted that in each case the vanishing of $H_{ij}(t)$ for certain or all values of t was involved. Hence it must be concluded that under the general non-singular projective transformations, the transformed $H_{ij}(t)$ vanishes for the same values of t that make $H_{ij}(t)$ vanish.

"(In mathematics) we behold the conscious logical activity of the human mind in its purest and most perfect form. Here we learn to realize the laborious nature of the process, the great care with which it must proceed, the accuracy which is necessary to determine the exact extent of the general propositions arrived at, the difficulty of forming and comprehending abstract concepts; but here we learn also to place confidence in the certainty, scope and fruitfulness of such intellectual activity." H. L. F. von Helmholtz, Über das Verhältnis der Naturwissenschaften zur Gesammtheit der Wissenschaft, Vorträge und Reden, vol. 1, 1896, p. 176.

INTEGRATION OF CERTAIN SIMPLE STEP FUNCTIONS

By H. P. DOOLE, University of Nebraska

The integration of simple step functions leads to some rather interesting results and to some applications in the summing of certain infinite series. These simple functions have been defined by F. Riesz [**1**, p. 196]* as follows: When the interval (ab) is divided into a finite number of segments and to each segment a constant value (ordinate) is assigned, the function is called a simple step function. The integral of a simple step function is then defined to be the sum of the products of the segments on (ab) and the corresponding constant value of the function on each segment. Step function integration is thus a finite summation process and for convenience we will use the notation due to E. T. Bell [**2**, p. 138] for such functions. For the step functions themselves the usual notation will be used in which $[x]$ always represents the largest integer contained in x; for example, for $5 \leqq x < 6$, $[x]=5$.

To handle the step functions to be considered it will be necessary to divide the interval (ab) into equal segments, first of length unity, and later with length $1/n$. Certain trigonometric series which of course are periodic, but for which the ordinary summation formulas apply only to the first period, are summed by means of step function integrals into formulas that cover all periods. To the writer this appears to be one of the most interesting results obtained.

In the course of some of our integrations a new type of step function will appear in which instead of the function always remaining constant in each subinterval it will fit a separate triangle or a parabolic curve on each segment. The areas under these parabolic curves could be replaced by regular step function rectangles as the area under $y=x^m$ from $x=0$ to $x=1$ is $1/(m+1)$ times the area of the circumscribing rectangle.

1. The simplest step function is $y=[x]$ where x varies from zero to some integral value. Its integral is merely the sum of the integral values of x from $x=0$ to $x=[x]-1$;

$$\int_0^{[x]} [x] = \frac{[x]([x]-1)}{2}.$$

If the interval of integration does not end at an integer, an added term for the interval from $[x]$ to x is necessary; in this case we have

$$\int_0^{x} [x] = \tfrac{1}{2}[x]([x]-1) + (x-[x])[x]. \tag{1}$$

To make a second integration of this function, it will be necessary to evaluate the integrals of these two terms. The first term gives us a step function with ordinates $\frac{1}{2}([x]-1)[x]$, whose area is summed by a known formula [**3**, p. 52], thus,

* Bold faced numbers in square brackets refer to references listed at the end of the paper.

$$\int_0^x [x]([x]-1) = \frac{([x]-2)([x]-1)[x]}{3} + (x-[x])([x]-1)[x], \tag{2}$$

the last term arising from the interval from $[x]$ to x as before. The second term to be integrated is a new type of step function in which the steps are triangles as in Figure 1. On summing the areas of the triangles we find that

$$\int_0^x (x-[x])[x] = \tfrac{1}{2}\{\tfrac{1}{2}([x]-1)[x] + (x-[x])^2[x]\}. \tag{3}$$

Adding the results of the integration of the terms in (1) and rearranging terms,

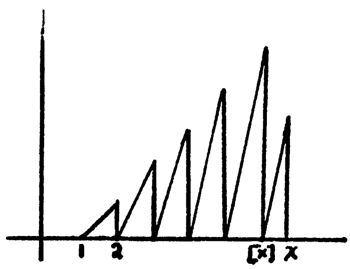

FIG. 1

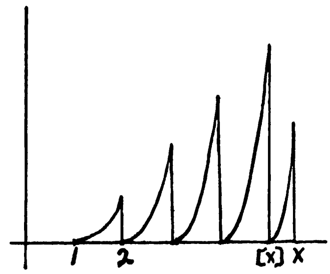

FIG. 2

we obtain

$$\begin{aligned}\int_0^x\int_0^x [x] &= \frac{([x]-2)([x]-1)[x]}{2\cdot 3} + \frac{([x]-1)[x]}{2\cdot 2} \\ &\quad + \frac{(x-[x])([x]-1)[x]}{2} + \frac{(x-[x])^2[x]}{2} \\ &= \frac{([x]-1)^3}{3!} + \frac{([x]-1)^2}{2\cdot 2!} + \frac{([x]-1)B_1}{2!} \\ &\quad + \frac{(x-[x])^2[x]}{2} + \frac{(x-[x])([x]-1)[x]}{2} \\ &= \frac{B_3([x])}{3!} + \frac{(x-[x])^2[x]}{2} + \frac{(x-[x])([x]-1)[x]}{2},\end{aligned} \tag{4}$$

where $B_n([x])$ represents a Bernoulli polynomial in $[x]$ and B_n represents a Bernoulli number [**4**, p. 304].

Further integration involves summing the step functions arising in the above terms. We will indicate the process and will list several of the more useful forms occurring in the general k-fold integral.

The integration of $(x-[x])^2[x]$ gives another example of the new step functions mentioned above and shown in Figure 2. Instead of the area of a rectangle

or of a triangle we now have the area under a parabolic curve in each interval. The result is

$$\begin{aligned}(5)\qquad \int_0^x (x-[x])^2[x] &= \sum_{[x]=0}^{[x]-1}\left([x]\int_0^1 z^2dz\right) + [x]\int_0^{x-[x]} z^2dz\\ &= \frac{1}{3}\left\{\frac{([x]-1)[x]}{2} + (x-[x])^3[x]\right\}.\end{aligned}$$

Other step function integrals follow:

$$\begin{aligned}\int_0^x [x]^p &= 1^p + 2^p + 3^p + \cdots + ([x]-1)^p + (x-[x])[x]^p\\ &= \frac{([x]-1)^{p+1}}{p+1} + \frac{([x]-1)^p}{2} + \frac{p([x]-1)^{p-1}B_1}{2!}\\ &\quad - \frac{p(p-1)(p-2)([x]-1)^{p-3}B_2}{4!} + \cdots + (x-[x])[x]^p\\ (6)\qquad & \qquad [\mathbf{5}, p.\ 138]\\ &= p!\left\{\frac{([x]-1)^{p+1}}{(p+1)!} + \frac{([x]-1)^p}{2p!} + \frac{([x]-1)^{p-1}B_1}{2!(p-1)!}\right.\\ &\quad \left. - \frac{([x]-1)^{p-3}B_2}{4!(p-3)!} + \cdots\right\} + (x-[x])[x]^p\\ &= \frac{B_{p+1}([x])}{p+1} + (-1)^{(p+1)/2}\left\{\frac{1+(-1)^{p+1}}{2}\right\}\frac{B_{(p+1)/2}}{p+1}\\ &\quad + (x-[x])[x]^p.\end{aligned}$$

$$\begin{aligned}\int_0^x (x-[x])[x]^p &= \tfrac{1}{2}\{1^p + 2^p + 3^p + \cdots + ([x]-1)^p + (x-[x])^2[x]^p\}\\ (7)\qquad &= \frac{1}{2}\left\{\frac{B_{p+1}([x])}{p+1} + (-1)^{(p+1)/2}\left[\frac{1+(-1)^{p+1}}{2}\right]\frac{B_{(p+1)/2}}{p+1}\right.\\ &\quad \left. + (x-[x])^2[x]^p\right\}.\end{aligned}$$

$$(8)\qquad \int_0^x (x-[x])^m[x] = \frac{1}{m+1}\left\{\frac{([x]-1)[x]}{2} + (x-[x])^{m-1}[x]\right\}.$$

$$\begin{aligned}\int_0^x (x-[x])^m[x]^p &= \frac{1}{m+1}\left\{\frac{B_{p+1}([x])}{p+1}\right.\\ (9)\qquad & + (-1)^{(p+1)/2}\left[\frac{1+(-1)^{p+1}}{2}\right]\frac{B_{(p+1)/2}}{p+1} + \left.(x-[x])^{m+1}[x]^p\right\}.\end{aligned}$$

$$\int_0^x (x-[x])[x]([x]-1)$$

$$(10)\qquad = \frac{1}{2}\left\{\frac{([x]-2)([x]-1)[x]}{3} + (x-[x])^2[x]([x]-1)\right\}$$

$$= \frac{([x]-1)^3}{3!} - \frac{([x]-1)}{3!} + \frac{(x-[x])^2[x]([x]-1)}{2}.$$

$$(11)\qquad \int_0^x ([x]-1) = -1 + \frac{([x]-2)([x]-1)}{2} + (x-[x])([x]-1).$$

$$\int_0^x ([x]-1)^p = (-1)^p + \frac{([x]-2)^{p+1}}{p+1} + \frac{([x]-2)^p}{2} + \cdots$$

$$+ (x-[x])([x]-1)^p$$

$$(12)\qquad = (-1)^p + \frac{B_{p+1}([x]-1)}{p+1}$$

$$+ (-1)^{(p+1)/2}\left[\frac{1+(-1)^{p+1}}{2}\right]\frac{B_{(p+1)/2}}{p+1}$$

$$+ (x-[x])([x]-1)^p.$$

It is now possible to proceed with the third and higher successive integrations of our step function $[x]$. Another integration of (4) gives

$$(13)\qquad \int_0^x\int_0^x\int_0^x [x] = \frac{B_4([x])}{4!} + \frac{B_2}{4!} + \frac{(x-[x])^3[x]}{3!} + \frac{(x-[x])^2[x]([x]-1)}{2\cdot 2!} + \frac{(x-[x])[x]([x]-1)(2[x]-1)}{2\cdot 3!}.$$

Finally a k-fold integration gives

$$(14)\qquad \int_0^x \cdots (k) \cdots \int_0^x [x] = \frac{B_{k+1}([x])}{(k+1)!} + (-1)^{(k+1)/2}\left[\frac{1+(-1)^{k+1}}{2}\right]\frac{B_{(k+1)/2}}{(k+1)!} + \sum_{s=1}^{k}\frac{(x-[x])^{k-s}}{(k-s)!}\frac{B_{s+1}([x])}{(s+1)!} + \frac{(x-[x])^k[x]}{k!}.$$

2. Certain types of non-periodic transcendental functions respond to this type of integration also. Thus the integration of $\log [x]$ gives

$$(15)\qquad \int_0^x \log [x] = \log ([x]-1)! + (x-[x]) \log [x],$$

and the integration of $e^{a[x]}$ gives

$$(16)\qquad \begin{aligned}\int_0^x e^{a[x]} &= 1 + e^a + e^{2a} + \cdots + e^{([x]-1)a} + (x - [x])e^{a[x]} \\ &= \frac{1 - e^{a[x]}}{1 - e^a} + (x - [x])e^{a[x]}.\end{aligned}$$

Also

$$(17)\qquad \begin{aligned}\int_0^x (x - [x])e^{[x]} &= \tfrac{1}{2}\{1 + e + e^2 + \cdots + e^{[x]-1} + e^{[x]}(x - [x])^2\} \\ &= \frac{1}{2}\left\{\frac{1 - e^{[x]}}{1 - e} + e^{[x]}(x - [x])^2\right\}.\end{aligned}$$

$$(18)\qquad \begin{aligned}\int_0^x [x]e^{[x]} &= e + 2e^2 + \cdots + ([x] - 1)e^{[x]-1} + [x]e^{[x]}(x - [x]) \\ &= \frac{e - [x]e^{[x]} + ([x] - 1)e^{[x]-1}}{(1 - e)^2} + [x]e^{[x]}(x - [x]). \quad [\mathbf{3}, p. 44]\end{aligned}$$

Other integrations may be performed similarly.

3. A more general case is considered where the interval (ab) is divided into equal divisions of length $1/n$ instead of unity. For the simple case of the integral of $[x]$ the ordinates of the successive steps will be $[nx]/n$. Several of the more useful integrals are the following:

$$(19)\qquad \int_0^x \frac{[nx]}{n} = \frac{[nx]([nx] - 1)}{2n^2} + \left(x - \frac{[nx]}{n}\right)\left(\frac{[nx]}{n}\right).$$

$$(20)\qquad \begin{aligned}\int_0^x \left(x - \frac{[nx]}{n}\right)^m \left(\frac{[nx]}{n}\right)^p &= \frac{1}{m+1}\,\frac{1}{n^{m+p+1}}\,\frac{1}{p+1}\Big\{B_{p+1}([nx]) \\ &+ (-1)^{(p+1)/2}\left[\frac{1 + (-1)^{p+1}}{2}\right]B_{(p+1)/2}\Big\} + \frac{([nx])^p}{n}\,\frac{(x - [nx]/n)^{m+1}}{m+1}.\end{aligned}$$

$$(21)\qquad \int_0^x \left(\frac{[nx]}{n} - 1\right) = \frac{1}{2}\left(\frac{[nx] - 1}{n} - 2\right)\left(\frac{[nx]}{n}\right) + \left(\frac{[nx]}{n} - 1\right)\left(x - \frac{[nx]}{n}\right).$$

4. The series

$$\sum_{m=1}^{\infty} \frac{\sin 2\pi m x}{m^k} \quad \text{and} \quad \sum_{m=1}^{\infty} \frac{\cos 2\pi m x}{m^k}$$

have been previously evaluated in terms of Bernoulli polynomials and numbers [**4**, p. 370; **6**, p. 317] but only for $0<x<1$. Bromwich [**4**, p. 355, $(1\cdot 2)$] gives the sum of the series

$$(22)\qquad \frac{1}{\pi}\sum_{m=1}^{\infty} \frac{\sin 2\pi m x}{m} = -(x - 1) - \tfrac{1}{2} + [x], \qquad 0 < x < \infty.$$

On integrating both sides of this equation we obtain

$$
(23)\qquad
\begin{aligned}
-\frac{1}{2\pi^2}\sum_{m=1}^{\infty}\frac{\cos 2\pi mx}{m^2} = &-\frac{(x-1)^2}{2!}-\frac{(x-1)}{2}-\frac{B_1}{2!}+\frac{([x]-1)^2}{2!}\\
&+\frac{([x]-1)}{2}+(x-[x])[x].
\end{aligned}
$$

Continued integration gives, if k is odd and $\geqq 3$,

$$
(24)\qquad
\begin{aligned}
\frac{(-1)^{(k-1)/2}}{2^{k-1}\pi^k}\sum_{m=1}^{\infty}\frac{\sin 2\pi mx}{m^k} = &-\frac{B_k(x)}{k!}-\frac{B_k([x])}{k!}\\
&+\sum_{m=1}^{k-2}\frac{(x-[x])^{k-m-1}}{(k-m-1)!}\frac{B_{m+1}([x])}{(m+1)!}+\frac{(x-[x])^{k-1}[x]}{(k-1)!}
\end{aligned}
$$

and if k is even and $\geqq 4$,

$$
(25)\qquad
\begin{aligned}
\frac{(-1)^{k/2}}{2^{k-1}\pi^k}\sum_{m=1}^{\infty}\frac{\cos 2\pi mx}{m^k} = &-\frac{B_k(x)}{k!}+\frac{B_k([x])}{k!}+\frac{(-1)^{k/2}B_{k/2}}{k!}\\
&+\sum_{m=1}^{k-2}\frac{(x-[x])^{k-m-1}}{(k-m-1)!}\frac{B_{m+1}([x])}{(m+1)!}+\frac{(x-[x])^{k-1}[x]}{(k-1)!}.
\end{aligned}
$$

Known formulas [6] for the summation of these series do not contain the step function terms which are necessary to make them valid beyond the first period. If a graph were made of the function

$$
\cos x+\frac{1}{2^2}\cos 2x+\frac{1}{3^2}\cos 3x+\cdots=\frac{1}{4}(x-\pi)^2-\frac{\pi^2}{12},
$$

it would be evident that the left side of this equation gives a periodic curve [4, p. 360, $(2\cdot 1)$] for all values of x while the right side of the equation is not periodic but gives a parabola that fits the graph of the left side of the equation only for $0\leqq x\leqq 2\pi$. The graph for both sides of (23) will be periodic and identical.

In addition to the references given, the writer has used L. B. W. Jolley's *Summation of Series* (Chapman and Hall, London, 1925) which consists entirely of series and their sums, over six hundred of them with references to their sources.

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AN ANALYTIC STUDY OF THE PASCAL HEXAGON

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Since most of the facts about the Pascal configuration have been arrived at by a repeated use of the Desargues theorem on perspective triangles, one wonders if an analytic treatment would not yield the same results much more readily, provided a way can be found by which we can quickly write down the equation of any particular Pascal line. In this paper we develop such a method, which seems to be new, for writing down at once the equation of the Pascal line corresponding to any desired ordering of the six points and then use this to obtain the 15 identities associated with six points of a conic. As another illustration of its application we prove Steiner's and Kirkman's theorems.

We shall use the symbol $\begin{pmatrix} a & e & c \\ d & b & f \end{pmatrix}$ to denote the hexagon* *abcdef*, whose diagonal points are (bc, ef), (ab, de), and (af, cd).

Suppose we desire the equation of the Pascal line arising from the six points as ordered above. Now systems of lines on the points (bc, ef), (ab, de), (af, cd) are $(cbx)+k_1(efx)=0$, $(abx)+k_2(dex)=0$, and $(afx)+k_3(cdx)=0$, respectively, where (cbx), for example, denotes the determinant of the homogeneous coordinates of the points c, b, and x. But since three points are on the same line all three of the systems must for the proper choice of the k's represent the same line. Hence we must have

$$(cbx) + k_1(efx) \equiv \rho[(abx) + k_2(dex)] \equiv \theta[(afx) + k_3(cdx)].$$

Setting successively $x=a, b, c, d, e, f$ in the identities above, we obtain six forms for this same Pascal line. However since the hexagon above is unaltered by all powers of the permutation $(abcdef)$, we could obtain one of the forms of the equation and then write the other five from it by use of the powers of this permutation. The value of k_2, and thus one form of the equation, is easily determined by setting in the identities above $x=e, f$. We have then as identical forms of the Pascal line for the hexagon *abcdef*

$$(abd)(afc)(bef)(dex) + (bde)(afd)(cef)(abx) = 0, \tag{1}$$

$$(acd)(abe)(bcf)(dex) + (cdf)(ade)(bce)(abx) = 0, \tag{2}$$

from either of which the remaining forms may be obtained by applying the permutation $(abcdef)$ and its powers.

From an observation of these results one may formulate a method of writing down at once the equation of any desired Pascal line. For example, (1) may be written in the symbolic form

$$\begin{pmatrix} c & & d & & x & & b & & c \\ & af & & ab & & de & & ef & \end{pmatrix} = 0.$$

* Friedrich Levi, Geometrische Konfigurationen, S. Hirzel, Leipzig, 1929, p. 178.

The method of formation may be illustrated by finding the equation of the Pascal line for the hexagon $\begin{pmatrix} a & e & c \\ b & d & f \end{pmatrix}$ in terms of the opposite sides ef and cd. We write first the lines with the variable x as $\begin{pmatrix} & x & \\ ef & & cd \end{pmatrix}$; then we write either of the vertices not already used on both ends of the bottom line, and the remaining vertex still not used at the extreme left and extreme right of the top line. We then have the form $\begin{pmatrix} a & & & x & & & a \\ & b & ef & & cd & b & \end{pmatrix}$. Now in the hexagon a is joined to d and f; so, in the top row we put d and f, d being on the side of ef, and f on the side of cd. Also in the hexagon ad is opposite be, and af is opposite bc; hence we attach to b in the lower line e on the left and c on the right. The complete equation is now

$$\begin{pmatrix} a & & d & & x & & f & & a \\ & be & & ef & & cd & & bc & \end{pmatrix} = 0.$$

Using the six forms of the same Pascal line, we obtain the 15 well known identities. For example, the two forms (1) and (2), since they represent the same line, require the coefficients of x_1, x_2, x_3 to be proportional; but this implies that the lines ab and de be coincident, or that

$$(adb)(ade)(bec)(bef)(cfd)(cfa) = (adc)(adf)(bea)(bed)(cfb)(cfe). \tag{3}$$

That there are 15 such identities is readily seen by noting that (3) is determined when the three pairs ad, be, cf are chosen. Or it might be noted that (3) belongs to a group of order 48 generated by the permutations (ad), (be), (cf), and $(abcdef)$, not all of which are independent.

Having these identities, we are able to prove Steiner's and Kirkman's theorems at once in one operation. For example, the Pascal lines of the hexagons $\begin{pmatrix} a & c & e \\ b & f & d \end{pmatrix}$, $\begin{pmatrix} a & c & e \\ d & b & f \end{pmatrix}$, $\begin{pmatrix} a & c & e \\ f & d & b \end{pmatrix}$ are respectively $\begin{pmatrix} e & & b & & x & & f & & e \\ & ad & & af & & bc & & cd & \end{pmatrix}$, $\begin{pmatrix} c & & d & & x & & f & & c \\ & ab & & af & & de & & be & \end{pmatrix}$, and $\begin{pmatrix} a & & d & & x & & b & & a \\ & cf & & bc & & de & & ef & \end{pmatrix}$, each equated to zero. Now the condition that these three equations be consistent is that the solutions of any pair satisfy the third; that is, if the variables are eliminated an identity must result, and conversely if an identity does result the three lines are concurrent. The elimination of the x's from the three equations is easily effected if half of each of the left members of the equations is transposed and the three right and left members multiplied in such a way that the product $(afx)(bcx)(dex)$ appears on both sides of the equation thus formed. If this be done, there results

$$(afd)(afe)(bcf)(bca)(deb)(dec) = (afb)(afc)(bce)(bcd)(dea)(def),$$

which is one of the 15 identities already established. These three Pascal lines

define the Steiner point $\begin{pmatrix} a & c & e \\ b & f & d \end{pmatrix} \cdot (bfd)$, which denotes the point of intersection of the Pascal line of the hexagon $\begin{pmatrix} a & c & e \\ b & f & d \end{pmatrix}$ with the Pascal lines of the hexagons formed from it by the powers of the permutation (bfd). It is clear then that the Steiner point $\begin{pmatrix} a & c & e \\ b & f & d \end{pmatrix} \cdot (ace)$ is identical with the Steiner point just written.

Similarly, eliminating x from the three Pascal lines of

$$\text{(S)} \qquad \begin{pmatrix} a & c & f \\ d & b & e \end{pmatrix}, \qquad \begin{pmatrix} a & e & d \\ f & b & c \end{pmatrix}, \qquad \begin{pmatrix} a & d & f \\ b & e & c \end{pmatrix},$$

we are led to a similar identity, and the three Pascal lines define the Kirkman point $\begin{pmatrix} a & c & f \\ d & b & e \end{pmatrix} \cdot (aec)(bfd)$, where the permutations apply as in the notation for the Steiner point. Thus any Pascal line has on it three Kirkman points. For example, the Pascal line of $\begin{pmatrix} a & c & f \\ d & b & e \end{pmatrix}$ has, in addition to the Kirkman point given above, the Kirkman points $\begin{pmatrix} a & c & f \\ d & b & e \end{pmatrix} \cdot (abf)(cde)$ and $\begin{pmatrix} a & c & f \\ d & b & e \end{pmatrix} \cdot (abe)(cdf)$. The permutations which give the additional pair of Pascal lines associated with any given Pascal line in the formation of a Kirkman point may be obtained by writing down any three vertices of the hexagon in order and the remaining vertices in reverse order. In this manner we obtained the three sets of permutations $(aec)(bfd)$, $(abf)(cde)$, and $(abe)(cdf)$ for the Pascal line of the hexagon $\begin{pmatrix} a & c & f \\ d & b & e \end{pmatrix}$ above.

That there are 60 such Kirkman points is immediately obvious from the discussion directly above or is seen by noting that the set of three hexagons (S) is unaltered by the dihedral group of order 12, generated by $(df)(ce)$ and $(afcbed)$.

By a continuation of this process one can show that the 20 Steiner points lie in 4's on 15 Steiner lines, each Steiner point being on 3 Steiner lines. That is, the Steiner points $\begin{pmatrix} a & e & c \\ d & f & b \end{pmatrix} \cdot (aec)$, $\begin{pmatrix} a & e & d \\ c & f & b \end{pmatrix} \cdot (aed)$, $\begin{pmatrix} a & b & c \\ d & f & e \end{pmatrix} \cdot (abc)$, and $\begin{pmatrix} a & b & d \\ c & f & e \end{pmatrix} \cdot (abd)$ are on a Steiner line which we shall denote by the symbol $\left[\begin{pmatrix} a & e & c \\ d & f & b \end{pmatrix} \cdot (aec)\right] \cdot (bdec)$. This Steiner line is easily seen to be the same if the permutation $(bdec)$ is replaced by either $(acfd)$ or $(aefb)$. To form this permutation of period four for the first Steiner point of the set above we may isolate any pair of vertices of the hexagon $\begin{pmatrix} a & e & c \\ d & f & b \end{pmatrix}$, one from each row, and then

write down a pair of opposite sides of the remaining quadrangle, taking care to order these in the same direction with respect to the hexagon and not to use those sides of the quadrangle which are already sides of the hexagon. Thus the existence of the three Steiner lines on any Steiner point is apparent.

Also the three Kirkman points $\begin{pmatrix} a & c & f \\ d & b & e \end{pmatrix} \cdot (aec)(bfd)$, $\begin{pmatrix} a & b & f \\ d & c & e \end{pmatrix} \cdot (eac)(bdf)$, and $\begin{pmatrix} a & b & e \\ d & c & f \end{pmatrix} \cdot (ace)(fdb)$ are points of a line, called the Salmon-Cayley line, for which we shall use the symbol $\left[\begin{pmatrix} a & c & f \\ d & b & e \end{pmatrix} \cdot (aec)(bfd)\right] \cdot (ace)(bfd)$. On this line lies also the Steiner point $\begin{pmatrix} a & c & e \\ b & f & d \end{pmatrix} \cdot (ace)$. From the formation of the permutations for the Salmon-Cayley line and its single Steiner point it is obvious that the 60 Kirkman points lie by 3's on 20 such lines and that the unique Steiner point for any Salmon-Cayley line may be written down merely by making rows in the representation of the Steiner point of the factors of the last permutation product.

In such a manner each of the properties of the Pascal figure can be shown to depend on one of the 15 identities, which is only the condition that the six points shall lie on a conic.

POHLKE'S THEOREM IN FOUR DIMENSIONS

By C. H. SISAM

Pohlke's theorem states that any three given line segments OP_1, OP_2, and OP_3, originating at O and lying in a plane π, are the parallel projections of three equal, mutually orthogonal line segments in space, $O^*P_1^*$, $O^*P_2^*$, and $O^*P_3^*$. This theorem, which is an important one in descriptive geometry, has been extended in various directions by a number of authors.† The following proof of an extension to four dimensions seems to be not without interest.

THEOREM: *Let OP_1, OP_2, OP_3, and OP_4 be any four given, real, non-infinite line segments, originating at O and lying in a three-space σ. There exist four equal, real, mutually orthogonal line segments $O^*P_1^*$, $O^*P_2^*$, $O^*P_3^*$, and $O^*P_4^*$, lying in a four-space that contains σ, from which the given segments may be obtained by a sequence of (at most) two parallel projections of which the first projects the equal orthogonal segments $O^*P_i^*$ on a three-space σ' and the second is orthogonal to σ'.*

It is the purpose of this note to prove the above theorem, to determine necessary and sufficient conditions that one parallel projection shall suffice, and to find conditions under which this one projection is orthogonal to σ.

† See, for example: Reye, Crelle's Journal, vol. 11, pp. 350–358; Schoute, Mehrdimensionale Geometrie, vol. 1, pp. 121–123; Emch, American Journal of Mathematics, 1918, vol. 40, pp. 366–374.

Let O be the origin of coördinates and σ the xyz-space. Let the coördinates of P_i be $(x_i, y_i, z_i, 0)$, $(i=1, 2, 3, 4)$.

Rotate the three coördinate axes that lie in σ (if necessary) in this space according to the equations

$$x' = l_1x + m_1y + n_1z, \qquad y' = l_2x + m_2y + n_2z, \qquad z' = l_3x + m_3y + n_3z, \tag{1}$$

(wherein l_i, m_i, n_i are the direction cosines of the new coördinate axes) in such a way that

$$\sum y_i'z_i' = 0, \quad \sum z_i'x_i' = 0, \quad \sum x_i'y_i' = 0, \quad \sum x_i'^2 \leqq \sum y_i'^2 \leqq \sum z_i'^2, \tag{2}$$

wherein $(x_i', y_i', z_i', 0)$ are the transformed coordinates of P_i.

That such a rotation exists is seen at once from the fact that it is precisely the rotation that transforms the equation of the quadric surface

$$(\sum x_i^2)x^2+(\sum y_i^2)y^2+(\sum z_i^2)z^2+2(\sum y_iz_i)yz+2(\sum z_ix_i)zx+2(\sum x_iy_i)xy=K,$$

into the equation

$$(\sum x_i'^2)x'^2 + (\sum y_i'^2)y'^2 + (\sum z_i'^2)z'^2 = K. \tag{3}$$

It further follows that $\sum x_i'^2$, $\sum y_i'^2$, and $\sum z_i'^2$ are the roots of the cubic equation in k

$$\begin{vmatrix} \sum x_i^2 - k & \sum x_iy_i & \sum z_ix_i \\ \sum x_iy_i & \sum y_i^2 - k & \sum y_iz_i \\ \sum z_ix_i & \sum y_iz_i & \sum z_i^2 - k \end{vmatrix} = 0. \tag{4}$$

I. If $\sum x_i'^2 = \sum y_i'^2 = \sum z_i'^2 = r^2$, we shall show that one orthogonal projection is adequate.

Let $O^*(0, 0, 0, w^*)$ be any given point on the w'-axis and consider the four points

$$P_i^*(x_i', y_i', z_i', w^* + rp_i), \qquad (i = 1, 2, 3, 4), \tag{5}$$

wherein p_1, p_2, p_3, and p_4 are four real numbers satisfying the equations

$$\sum p_ix_i' = 0, \qquad \sum p_iy_i' = 0, \qquad \sum p_iz_i' = 0, \qquad \sum p_i^2 = 1. \tag{6}$$

The segments OP_i are obviously the orthogonal projections of $O^*P_i^*$ on $w'=0$.

If $r=0$, the (real) points P_i obviously coincide with O and the points P_i^* coincide with O^*. The theorem follows at once if the directions of the (zero) segments $O^*P_i^*$ are thought of as any four mutually orthogonal directions through O^*.

If $r\neq 0$, let l_i, m_i, and n_i $(i=1, 2, 3, 4)$ be defined by the equations

$$x_i' = rl_i, \qquad y_i' = rm_i, \qquad z_i' = rn_i, \qquad (i = 1, 2, 3, 4). \tag{7}$$

From (2), (6), and (7), with the hypothesis of the present case, it follows at once that l_1, l_2, l_3, l_4; m_1, m_2, m_3, m_4; n_1, n_2, n_3, n_4; and p_1, p_2, p_3, p_4 are the direc-

tion cosines of four mutually perpendicular lines. Hence, also, from the fundamental theorem of the inverse of a rotation of axes, it follows that

$$l_i,\ m_i,\ n_i,\ p_i, \qquad (i = 1, 2, 3, 4),$$

are also the direction cosines of four such lines.

But these numbers are, from (5) and (7), the direction cosines of the four lines $O^*P_i^*$. Hence, these four lines are mutually orthogonal. Moreover, we have

$$O^*P_i^{*2} = x_i'^2 + y_i'^2 + z_i'^2 + rp_i^2 = r^2(l_i^2 + m_i^2 + n_i^2 + p_i^2) = r^2,$$

so that the four segments are also equal in length.

Since the rotation (1) may, in this case, be taken as the identical rotation, we have the theorem: *The sufficient conditions that the given segments OP_i are the orthographic projections of four equal, mutually orthogonal line segments originating at O^* are that*

$$\sum x_i^2 = \sum y_i^2 = \sum z_i^2, \qquad \sum y_i z_i = 0, \qquad \sum z_i x_i = 0, \qquad \sum x_i y_i = 0.$$

These conditions are also necessary. For, let l_i, m_i, n_i, p_i be the direction cosines of any four given, mutually perpendicular lines through O^* and let the lengths of the equal segments $O^*P_i^*$ on these lines be r. Then the coördinates of P_i^* are $(rl_i, rm_i, rn_i, w^*+rp_i)$ and the coördinates $(x_i, y_i, z_i, 0)$ of the projection of P_i^* on $w=0$ are

$$x_i = rl_i, \qquad y_i = rm_i, \qquad z_i = rn_i, \qquad w_i = 0, \qquad (i = 1, 2, 3, 4),$$

so that the above conditions are satisfied.

II. Let $\sum x_i'^2 = \sum y_i'^2 = r^2, \sum z_i'^2 > r^2$. Rotate the z' and w' axes, in their plane, through an angle θ such that $\cos^2\theta = r^2/\sum z_i'^2$. Let $P_i''(x_i'', y_i'', z_i'', 0)$ be the orthogonal projection of P_i on $w''=0$. We have

$$x_i'' = x_i', \qquad y_i'' = y_i', \qquad z_i'' = z_i' \cos\theta, \qquad w_i'' = 0, \qquad (i = 1, 2, 3, 4).$$

Since the coördinates of the points P_i'' satisfy the conditions of the theorem of Case I, the segments OP_i'' are the orthogonal projections on $w''=0$ of four equal, orthogonal segments $O^*P_i^*$, originating at a point O^* on the w''-axis. Moreover, the points P_i, P_i'', and P_i^* are collinear. Hence we have the theorem: *The sufficient condition that the four given segments OP_i are the parallel projections of four equal, mutually orthogonal segments originating at a point O^* is that the smaller two roots of the cubic equation* (4) *in k are equal. This condition is also necessary.* For, let there be given such a parallel (non-orthographic) projection. Let O be the origin, σ the xyz-space, and let the xy-plane be perpendicular to the generators of the projection. Let the coördinates of O^* be $(0, 0, a \sin\theta, a\cos\theta)$, (wherein $0<\theta<\pi/2$) and let the direction cosines of $O^*P_i^*$ be l_i, m_i, n_i, p_i. Then the coördinates of P_i^* are $(rl_i, rm_i, a\sin\theta + rn_i, a\cos\theta + rp_i)$ and those of P_i are

$$x_i = rl_i, \qquad y_i = rm_i, \qquad z_i = r(n_i - p_i \tan\theta), \qquad w_i = 0, \qquad (i = 1, 2, 3, 4).$$

Since the four lines $O^*P_i^*$ are mutually orthogonal, so also are the lines whose

direction cosines are l_1, l_2, l_3, l_4; m_1, m_2, m_3, m_4; n_1, n_2, n_3, n_4; and p_1, p_2, p_3, p_4. It follows that

$$\sum x_i^2 = \sum y_i^2 = r^2, \qquad \sum y_i z_i = 0, \qquad \sum z_i x_i = 0, \qquad \sum x_i y_i = 0,$$

and

$$\sum z_i^2 = r^2(1 + \tan^2\theta) > r^2.$$

We have supposed in the foregoing that $\theta < \pi/2$. If $\theta = \pi/2$, (then the segments being all supposed finite) $r = 0$, the points P_i^* all coincide with O^*, and O^*, O, and P_i all lie on the z-axis.

III. Let $r^2 = \sum x_i'^2 < \sum y_i'^2 < \sum z_i'^2$. We rotate the y' and w' axes in their plane through an angle θ such that $\cos^2\theta = r^2/\sum y_i'^2$. Then the coördinates of P_i become $(x_i'', y_i'', z_i'', w_i'')$, wherein

$$x_i'' = x_i', \quad y_i'' = y_i' \cos\theta, \quad z_i'' = z_i', \quad w_i'' = \pm\, y_i' \sin\theta, \quad (i = 1, 2, 3, 4).$$

Let P_i'' be the orthographic projection of P_i on $w'' = 0$. The coördinates of P_i'' are $(x'', y'', z'', 0)$ and we have

$$\sum x_i''^2 = \sum y_i''^2 = r^2, \sum z_i''^2 > r^2, \sum y_i'' z_i'' = 0, \sum z_i'' x_i'' = 0, \sum x_i'' y_i'' = 0.$$

By the preceding theorem, it follows that OP_1'', OP_2'', OP_3'', and OP_4'' are the parallel projections on $w'' = 0$ of four equal, mutually orthogonal segments $O^*P_1^*$, $O^*P_2^*$, $O^*P_3^*$, and $O^*P_4^*$. Hence the given segments OP_i may be obtained from these orthogonal segments $O^*P_i^*$ by a sequence of two parallel projections, first on $w'' = 0$ and then, by a projection perpendicular to $w'' = 0$, on the three-space σ.

By an immediate extension of the foregoing discussion, it is readily seen that a set of n segments OP_i lying in a space of $n-1$ dimensions may be obtained by a sequence of (at most) $n-2$ parallel projections from a set of n equal, orthogonal segments lying in a space of n dimensions.

NOTE ON POHLKE'S THEOREM

By ARNOLD EMCH, University of Illinois

Professor Sisam has given in the preceding paper an interesting proof of an extension of Pohlke's Theorem.* Some years ago, I showed† that Pohlke's original theorem, its known and new generalizations, and related theorems may be comprehensively proved by an application of affine collineations in a euclidean ordinary space of three dimensions, E_3. By the same principle of affine collineations in higher spaces E_n, $n > 3$, Pohlke's theorem may be proved for any E_n. A synthetic proof of Pohlke's theorem for any E_n may be found in Schoute's *Mehrdimensionale Geometrie*, vol. 1, pp. 121–123.

* A precise statement of Pohlke's theorem is: *Three straight line segments of arbitrary length in a plane, drawn from a point and making arbitrary angles with each other, form a parallel projection of three equal mutually perpendicular segments drawn from a point; however, only one of the original segments, or one of the angles can vanish.* See an article by Reye in Crelle's Journal, vol. 11, pp. 350–358.

† American Journal of Mathematics, vol. 40, 1918, pp. 366–374.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions, and Notes in the Monthly is open to all forms of activity in collegiate mathematics including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

REMARKS ON THE DEFINITION OF CONTINUITY

By GLENN JAMES, University of California at Los Angeles

The classic definition that a function is continuous over a given interval provided it is continuous at every point on the interval cannot, it seems to the writer, be applied to identify the sort of property we ordinarily attribute to a function when we say that it is continuous over a given interval. One certainly cannot test for a property over a continuous interval by examining individual points. What we actually do is to make use of some "general" procedure to study the function; and this "general" procedure interpolates into our definition a concept which is essentially uniform continuity over each one of some denumerable set of subintervals.

It would, it seems, be preferable to define uniform continuity first and base the definition of simple continuity upon it. A function would then be said to be *continuous over a given interval provided it is uniformly continuous over some set of subintervals which cover the given interval.* (Obviously such a definition does not imply uniform continuity over the entire interval unless the interval is closed.) More generally we would say that *a function is continuous over a given set of points provided it is uniformly continuous over each one of some selection of sets which contain as interior points all the points of the given set.*

On the pedagogical side, especially, there is a lot to be said in favor of this change in our definition of continuity. When a student takes up the study of continuity of functions he has usually learned in analytical geometry the limitation of point by point plotting, namely, that no matter how many points he plots he cannot be sure of how to draw the tiniest arc of a curve until he has some "general" knowledge of the function which it represents. The analogy between this situation and that which he meets when trying to apply the classic definition of continuity, is immediate. The fundamental difficulty in both cases is, of course, the impossibility of constructing a dense set by enumerating its elements. The student, having been introduced to this difficulty in a graphic way in his analytics, is prepared to give it explicit algebraic treatment when he takes up the consideration of continuity in his calculus. Furthermore, if both the definitions of continuity at a point and that of uniform continuity are expressed in the ϵ, δ form their similarity makes it easy for the student to pass from the former to the latter.

Note by the Editor: This very interesting discussion introduces a lively topic for debate, for the proposal seems certain to encounter opposition, but Professor James will find plenty of supporters for his position—at least in the opinion of the editor of this department. R.E.G.

$$I = \int_{x_1}^{x_2} F dx$$

would contain $2(n+m)$ parameters. However, owing to the absence of these derivatives the solutions contain only $2(n+m)-2m$ parameters. This leaves just $2n$ parameters which may be assigned to the y's for the initial conditions at P_1 and P_2. The initial values of the λ's will then not be assignable. This is not a hardship, for it is the y's in which one is primarily interested and not the λ's.

The choice $F=1f+\lambda_1\phi_1+\cdots+\lambda_m\phi_m$ is sufficient in order that $F=f$ along L. The Euler conditions thus become

$$F_{y_i} - \frac{dF_{y_i'}}{ax} = 0, \qquad F_{\lambda_\alpha} = 0,$$

and the last m equations are identical with the auxiliary differential equations $\phi_\alpha=0$. This line of argument is subject to the assumption that there will exist variations of the type necessary for the variation of the value of the integral I for the case where the multiplier of f in F is 1.

Such considerations might well explain the existence of the Euler-Lagrange Multiplier Rule as stated by Lagrange long before the rigorous demonstrations which have been more recently given by Bliss and Bolza.*

A Note on Wilson's Quotient

By Emma Lehmer, Bethlehem, Pennsylvania

In example 15, on page 318 in his *Theory of Numbers*, Matthews gives the congruences

$$(1) \qquad (p-1)!+1 \equiv pw_p \pmod{p^2}$$

for p a prime $\leqq 41$, and adds: "Can any rule be discovered for finding primes p such that

$$(2) \qquad (p-1)!+1 \equiv 0 \pmod{p^2}?"$$

In his table two solutions of (2) appear for $p=5$ and $p=13$.

Of course (1) implies that $(p-1)!+1\equiv 0 \pmod{p}$, if p is a prime, which is Wilson's theorem, and the number w_p may be called "Wilson's Quotient"; it is analogous to and connected with the well known Fermat's quotient q_a defined by

$$a^{p-1} - 1 \equiv pq_a \pmod{p^2}.$$

Wilson's quotient was considered by Glaisher [1] and Lerch [2] both of whom gave the fundamental congruence

$$(3) \qquad (p-1)! \equiv p(B_{p-1}-1) \pmod{p^2},$$

* An interesting historical note appears in Oskar Bolza's Vorlesungen über Variationsrechnung, pp. 566–569.

where B is a Bernoulli number defined symbolically by $(B+1)^n=B_n$ and $B_1=1/2$. In 1913 N. G. W. H. Beeger [3] gave an independent proof of (3) based on Euler's identity $\sum_{\nu=1}^{n}(-1)^\nu\binom{n}{\nu}\nu^n=(-1)^n n!$, and used (3) together with Adams' [4] table of Bernoulli numbers to show that $p=5$ and $p=13$ are the only solutions of (2) for $p\leqq 113$.

Since the table of Bernoulli numbers has been recently extended by D. H. Lehmer [5] to B_{220}, it seemed worth while to find B_{p-1} (mod p^2) for each $p\leqq 211$, and to calculate w_p of equation (1) in each case. It follows from the subjoined table of w_p that $p=5$ and $p=13$ are the only solutions of (2) for $p\leqq 211$.

A table of $w_p\equiv((p-1)!+1)/p$ (mod p) follows:

p	w_p	p	w_p	p	w_p	p	w_p	p	w_p
2	1	31	19	73	11	127	71	179	48
3	1	37	7	79	73	131	119	181	72
5	0	41	16	83	20	137	56	191	159
7	5	43	13	89	70	139	67	193	35
11	1	47	6	97	70	149	94	197	147
13	0	53	34	101	72	151	86	199	118
17	5	59	27	103	57	157	151	211	173
19	2	61	56	107	1	163	108		
23	8	67	12	109	30	167	21		
29	18	71	69	113	95	173	106		

References

1. Proceedings of the London Mathematical Society, vol. 32, 1900, p. 172.
2. Mathematische Annalen, vol. 60, 1905, pp. 471–490.
3. Messenger of Mathematics, vol. 43, 1913–14, pp. 83–84.
4. Journal für die reine und angewandte Mathematik, vol. 85, 1879, pp. 269–272.
5. Duke Mathematical Journal, vol. 2, 1936, pp. 460–464.

Some Unfamiliar Ordinals

By W. B. Campbell, Cornell University

Our usual *words* for 11, 12, 11th, 12th, are relics of a duodecimal number system, and are in a certain sense independent of the digits employed or their numerical representaton; the teen numbers, while based on the digit representation, follow a different law from that used in naming the larger numbers.

But some native sign-painter in Mandalay, the ancient capital of Burma, has apparently worked out in his own mind a self-consistent system for naming the ordinal numbers. In that city, one finds streets marked 11st, 12nd, 13rd. Presumably these are to be read as something like onety-first, onety-second, onety-third, analogous to twenty-first, etc. They conform to a scheme which presents fewer exceptions than the nomenclature in common use in the West. Still simpler would be a system in which the special ordinals, first, second, third, were abandoned, and we said and wrote oneth, twoth, threeth; 1th, 2th, 3th, 11th, 12th, 13th, 21th, 22th, 23th, etc., and the nth ordinal was always obtained by affixing th to the word corresponding to n.

RECENT PUBLICATIONS

EDITED BY W. R. LONGLEY, Yale University

All books for review should be sent directly to the editor of this department, at the American Mathematical Monthly, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Analytic Geometry. By J. W. Young, Tomlinson Fort, and F. M. Morgan. Cambridge, Mass., Houghton Mifflin Co., 1936, x+347 pages. $2.25.

The manuscript of this book was begun by the late Professor Young, and was completed after his death by Professor Fort and Dr. Morgan. Such authorship naturally ensures a well written, well constructed book of high mathematical calibre. While the treatment of certain topics is slightly more rigorous than is the case in the usual freshman textbook, the explanation is sufficiently detailed and there is a sufficient number of illustrative examples to make the book entirely suitable for the average freshman class. The exercises are numerous and seem to be well chosen. No answers are given but a separate answer book is available. Numerous well drawn figures help to make the book suitable for teaching purposes.

The text is designed to meet the requirements of the usual course in plane and solid analytic geometry, including a brief introduction to the differential calculus. The book is divided into sixteen chapters. An introductory chapter contains a review of quadratic equations, determinants, and trigonometry, and a discussion of directed line segments and projections. Then follow eight chapters on the usual topics of plane analytic geometry in rectangular coördinates, a chapter on polar coördinates, and a chapter on parametric equations. Three chapters are devoted to solid analytic geometry. The treatment is brief, but the essential parts of solid geometry are covered. The two concluding chapters are devoted to differential calculus, including applications to tangents, points of inflection, maxima and minima, and to velocities and rates.

As is natural in a first edition, there are a few misprints, none serious, and a few editorial lapses. Certain topics are omitted entirely. For example, no mention is made of oblique coördinates. Very few loci are considered except straight lines and conics. Some cubics occur in the exercises of chapter 3, but other curves occur only in the chapters on polar coördinates and parametric equations.

The reader is given an entirely incorrect impression of the power of the methods of analytic geometry in example 2, pages 31–2. This example should be corrected somewhat as follows: "From $a^2=b^2$, we conclude that either $a=b$ or $a=-b$. If $a=b$, the points P_1 and P_2 coincide and the three points are not the vertices of a triangle. Hence $a=-b$, $OP_1=-OP_2$, $P_1O=OP_2$, etc." The second sentence in the footnote on page 32 should be deleted, as no assumption is made, nor is it needed, that P_1 and P_2 are on opposite sides of the origin.

The angle of inclination of a line is so defined (page 25) that it is not true

that if two lines are parallel, their angles of inclination are equal (page 29); instead, they may differ by 180°. Also on page 29, the case where the lines have no slope is ignored under parallel lines, although it is considered when perpendicular lines are discussed. This entire section needs considerable editorial revision. Throughout the book, the case where a line has no slope is usually ignored, even in discussing topics where this case should be considered, e.g., diameters, page 208 ff.

Although it is implied, it is not stated explicitly that only points with real coördinates are to be considered in this text. The possible existence of points with complex coördinates is ignored until such points arise in plotting the locus of a given equation. It should be noted that the theorems on pages 208 and 210 are not correct unless imaginary points on the conic are considered. These theorems are not consistent with the conclusion of example 3, pages 203–4. Ignoring the existence of imaginary points also requires the polar of a point outside a conic to be defined as a chord segment (page 216), which limits the generality of exercise 3, page 217.

Determinants of orders 3 and 4 are expanded on page 3 according to the elements of the first column only. It would be better for the use that is made of them later (e.g., pages 60, 109, 268 ff.), to expand them according to the elements of the first row instead.

In certain places, terms are introduced without adequate definition. The footnote on page 36 is somewhat ambiguous. On page 49, "open branch curve" should be defined, as should "approaches coincidence with." The student on the basis of his previous mathematical experience can conceive of two congruent geometric figures approaching coincidence, but he may have difficulty in conceiving of a curved locus approaching coincidence with a straight line. Under calculus, all functions considered are tacitly assumed to be continuous. However, no definition of continuity is given, although it is mentioned on pages 326–9. On page 328, the second derivative is introduced without any previous definition, or explanation of the notation used.

At the top of page 126 occurs a combination of careless editing and proofreading. Obviously (5) cannot be equivalent to (1) unless $x+p=0$.

In proving that an equation is the equation of a given locus, the authors show that if the point is on the locus, its coördinates satisfy the equation, and are equally careful to show conversely that if the coördinates of a point satisfy the equation, the point is on the locus. The converse is ignored by most other authors. In certain cases, it might have been easier to have proved the converse by showing that if a point is not on the locus, its coördinates do not satisfy the equation. While this alternative method of proving the converse is the only one mentioned on page 36, it is not used in any of the illustrative examples.

As can be seen, the above criticisms are directed chiefly at features of minor importance and do not detract from the value of the book as a whole.

H. M. GEHMAN

The Mathematical Theory of Finance. By Kenneth P. Williams, New York, The Macmillan Co., 1935, 280 pages. $2.75.

The large number of books dealing with this subject that have appeared in this country in the last fifteen or twenty years points unmistakably to a heavy demand for a course in the subject in American colleges and universities. Insofar as they have come to the attention of the reviewer, they follow fairly closely the pattern of the book by Skinner, which seems to have been the first book of this sort published in this country.

The mathematical preparation required for the present text is not explicitly stated, but is apparently assumed to be algebra to and including the binomial theorem, the progressions, and logarithms, as there are brief treatments of these subjects in an Appendix. If an acquaintance with them could be assumed by an instructor, the major portion of the book, excluding probably the two final chapters on Life Annuities and Insurance, might be covered in a single semester, although the lists of problems are so ample and varied that an instructor preferring to proceed more leisurely might do so without difficulty.

As one would expect from an author of Professor Williams's standing, the book is very clearly written, and the publishers have presented it in a very attractive form. The tables at the end contain, in addition to the usual interest functions, logarithms of numbers to six places and both the American Experience and the American Annuitants' Tables of Mortality.

Perhaps the chief criticism that may be offered to books of this sort is the meagre amount of attention given to underlying mathematical principles. It is the feeling of the reviewer that a discussion of the general concept of functions together with the associated ideas of graphical representation and of interpolation would have been of much advantage. These would illumine particularly the type of problem in which the rate of interest is the unknown. There is considerable to be said also in favor of the inclusion of a general discussion of probability as a preliminary to the treatments of life annuities and insurance in spite of the fact that the author has handled these subjects in such a way that it is unlikely that a student would feel that anything is missing.

The book seems pretty free from errors and statements to which it is possible to take exception, although a few of these did come to the reviewer's attention. In view of the fact that the process of dividing 1 by $1 \pm x$ provides the successive remainders, it is somewhat surprising to read (p. 188) that these developments "can be found directly by division, but the result is purely formal, that is, its actual validity is not thereby established." One can easily disagree with the statement (p. 29) that "the proof can be found in any text on calculus," the reference being to the establishment of the limit defining the Naperian base.

Example 2 (p. 62) is incorrectly stated and solved, as a present payment of $3500 and ten future payments of $500 each are obviously not equivalent to a present payment of $10,000 at any rate of interest. In several places on pp. 99–101 the bond interest rate per period should read r/p instead of r/j.

H. H. Mitchell

Introductory College Mathematics. By F. E. Johnston. New York, Farrar and Rinehart, 1936, 270 pages of text, 44 pages of tables, index and a complete set of answers. $2.60.

This text treats the topics ordinarily included in the courses of plane trigonometry, college algebra and analytic geometry and briefly develops the concept of the derivative in considering the slope of the curve. It is designed for a four-hour or a five-hour year course or, with the omission of certain sections, for a three-hour year course.

The first part of the book develops the study of trigonometry in a highly satisfactory manner. In the treatment of logarithms, however, there are evidences of mechanization of rules in such form that the student's understanding of the meaning of logarithms truly suffers. Some of these rules, furthermore, are mathematically unsatisfactory. The chapter on the straight line provides an excellent introduction to the spirit of analytic geometry, for the pupil cannot fail to appreciate the union of algebra and geometry.

In the chapter on conic sections the various curves are considered individually, each with its special definition. Then the results are summarized by giving one definition for the general conic section. The author makes it clear to the student that degenerate conics are considered as conics by convention for the sake of generality. The discussion of each conic is presented with both rectangular coördinates and polar coördinates.

The chapter on tangents, normals, and diameters gives the student fundamental concepts and methods of the differential calculus which will be of direct use in his more advanced mathematical studies. He will not have need to "unlearn" his elementary ideas at a later stage for his future study will come from a direct extension of the elementary fundamental principles developed here.

In the presentation of the general theory of determinants, the accuracy of the statements will be appreciated although few proofs are supplied. The transition to the general method of expanding a determinant is simplified by the fact that only the "expansion by minors" was presented in the earlier treatment of determinants of the third order. Every opportunity is grasped for the application of determinants in the study of analytic geometry.

The illustrative problems and the numerous exercises appear quite satisfactory from the viewpoint of a classroom teacher. Included among the exercises are several important formulas and theorems. The physical construction of the book both in details and as a whole is excellent. The text should not be criticized strongly for its lack of direct motivation or for its failure to take advantage of the numerous opportunities to improve explicitly the student's functional thinking since the classroom teacher can supply these phases of the instruction more efficaciously than can a textbook.

Considering the book as a whole, both pupil and teacher will be thoroughly impressed by the author's success in producing an elementary text in which the presentation is simple, direct, unambiguous, and accurate.

JACK WOLFE

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All reports of club activities should be sent to F. W. Owens, 462 East Foster Ave., State College, Pennsylvania.

LOAN LIBRARY OF STUNTS

As outlined in the letter sent to all clubs on our lists in December, the Loan Library of Stunts is ready for active service to undergraduate mathematics clubs. The material on hand is, as yet, limited but includes games, cross word puzzles, songs, missing word stories, and a few unclassified stunts, all with a mathematical slant. It is sincerely to be hoped that any person or club having material or suggestions will send them to this department, in order that this new venture may be of real value.

Those wishing the use of any item from the collection should write to this department, enclosing return postage and pledging the return of manuscripts within two weeks of their receipt.

A CORRECTION

On page 102 in the February number of this MONTHLY exponents were omitted in the numerators of the formulas for $2K$ and $6V$; the determinants should have exponents 2 and 3 respectively.

NATIONAL CONVENTION OF KAPPA MU EPSILON

The Third National Convention of Kappa Mu Epsilon will be held at State College, Mississippi, April 30 and May 1, 1937. Details of these meetings can be secured from the President of the organization, Professor J. A. G. Shirk, Kansas State Teachers College, Pittsburg, Kansas.

CLUB REPORTS 1935–1936

Correction. In the report of the Mathematics Club of the Cooper Union Institute of Technology in the December, 1936, issue of this department, credit for a paper on "π and probability" was given to C. Pinzka. The paper was the work of H. Turbin, with C. Pinzka acting as calculator.

Kappa Mu Epsilon, Northeastern Teachers College, Oklahoma

President, P. Bohart; Vice-President, A. Rose Machesney; Secretary, Lorene Williams; Treasurer, W. H. Ward.

The club held two picnics each semester and studied constellations after each. At the bimonthly meetings the topics discussed concerned history of mathematics and mathematicians, important review topics, and methods of teaching mathematics. This year the club is collecting material for a directory of all past and present members of the club, hoping thereby to foster the friendly relations among students of mathematics.

Mathematics Club, Creighton University

President, J. Martin; Vice-President, J. Croft; Secretary-Treasurer, Mary Caroline Kull. Meetings were held twice a month. At least once a month an historical or biographical paper was presented. Two meetings had guest speakers. All other speakers were members of the club. Topics discussed were: Diophantine equations; Origin and development of logarithms; Mathematical fallacies; An original method of calculating π by chance. The Rigge compound harmonic motion machine was demonstrated and a tour was conducted through the observatory. The year closed with a picnic.

Mathematics Club, Oklahoma Agricultural and Mechanical College

President, S. Spargo; Vice-President, D. Bowers; Secretary-Treasurer, Margaret Messler; Faculty Adviser, Dr. R. Allen. The first year's work of this group was based on material found in this MONTHLY but one evening was devoted to astronomy. The club got to an early start for the current year, hoping for an even more successful program.

Pi Mu Epsilon, University of Pennsylvania

Director, Professor S. P. Shugert; Treasurer, A. Milgram; Secretary, Lillian Pechero. Monthly meetings were held with talks on: Arithmetic recreations; Squaring the circle; Continued fractions; Hypergeometric series; Stirling's approximation formula; Non-euclidean geometry; and Abstract space. A paper by D. L. Herr on "The differential analyzer" was voted the best of the year's papers by undergraduates. Entered in a contest in the district convention of student branches of the American Institute of Electrical Engineers, this paper was awarded a first prize.

Kappa Mu Epsilon, Mississippi State College

President, R. L. Wilson; Vice-President, F. B. Wylie, Jr.; Recording Secretary, H. W. Webster; Treasurer, M. Y. Mullen; Corresponding Secretary, C. R. Stark. The club held luncheon meetings with talks on mathematical topics. A joint meeting with the chapter at Mississippi State Teachers College for Women was held at State College. A unique picnic began the year's activities. It might have been called the Fives Picnic. Beginning at 5 o'clock, the participants went in 5 cars to 5 Oaks, had a 5 point star campfire, 5 refreshments, listened to 5 talks by officers and gave a vote of thanks to the 5 sponsors for a 5 way successful evening. The annual mimeographed report is a booklet entitled "Magnolia." It includes detailed programs, lists of officers and members, abstracts of principal papers of the year, and accounts of honors won by chapter members.

The Mathematics Club, Rutgers University

President, J. B. McIlroy; Vice-President, E. A. Darby; Secretary-Treasurer, P. F. Stryker; Faculty Advisor, Professor E. P. Starke. A spring banquet concluded a year of interesting programs. The following topics were discussed: Generalizations in mathematics; The photo-electric number sieve; Probability; and Mathematical series. They were supplemented by solutions of problems appearing in this MONTHLY, by mathematical puzzles and original problems. Three meetings were devoted to the lives and works of former Rutgers mathematicians, G. W. Hill, T. Strong, and M. Bowser.

The Mathematics Club, University of Rochester

President, E. O. Stephany; Secretary-Treasurer, L. E. Moss; Advisors, Dr. G. Baley, Dr. J. J Gergen. The club, enjoying its first year of regular organization after two years of informal work, held one supper meeting and one May outing besides several regular meetings with papers by students and faculty members. A first prize of $10 and a second prize of $5 were offered for the best student papers presented at a meeting. This year the prizes were combined and divided equally among three students, E. O. Stephany, J. Goldberg, and P. Froeschle. Their respective papers were entitled: Diophantine analysis; Fundamentals in n-dimensional space; Perfect numbers.

Kappa Mu Epsilon, University of New Mexico

President, L. Koch; Vice-President, L. Medveson; Secretary, W. Biddle; Treasurer, Professor E. F. Smellie; Corresponding Secretary, Professor C. A. Barnhart; Sponsor, Professor C. V. Newsom. The chapter enjoyed frequent afternoon teas and two banquets. Topics discussed included: Magic squares; Vector analysis; Hyperbolic functions and their applications; Heaviside calculus. The club also sponsored a lecture by Professor F. W. Sparks of Texas Technological College on "Some interesting problems in number theory."

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 271. *Proposed by V. Thébault, Le Mans, France.*

Find the base B of a system of notation in which there exists a number of the form, $abba$, which is the square of the number, bb. Show that $b^2+B=aa$.

E 272. *Proposed by W. B. Campbell, Ithaca, New York.*

A certain railroad has n stations along the main line. How many different printed forms must be provided to take care of all possible one-way journeys, including provision by name for any or all stopovers which might be desired?

E 273. *Proposed by W. B. Clarke, San Jose, California.*

Last year three brothers discovered that if the age of each were deducted from the sum of the ages of the other two, and the three numbers so obtained were multiplied together, the result would equal sixteen times their combined ages. They went to tell their three sisters about this, but could only find Ida, who was the youngest of the six. She made a rapid calculation and announced that the same thing was true of the ages of herself and two sisters. In what years were the different boys and girls born?

E 274. *Proposed by R. P. Agnew, Cornell University.*

Let r be any positive rational number and n any positive integer. From r subtract the first term of the harmonic series, $1/n,\ 1/(n+1),\ 1/(n+2), \cdots$ which is $\leqq r$. From that difference deduct the next following term which is $\leqq$ that difference. Prove that a continuation of this process must produce a difference of zero in a finite number of subtractions.

E 275. *Proposed by J. A. Benner, Lafayette College, Easton, Pa.*

In a certain town it began snowing before noon and continued at a constant rate until dark. At noon a crew of men set out along the highway, clearing the snow from it as they went. They cleared two miles in the first two hours, but only one mile in the next two hours. If the crew clears equal volumes of snow in equal times, at what time did it begin to snow?

E 276. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

The inside dimensions of a rectangular box with a lid on it, are three feet, four feet, and five feet. A post in the form of a right circular cylinder nine inches in diameter just fits diagonally in the box, touching all six inner faces. How long is the post? (Note that the axis of the post, if prolonged, would miss the corners of the box.)

SOLUTIONS

E 229 [1936, 432]. *Proposed by N. A. Court, University of Oklahoma.*

Given a sphere S and three fixed points A, B, and C; determine a point M on S such that the three given points and the points in which MA, MB, and MC meet S (other than at M), may all six lie on one sphere.

Solution by L. M. Kelly, Lawrence, Massachusetts.

If A', B', and C' are the points in which MA, MB, and MC meet the sphere S, we have only to note that the plane of A', B', and C' is antiparallel to the tangent plane at M. Now if the tangent plane at M is parallel to the plane of A, B, and C, then the plane of A', B', and C' is antiparallel to the plane of A, B, and C.

Now the plane of A', B', and C' intersects the sphere in a circle, and this circle is also a section of the oblique circular cone with M as its vertex and the circle through A, B, and C as its base. [N. A. Court's *Modern Pure Solid Geometry*, pp. 226–7.] Hence the two circular sections of the cone will be cospherical.

We therefore need only to locate the point M on the sphere so that the plane tangent to the sphere at M is parallel to the plane of A, B, and C, and this may be done by dropping a perpendicular from the center of S onto the plane of A, B, and C, which cuts S at the two possible locations for M. Consequently, there are always two solutions.

Note by the proposer. The corresponding problem in the plane was considered in the *Bulletin des sciences mathématiques et physiques élémentaires*, vol. 10, p. 72 (1904–5).

Also solved by Simon Vatriquant.

E 230 [1936, 432]. *Proposed by Meyer Karlin, Yeshiva College, New York City.*

Prove that the sum of the series:

$$S_n = C_0^n - C_1^{n-1} + C_2^{n-2} - C_3^{n-3} + \cdots$$

will be $+1$, 0 or -1 according as n is a positive integer of the forms $6m$ or $6m+1$, $6m-1$ or $6m+2$, $6m+3$ or $6m+4$ respectively.

Solution by A. V. Richardson, Bishop's College, Quebec.

If ω and ω^2 are the complex cube roots of unity, then $1+x+x^2 = (1-\omega x)(1-\omega^2 x)$. Therefore, using partial fractions,

$$(1+x+x^2)^{-1} = \frac{1}{(1-\omega)(1-\omega x)} - \frac{\omega}{(1-\omega)(1-\omega^2 x)}.$$

The left side of this equation equals $\sum_{r=1}^{\infty}(-1)^r(x+x^2)^r$, in which the coefficient of x^n is $(-1)^n[C_0^n - C_1^{n-1} + \cdots] = (-1)^n S_n$. Consequently, $(-1)^n S_n$ will equal the coefficient of x^n on the right side. But this coefficient must be $\omega^n/(1-\omega) - \omega^{2n+1}/(1-\omega)$, or $\omega^n(1-\omega^{n+1})/(1-\omega)$. Now we know that $\omega^{3r}=1$. Hence we may write

if $n=6m$, $S_{6m}=\omega^{6m}(1-\omega)/(1-\omega)=+1$,
if $n=6m+1$, $-S_{6m+1}=\omega(1-\omega^2)/(1-\omega)=\omega(1+\omega)=-1$,
if $n=6m+2$, $S_{6m+2}=\omega^2(1-\omega^3)/(1-\omega)=0$,
if $n=6m+3$, $-S_{6m+3}=\omega^3(1-\omega)/(1-\omega)=+1$,
if $n=6m+4$, $S_{6m+4}=\omega(1-\omega^2)/(1-\omega)=-1$, and
if $n=6m+5$, $-S_{6m+5}=\omega^2(1-\omega^3)/(1-\omega)=0$.

Also solved by E. P. Starke, Herbert Tate, Simon Vatriquant, and the proposer.

E 231 [1936, 432]. *Proposed by A. A. Bennett, Brown University.*

In a certain bank there were eleven distinct positions; namely, in decreasing rank, President, First Vice-President, Second Vice-President, Third Vice-President, Cashier, Teller, Assistant Teller, Bookkeeper, First Stenographer, Second Stenographer, and Janitor. These eleven positions are occupied by the following, here listed alphabetically, Mr. Adams, Mrs. Brown, Mr. Camp, Miss Dale, Mr. Evans, Mrs. Ford, Mr. Grant, Miss Hill, Mr. Jones, Mrs. Kane, Mr. Long. Concerning them the following facts only are known:

1. The Third Vice-President is the pampered grandson of the president, but is disliked by both Mrs. Brown and the Assistant Teller.

2. The Assistant Teller and the Second Stenographer shared equally in their father's estate.

3. The Second Vice-President and the Assistant Teller wear the same style of hats.

4. Mr. Grant told Miss Hill to send him a stenographer at once.

5. The President's nearest neighbors are Mrs. Kane, Mr. Grant, and Mr. Long.

6. The First Vice-President and the Cashier live at the exclusive Bachelors' Club.

7. The janitor has occupied the same garret room since boyhood.

8. Mr. Adams and the Second Stenographer are leaders in the social life of the younger unmarried set.

9. The Second Vice-President and the Bookkeeper were once engaged to be married to each other.

10. The fashionable Teller is son-in-law of the First Stenographer.

11. Mr. Jones regularly gives Mr. Evans his discarded clothing to wear, without the elderly Bookkeeper knowing about it.

Show how to match correctly the eleven names against the eleven positions occupied.

Solution by Mary L. Constable, Philadelphia High School for Girls.

In seeking to determine which offices are held by the six men, we note five of them to be; First Vice-President by 6, Third Vice-President by 1, Cashier by 6, Teller by 10 and Janitor by 7. From 3 the Second Vice-President and the Assistant Teller are of the same sex, and since there are only six men, these must both be women. Therefore, as the Second Vice-President is a woman, the Bookkeeper is a man by 9.

Women must then hold the remaining five offices of President, Second Vice-President, Assistant Teller, First Stenographer and Second Stenographer.

Since the President is married and is not Mrs. Brown by 1, nor Mrs. Kane by 5, she is Mrs. Ford. Miss Hill is not a stenographer by 4, and the second stenographer is unmarried by 8. Therefore the Second Stenographer is Miss Dale, and then by 2, the Assistant Teller is married. Since the Assistant Teller is not Mrs. Brown by 1, she is Mrs. Kane. The First Stenographer is married by 10, and is therefore Mrs. Brown. So Miss Hill is Second Vice-President.

Mr. Grant gives Miss Hill orders in 4, so his position must be higher than hers, and he is the First Vice-President. He and the Cashier live together by 6, so by 5 the Cashier must be Mr. Long. The young Mr. Adams of 8 is not the elderly bookkeeper of 11, and neither is Mr. Jones nor Mr. Evans, by 11. Therefore Mr. Camp is the bookkeeper. Mr. Adams, the unmarried socialite of 8, is not the married teller of 10, nor the janitor who lives in a garret according to 7. Therefore Mr. Adams is the Third Vice-President.

The fashionable Teller of 10 would not regularly receive discarded clothing to wear. Therefore he is not Mr. Evans by 11, and must be Mr. Jones. Finally, Mr. Evans of 11 must be the janitor, which is consistent with his garret residence of 7.

Also solved by J. A. Benner, Ruth Campbell, W. B. Carver, Wm. Douglas, George Freier, R. A. Good, C. H. Graves, Elmer Latshaw, C. E. Springer, E. P. Starke, J. E. Trevor, and Simon Vatriquant.

E 232 [1936, 494]. *Proposed by V. Thébault, Le Mans, France.*

Prove that no perfect square can be written in the scale of ten with just five digits which are distinct, but congruent modulo two.

Solution by C. W. Trigg, Cumnock College, Los Angeles.

There are but two sets of five digits, 0 2 4 6 8 and 1 3 5 7 9, which are distinct and congruent modulo two. The sum of the digits of every perfect square must be congruent modulo nine to 0, 1, 4, or 7. The sum of the digits of the first set is congruent to two modulo nine, so no permutation of this set can be a square. If the last digit of a perfect square is odd, the penultimate digit must be even. Since the second set contains no even digit, none of its permutations can be a square. Therefore no square exists which meets the given conditions.

Also solved by W. E. Buker, R. F. Schnepp, E. P. Starke, W. R. Talbot, Simon Vatriquant, and the proposer.

E 234 [1936, 494]. *Proposed by Leon Recht, College of the City of New York.*

Ten years from now Tim will be twice as old as Jane was when Mary was nine times as old as Tim. Eight years ago Mary was half as old as Jane will be when Jane is one year older than Tim will be at the time when Mary will be five times as old as Tim will be two years from now. When Tim was one year old, Mary was three years older than Tim will be when Jane is three times as old as Mary was six years before the time when Jane was half as old as Tim will be when Mary will be ten years older than Mary was when Jane was one-third as old as Tim will be when Mary will be three times as old as she was when Jane was born. How old are they now?

Solution by A. C. Maddox, Louisiana State Normal School.

Letting the respective ages of Mary, Jane, and Tim be x, y, and z years, the stated conditions produce the three equations,

$$(1)\qquad \frac{z+10}{2} = 9\,\frac{x-z}{8} - x + y,$$

$$(2)\qquad 2(x-8) = 5(z+2) - x + z + 1,$$

$$(3)\qquad x - z - 2 = 3\left[\frac{\dfrac{3(x-y)-x+z}{3} - y + 10 + z}{2} - y + x - 6\right] - y + z.$$

This system is equivalent to the system

$$\begin{aligned} x + 8y - 13z &= 40, \\ x \qquad\ \ - 2z &= 9, \\ 3x - 7y + 4z &= 1, \end{aligned}$$

which yields by elementary methods the unique solution $x=15$, $y=8$, $z=3$. That is, Mary is fifteen, Jane is eight, and Tim is three years old.

Also solved by Wm. Douglas, Robert Gaskell, R. A. Good, H. O. Hanson, D. W. Hein, E. P. Starke, J. E. Trevor, and the proposer.

E 235 [1936, 495]. *Proposed by Cezar Coşniţă, Roumanian Mathematical Institute.*

Prove that the following system of equations is consistent, and solve it:

$$\begin{aligned} (ad+be)x + (ae+bf)y + (af+bd) &= 0, \\ (bd+ce)x + (be+cf)y + (bf+cd) &= 0, \\ (cd+ae)x + (ce+af)y + (cf+ad) &= 0. \end{aligned}$$

Solution by W. R. Talbot, Lincoln University, Jefferson City, Mo.

Numbering the given equations as (1), (2), and (3) respectively, it is to be noticed that they may be written as

$$aL + bM = 0, \tag{1}$$

$$bL + cM = 0, \tag{2}$$

$$cL + aM = 0, \tag{3}$$

where $L=dx+ey+f$ and $M=ex+fy+d$. For given values of a, b, and c, these equations give particular lines through the intersection of the lines, $L=0$ and $M=0$. Solving $L=0$ and $M=0$ simultaneously, we get

$$x = \frac{de - f^2}{df - e^2}, \qquad y = \frac{ef - d^2}{df - e^2}.$$

The classical special cases arise when the lines, $L=0$ and $M=0$, are parallel (when no finite solution exists), when these two lines are coincident (producing an unlimited number of solutions), and when lines (1), (2), and (3) coincide (again producing an unlimited number of solutions).

Also solved by Lois E. Bell, K. W. Crain, J. M. Feld, Robert Gaskell, R. F. Schnepp, E. P. Starke, J. L. Stearn, J. E. Thompson, C. W. Trigg, Simon Vatriquant, and J. A. Ward.

E 236 [1936, 495]. *Proposed by J. M. Feld, New York City.*

On the sides of triangle $A_1B_1C_1$ points A_2, B_2, and C_2 are chosen so that $A_1B_1=nA_1A_2$, $B_1C_1=nB_1B_2$, $C_1A_1=nC_1C_2$. On the sides of triangle $A_2B_2C_2$ points A_3, B_3, and C_3 are chosen so that $A_2B_2=nA_2A_3$, $B_2C_2=nB_2B_3$, and $C_2A_2=nC_2C_3$. This process is repeated indefinitely. Show that the vertices of the sequence of triangles approach the centroid of $A_1B_1C_1$ as a limit.

Solution by K. W. Crain, Purdue University.

The triangles $A_iB_iC_i$ ($i=1, 2, 3, \cdots$) of this problem have a common centroid, since "if the vertices of a triangle lie on the sides of another, and divide them in a fixed ratio, the triangles have the same median point." [Cf. R. A. Johnson, *Modern Geometry*, page 175.] From the definition of a limit, it follows that the vertices of this sequence of triangles approach the centroid of $A_1B_1C_1$ as a limit, provided only that the given ratio, n, is greater than unity.

Also solved by W. B. Clarke, T. C. Esty, D. L. MacKay, J. Rosenbaum, Simon Vatriquant, and the proposer.

E 237 [1936, 495]. *Proposed by C. W. Trigg, Cumnock College, Los Angeles.*

Prove that there exists just one six-place, palindromic square, and determine its value (palindromic means reading backwards and forwards alike).

Solution by R. F. Schnepp, St. Mary's University of San Antonio, Texas.

Every palindromic number with an even number of digits is divisible by eleven. Since eleven is prime, every palindromic square with an even number of digits is divisible by 121.

Denote the required square by $n^2=abccba$. Then

$$n^2 = 100001a + 10010b + 1100c = 11(9091a + 910b + 100c).$$

Since this is divisible by 121, $9091a+910b+100c\equiv 0 \pmod{11}$, or $5a+8b+c\equiv 0 \pmod{11}$.

If a is odd, b is even, being the penultimate digit of an odd square. If a is odd, $a=1$, 5, or 9. In particular, if $a=5$, b must be 2. On the other hand, if a is even, then either $a=4$ and b is even, or else $a=6$ and b is odd.

Furthermore, since $n^2\equiv 0$, 1, 4, or 7 (mod 9), it follows that $2(a+b+c)\equiv 0$, 10, 4, or 16 (mod 9), or $a+b+c\equiv 0$, 5, 2, or 8 (mod 9).

The only numbers, abc, satisfying all these conditions are found by trial to be 162, 188, 428, 443, 657, 698, and 981, but of these only 698 yields a palindromic square: $698896=836^2$.

Also solved by W. E. Buker, Mary L. Constable, Daniel Finkel, S. N. Fletcher, Jr., E. P. Starke, W. R. Talbot, Simon Vatriquant, and the proposer.

Advanced Problems

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3824. *Proposed by F. A. Lewis, University of Alabama.*

Determine the roots of the characteristic equation of the matrix

$$V = (v_{rc}) = (\epsilon^{(r-1)(c-1)}), \qquad \text{where} \qquad \epsilon = e^{2\pi i/n}.$$

3825. *Proposed by H. T. R. Aude, Colgate University.*

A number, written in the scale of 10, has the digit d ($d=2, 3, \cdots, 9$) in the position on the extreme right. A second number is formed by moving the digit d to the position on the extreme left. If the second number is d times the first, find the least number of digits n of the numbers for the various values of d. Also find these eight numbers, or obtain a formula for them.

3826. *Proposed by W. Macray, Clark Academy, N. Y.*

Solve the system of partial differential equations,

$$U\frac{\partial W}{\partial x} + 2W\frac{\partial U}{\partial x} = 0, \qquad U\frac{\partial W}{\partial x} + 2W\frac{\partial V}{\partial y} = 0, \qquad \frac{\partial U}{\partial y} - \frac{\partial V}{\partial x} = 0.$$

3827. *Proposed by V. Thébault, Le Mans, France.*

With three consecutive integers taken from 0, 1, 2, $\cdots$, 9 form a number of five figures such that its square is formed from the ten given integers. The solution is unique.

3828. *Proposed by V. Thébault, Le Mans, France.*

(a) The perpendiculars from a point P to the straight lines AQ, BQ, CQ, which join the vertices of a triangle to an arbitrary point Q, cut the sides of the triangle $A_1B_1C_1$, determined by the perpendiculars to the lines PA, PB, PC, drawn from the inverse points of A, B, C in an inversion (P, k), in three points of a straight line Δ perpendicular to PQ.

(b) If the point P remains fixed and Q describes a given straight line d, the straight line Δ passes through a fixed point.

SOLUTIONS

3740 [1935, 396]. *Proposed by Paul Erdös, The University, Manchester, England.*

From a point O inside a given triangle ABC the perpendiculars OP, OQ, OR are drawn to its sides. Prove that

$$OA + OB + OC \geqq 2(OP + OQ + OR).$$

I. *Solution by L. J. Mordell, The University, Manchester, England.*

Denote by x, y, z the lengths of the perpendiculars OP, OQ, OR to the sides BC, CA, AB of the triangle ABC, and by a, b, c the lengths of OA, OB, OC. We then have

$$(1) \qquad QR = (y^2 + z^2 + 2yz \cos \alpha)^{1/2}, \qquad a = QR/\sin \alpha,$$

where α, β, γ are the angles of the triangle. We have now in turn

$$\begin{aligned} a + b + c &= \sum_{\alpha,\beta,\gamma} (y^2 + z^2 + 2yz \cos \alpha)^{1/2}/\sin \alpha \\ &= \sum [(y \sin \gamma + z \sin \beta)^2 + (y \cos \gamma - z \cos \beta)^2]^{1/2}/\sin \alpha \\ (2) \qquad &\geqq \sum (y \sin \gamma + z \sin \beta)/\sin \alpha = \sum x\left(\frac{\sin \beta}{\sin \gamma} + \frac{\sin \gamma}{\sin \beta}\right) \\ &\geqq 2(x + y + z). \end{aligned}$$

This is the desired result.

II. *Solution by David F. Barrow, University of Georgia.*

The perpendiculars are shorter than any other lines drawn from O sides of the triangle, so the proposed inequality will be proved if we establish the more stringent one of the following

THEOREM. *Twice the sum of the bisectors of the three angles formed by joining any point inside a triangle to the vertices is less than or equal to the sum of the distances of the point from the vertices.*

Let ABC be the given triangle, O the interior point, and U, V, W the points where the bisectors of angles BOC, COA, AOB cut the sides BC, CA, AB respectively. Then we wish to prove

$$2(OU + OV + OW) \leqq OA + OB + OC. \tag{1}$$

The following lemmas will be used in the proof:

LEMMA 1. *If α, β, γ are real angles subject to the condition*

$$\alpha + \beta + \gamma = 180°, \tag{2}$$

and a, b, c are positive constants, then

$$a \cos \alpha + b \cos \beta + c \cos \gamma \leqq \frac{ab}{2c} + \frac{bc}{2a} + \frac{ca}{2b}. \tag{3}$$

To prove this, we consider the left member of (3) as a function of two real, independent variables, α, β, and examine this function for maxima. The work is a little tedious but follows standard lines, and the result is that for certain values of a, b, c the maximum of our function is the right member of (3), while for other values of a, b, c the maximum value of the function will be one of the three expressions:

$$a + b - c, \qquad a - b + c, \qquad -a + b + c.$$

The right member of (3) exceeds each of these expressions: indeed it exceeds the first by $(-ab+bc+ca)^2/2abc$. Hence our lemma is proved.

LEMMA 2. *The following algebraic identity may be verified directly*

$$\begin{aligned} &\frac{2x^2(y+z)}{(x+y)(z+x)} + \frac{2y^2(z+x)}{(x+y)(y+z)} + \frac{2z^2(x+y)}{(y+z)(z+x)} \\ &\qquad \equiv x + y + z - \frac{xy(x-y)^2 + yz(y-z)^2 + zx(z-x)^2}{(x+y)(y+z)(z+x)}. \end{aligned} \tag{4}$$

LEMMA 3. *In any triangle the bisector of an angle equals the cosine of half this angle multiplied by twice the product of the including sides and divided by the sum of the including sides.*

For instance in triangle BOC, the bisector OU is given by

$$OU = \frac{2(OB)(OC)}{OB + OC} \cos \tfrac{1}{2}(BOC). \tag{5}$$

In deriving this the writer made a roundabout use of trigonometry and the fact

that the bisector divides the opposite side in segments proportional to the adjacent sides. It looks as if there should be some shorter way to get it.

Proof of the theorem: Let angles BOC, COA, AOB be denoted by 2α, 2β, 2γ respectively. Then since these angles form a perigon, it is evident that α, β, γ satisfy (2). Also let OA, OB, OC be denoted by x, y, z respectively. Then by use of lemma 3, we have

$$(6) \qquad 2(OU + OV + OW) = \frac{4yz}{y+z}\cos\alpha + \frac{4zx}{z+x}\cos\beta + \frac{4xy}{x+y}\cos\gamma.$$

Then by making use of (3) of lemma (1), with obvious substitutions for a, b, c, we find that the right member of (6) is not greater than the left member of (4), and this latter identity makes the theorem immediately obvious.

COROLLARY. *The equality sign in* (1) *holds only when ABC is an equilateral triangle and O is its center.*

Note by the Editor. Formula (5) is given in Loney's *Plane Trigonometry*, p. 247.

3741 [1935, 396]. *Proposed by H. D. Ruderman, James Madison High School, Brooklyn, N. Y.*

Find the value of the sum

$$\sum_{i=1}^{n} \tan^2 \frac{i\pi}{2n+1},$$

where n is a positive integer.

Solution by L. S. Johnston, University of Detroit.

We shall consider the more general problem of finding the value of the sum

$$\sum_{i=1}^{[(m-1)/2]} \tan^2 \frac{i\pi}{m}$$

(where the bracketed expression means, as usual, that the greatest integer contained therein is to be used), distinguishing the cases according as m is odd or even. It is obvious that the proposed problem is the case for which m is odd.

The following formulas are easily established by the De Moivre Theorem or by induction:

$$\tan(2n+1)\theta = \frac{\tan^{2n+1}\theta - {}_{2n+1}C_2 \tan^{2n-1}\theta + \cdots}{{}_{2n+1}C_1 \tan^{2n}\theta - {}_{2n+1}C_3 \tan^{2n-2}\theta + \cdots},$$

$$\tan 2n\theta = \frac{{}_{2n}C_1 \tan^{2n-1}\theta - {}_{2n}C_3 \tan^{2n-3}\theta + \cdots}{{}_{2n}C_0 \tan^{2n}\theta - {}_{2n}C_2 \tan^{2n-2}\theta + \cdots}.$$

Both these numerators and both denominators are, of course, rational integral

functions of $\tan\theta$ for any given integral value of n, and could there be displayed *in toto*, but we shall find that we need only the first two terms of each of the expressions. Furthermore, each of the expressions is finite in value for every value of θ except odd multiples of $\pi/2$.

We now consider the two cases separately.

Case I. $m=2n+1$.

Consider the two equations

$$(2n+1)\theta = p\pi, \tag{1}$$

$$\tan(2n+1)\theta = 0, \qquad (p = 0, 1, 2, 3, \cdots, 2n). \tag{2}$$

The values of θ satisfying these equations are

$$\theta_p = \frac{p\pi}{2n+1}, \qquad (p = 0, 1, 2, 3, \cdots, 2n).$$

Now substituting in (2) above the value of $\tan(2n+1)\theta$ in terms of $\tan\theta$ which has already been displayed, factoring out $\tan\theta$ (corresponding to the root $\tan\theta=0$) and clearing of fractions, we have

$$\tan^{2n}\theta - {}_{2n+1}C_2 \tan^{2n-2}\theta + \cdots = 0, \tag{3}$$

which holds for every value of θ_p shown above. But

$$\theta_{n+j} = \pi - \theta_{n-j+1},$$

$$\tan\theta_{n+j} = -\tan\theta_{n-j+1}, \qquad (j = 1, 2, 3, \cdots, n).$$

Hence the values of $\tan\theta$ satisfying (3) are $\pm\tan k\pi/(2n+1)$ $(k=1, 2, 3, \cdots, n)$ and we may write (3) in the form

$$\prod_{k=1}^{n}\left(\tan^2\theta - \tan^2\frac{k\pi}{2n+1}\right) = 0. \tag{4}$$

Expanding (4), we note that the coefficient of the second term of the expanded form—that is, the coefficient of the term in $\tan^{2n-2}\theta$—is the negative of the sum whose value the problem seeks. Comparing this expansion with (3), we find

$$\sum_{k=1}^{n}\tan^2\frac{k\pi}{2n+1} = {}_{2n+1}C_2 = n(2n+1).$$

Case II. $m=2n$.

Using the equations

$$2n\theta = p\pi, \qquad (p = 0, 1, 2, 3, \cdots, 2n-1),$$

$$\tan 2n\theta = 0,$$

and the formula for $\tan 2\theta$ in terms of $\tan\theta$ which has already been displayed, we follow a line of argument exactly like that already followed, and derive without much difficulty the formula

$$\sum_{i=1}^{n-1} \tan^2 \frac{i\pi}{2n} = \frac{(n-1)(2n-1)}{3}.$$

It may be remarked that a particular example under Case I is mentioned by Professor Cairns as an examination exercise at Oxford and Cambridge [this MONTHLY, January 1935, p. 23], in which the candidate is asked to prove the equation

$$\sum_{i=1}^{8} \tan^2 \frac{i\pi}{17} = 136.$$

Professor Cairns also quotes as the exercise just preceding the one mentioned, and in the same examination, the exercise "Establish the formulas for tan $n\theta$ in terms of tan θ, distinguishing the cases according as n is odd or even." The formulas exhibited in this note are the formulas sought, as derived by the writer, who does not recall seeing such formulas in any text.

In connection with this problem and with the reference just made to Professor Cairns's report, it will be illuminating, to say the least, to reread the entire Cairns report and the discussion thereof by Professor Watkeys, which immediately follows the Cairns report. Particularly striking is the sentence in the Cairns report on page 22—"Keep in mind that this examination is taken at the age of eighteen and frequently earlier." After reading that sentence and the specimen examinations immediately following it, the reader will fully agree with the sentiments expressed in the first paragraph of the Watkeys discussion.

Solved also by J. Rosenbaum, F. Underwood, and the proposer.

3742 [1935, 396]. *Proposed by Maud Willey, Long Beach, Miss.*

Prove that the group of movements into itself, in space of n dimensions, of a regular solid with $n+1$ vertices is the alternating group of degree $n+1$; and that, in space of $n+1$ dimensions, the group for the same solid is the symmetric group of degree $n+1$.

Solution by the Proposer.

For any value of $n \geqq 2$, it can be shown that a rotation interchanging any three vertices cyclically and leaving the other vertices fixed is possible; that a rotation interchanging any two vertices and leaving the others fixed is possible in space of $n+1$ dimensions but impossible in space of n dimensions; that a group which includes all transpositions (each transposition representing the interchange of two vertices) is the symmetric group, and that a group which includes all permutations of the form (ABC) but no transpositions is the alternating group.

In space of n (or $n+1$) dimensions, if $n-2$ of the vertices are fixed, the locus of the three remaining vertices is a sphere (or hypersphere). On this sphere (or hypersphere) the three points can be moved and interchanged cyclically without altering the lengths of the segments joining them. If $n-1$ of the vertices

are fixed, the locus of the two remaining vertices is a circle (sphere). It can be shown that the diameter of this circle is not equal to the distance between the two points; therefore they cannot be interchanged without either moving them off the circle or altering the distance between them, but they can be interchanged on the sphere.

To find the diameter of this circle, select the coördinate system so that the $n-1$ fixed vertices are the unit points on $n-1$ of the n rectangular axes. Then the distance between any two of these vertices is $2^{1/2}$. The locus of the two remaining vertices is the intersection of the $n-1$ hyperspheres with centers at the fixed vertices and radii $2^{1/2}$. Their equations are

$$-2x_i + x_1^2 + x_2^2 + \cdots + x_n^2 = 1, \qquad (i = 1, 2, \cdots, n-1).$$

This locus lies in the plane $x_i = x_1$, and may be represented by the equations

$$x_i = x_1, \qquad x_n = \pm [1 + 2x_1 - (n-1)x_1^2]^{1/2}, \qquad (i = 1, 2, \cdots, n-1).$$

The function x_n has the maximum value of $[n/(n-1)]^{1/2}$ for $x_i = x_1 = 1/(n-1)$, and a minimum which is the negative of the maximum for the same values of x_i. The locus is a circle with center at $x_i = x_1 = 1/(n-1)$, $x_n = 0$, and radius $[n/(n-1)]^{1/2}$. To check this statement, notice that this center is in the plane of the locus and that the point

$$x_i = x_1 = a, \qquad x_n = \pm [1 + 2a - (n-1)a^2]^{1/2}, \qquad (i = 1, 2, \cdots, n-1).$$

which represents any point on the locus as a varies, is at the distance which has been indicated as the radius from the point which has been called the center. It is easily verified that the diameter is greater than the distance between the vertices $2^{1/2}$. This completes the proof.

Editorial Note. It is desirable to show first the analytical existence and construction of such regular configurations and to determine the least order of the space which contains them. We shall do this in proving the following theorem and obtaining at the same time some important related facts:

If $A_1, A_2, \cdots, A_m$ are m points such that the distance between any two is $e>0$, the configuration determines a space of $m-1$ dimensions. It is a regular simplex whose centroid is its orthocenter and circumcenter. No other finite point in this space is equally distant from the m points.

Take the centroid G_m as origin of vectors $\mathbf{a}_i$ to the vertices A_i. Then

$$(1) \qquad (\mathbf{a}_i - \mathbf{a}_j)^2 = \mathbf{a}_i^2 + \mathbf{a}_j^2 - 2\mathbf{a}_i \cdot \mathbf{a}_j = e^2, \qquad i \neq j;$$

and, after summing for i fixed but arbitrary, we have

$$(2) \qquad m\mathbf{a}_i^2 + \sum_{j=1}^{m} \mathbf{a}_j^2 = (m-1)e^2,$$

noting that $\mathbf{a}_1 + \mathbf{a}_2 + \cdots + \mathbf{a}_m = 0$. Hence $\mathbf{a}_1^2 = \mathbf{a}_2^2 = \cdots = \mathbf{a}_m^2 = r_m^2$, where r_m is the distance of each point from the centroid G_m, and

$$r_m = \sqrt{\frac{m-1}{2m}}\, e. \tag{3}$$

It is also easily found from (1) that

$$\mathbf{a}_i \cdot \mathbf{a}_j = -\frac{1}{2m} e^2, \qquad i \neq j. \tag{4}$$

Since for any three distinct subscripts i, j, k, these results give

$$\mathbf{a}_i \cdot (\mathbf{a}_j - \mathbf{a}_k) = 0, \tag{5}$$

A_iG_m is perpendicular to the face opposite A_i, and G_m is therefore the orthocenter. If $G_{m-1,i}$ is the centroid of this face, its vector is $-\mathbf{a}_i/(m-1)$; and hence the altitude from A_i has the length

$$h_m = \frac{m}{m-1} r_m = \sqrt{\frac{m}{2(m-1)}}\, e, \qquad 2r_m h_m = e^2. \tag{6}$$

We now show that no $m-1$ of the vectors are linearly dependent. Suppose, for instance, that

$$\sum_{i=1}^{m-1} C_i \mathbf{a}_i = 0,$$

where at least one coefficient, say C_1 is not zero. Then we have in turn

$$C_1 \mathbf{a}_1^2 - \frac{e^2}{2m} \sum_{i=2}^{m-1} C_i = 0, \qquad C_1 \mathbf{a}_1 \cdot \mathbf{a}_m - \frac{e^2}{2m} \sum_{i=2}^{m-1} C_i = 0, \qquad \frac{e^2}{2} C_1 = 0, \qquad C_1 = 0.$$

Since this contradicts our assumption, no such linear relation exists unless all the C's are zero. This leads to the fact that the only linear relation between the m vectors is one in which all the coefficients are equal, or unity, if they are not all zero.

The centroid G_m is the only point in this space of $m-1$ dimensions which is at the same distance from the vertices. For, if C is another such point with vector $\mathbf{c}$ in this space, we must have in turn

$$\mathbf{c} = \sum_{i=2}^{m} B_i(\mathbf{a}_i - \mathbf{a}_1), \qquad (\mathbf{c} - \mathbf{a}_1)^2 - (\mathbf{c} - \mathbf{a}_i)^2 = 0, \qquad \mathbf{c} \cdot (\mathbf{a}_i - \mathbf{a}_1) = 0,$$

$$\mathbf{c}^2 = 0,$$

where the last equation results from the scalar product of the first by $\mathbf{c}$. Hence C is not distinct from G_m.

If V_{m-1} is the content in rectangular units of the face opposite A_1, then that of the simplex is

$$V_m = h_m V_{m-1}/(m-1). \tag{7}$$

It is then easily shown that

$$V_m = \left(\frac{m}{2^{m-1}}\right)^{1/2} \frac{e^{m-1}}{(m-1)!}. \tag{8}$$

The content of the parallelopiped, with 2^{m-1} vertices, $m-1$ of whose edges are the $m-1$ vectors, is obtained by omitting in (8) the $(m-1)!$. The square of this content is important and more convenient in considerations of the linear independence of the vectors. This squared content is given by a special case of the "striped" determinants considered in the solution of 3705 [1936, 248]. Here in the determinant of order $m-1$ the elements in the principal diagonal are e^2, while in the parallels to the diagonal the elements are $e^2/2$.

It follows that, if we have such a regular simplex with m vertices and wish to add an additional vertex A_{m+1}, the new vertex must be taken outside of the space of $m-1$ dimensions of the simplex. If the simplex lies in a space of n dimensions, $n \geqq m$, there are ∞^{n-m} vectors with origin at G_m perpendicular to the space of the simplex, where for $m=n$ there is just one. On any such chosen vector lay off from G_m in either direction G_mA_{m+1}, so that

$$h_{m+1} = G_mA_{m+1} = \sqrt{\frac{m+1}{2m}}\, e.$$

It is then easily shown that A_{m+1} is at the distance e from each of the m vertices of the original simplex. Thus, starting with two points A_1 and A_2, a simplex with $n+1$ vertices may be constructed in many ways in a space of n dimensions. If we take m vertices, say $A_1, A_2, \cdots, A_m$, of this regular simplex, they form a regular simplex in a space of $m-1$ dimensions which contains the centroid G_m. Then G_m and the remaining $n-m+1$ vertices determine a space of $n-m+1$ dimensions perpendicular to the space of the m simplex at G_m. The remaining vertices form also a regular simplex in a space of $n-m$ dimensions containing its centroid G_{n-m+1}. If G_{n+1} is the centroid of the complete regular simplex, it is easily shown by vectors that

$$\begin{gathered} G_{n+1}G_m = \left[\frac{n-m+1}{2m(n+1)}\right]^{1/2} e, \qquad G_{n-m+1}G_m = \left[\frac{n+1}{2m(n-m+1)}\right]^{1/2} e, \\ (G_{n+1}G_m)(G_{n-m+1}G_{n+1}) = \frac{e^2}{2(n+1)}. \end{gathered} \tag{9}$$

The three centroids are obviously collinear.

The determination of the group of motions of the regular simplex is now quite simple. If $m=2$, and if G_{n+1} is fixed as are also the remaining $n-1$ vertices, the simplex is rigid in its space of n dimensions. For A_1 and A_2 must lie on the unique straight line perpendicular at the fixed point G_2 to the space of $n-1$ dimensions determined by G_2 and the remaining $n-1$ vertices. Since the dis-

tances of A_1 and A_2 from G_2 must be $e/2$, no motion is possible in the n dimensional space of the simplex. If $m>2$, the m vertices $A_1, A_2, \cdots, A_m$ lie on a sphere with center G_m and radius r_m in the space of the m simplex. Let P be any point on this sphere, and A_j any one of the remaining vertices, then the triangles PG_mA_j and $A_1G_mA_j$ are right angled at G_m and hence congruent. Thus $PA_j=A_1A_j=e$. Hence, if G_{n+1} and the remaining vertices are fixed, the m simplex may be rotated as a rigid figure about G_m without altering the distance of any one of its vertices from any one of the remaining vertices. Hence the group of motions of the complete simplex has as a subgroup the group of motions of the m simplex in its space about G_m. For $m=2$, this subgroup is the identity as shown above. For $m=3$, the subgroup is the cyclic group of the equilateral triangle, where the sphere is simply the circumcircle of the triangle. The elements of the group are isomorphic with the permutations on the $n+1$ letters which name the vertices. Since these permutations contain every cycle of three letters and no permutations of two, we know that the group is the alternating group of degree $n+1$.

If on the other hand the complete simplex lies in a space of $n+1$ dimensions, this space is completely determined by a space of two dimensions, an ordinary plane, perpendicular at G_2 to the space of $n-1$ dimensions of G_2 and the remaining vertices. A similar reasoning to the above will show that A_1A_2 may be rotated in the plane as a rigid segment about its mid-point G_2 without altering the distances of A_1 or A_2 from the remaining $n-1$ vertices. Hence the group of motions is isomorphic in this case with the symmetric group of degree $n+1$.

The proposer considers the locus in n dimensions of the two remaining vertices A_1 and A_2 when the other $n-1$ vertices are fixed. It is obvious that this locus must be in the plane (of two dimensions) perpendicular at G_{n-1} to the space of the $n-1$ vertices, and that it must be a circle in the plane with G_{n-1} as center and radius $h_n=[n/2(n-1)]^{1/2}e$. As the chord A_1A_2 of fixed length e moves with its extremities on the circle, the centroid G_{n+1} describes in the same plane a concentric circle of radius $e/(n^2-1)^{1/2}$.

3744 [1935, 453]. *Proposed by R. E. Gaines, University of Richmond.*

Three parallel tangents are drawn to the cardioid $\rho=a(1+\cos\theta)$, and also another set of three tangents perpendicular to these. The locus of three of the nine intersections of the tangents is a circle, and that of the other six is a limaçon whose equation is of the form $\rho=b+c\cos\theta$, with a suitable change of origin.

Solution by W. T. Short, Oklahoma Baptist University.

If α is the angle that the tangent makes with the x axis

$$\tan\alpha=\frac{\sin\theta d\rho+\rho\cos\theta d\theta}{\cos\theta d\rho-\rho\sin\theta d\theta},\qquad d\rho=-a\sin\theta d\theta;$$

and we find after reductions that

$$\tan\alpha=-\cot 3\theta/2,\qquad \alpha=3\theta/2\pm\pi/2.$$

Let P_1, P_3, P_5 be the points of contact for the three tangents to the curve which make the angle α with the x axis. Let P_2, P_4, P_6 be the points of contact for the three tangents which make the angle $\alpha+\pi/2$ with the x axis. Let θ_i be the angle of the radius vector for the point P_i and let t_i be the tangent at the point P_i. We shall so select α that $\alpha=3\theta_1/2+\pi/2$, then $\alpha+\pi/2=3\theta_2/2+\pi/2$. From this it follows that $\theta_2=\theta_1+\pi/3$. Thus we may so arrange our P's that $\theta_{i+1}=\theta_i+\pi/3$, $i=1, 2, \cdots, 6$. We consider first the intersection of t_1 and t_2.

The equation of the tangent for a given value of θ is

$$(1)\qquad R\cos\left(\phi-\frac{3}{2}\theta\right)=2a\cos^3\frac{\theta}{2}=\frac{a}{2}\left(\cos\frac{3}{2}\theta+3\cos\frac{\theta}{2}\right).$$

After certain reductions we obtain for the equations of t_1 and t_2

$$(t_1):\qquad R\cos\phi\cos\frac{3}{2}\theta_1+R\sin\phi\sin\frac{3}{2}\theta_1=\frac{a}{2}\left(\cos\frac{3}{2}\theta_1+3\cos\frac{\theta_1}{2}\right),$$

$$(t_2):\quad -R\cos\phi\sin\frac{3}{2}\theta_1+R\sin\phi\cos\frac{3}{2}\theta_1$$

$$=\frac{a}{2}\left(-\sin\frac{3}{2}\theta_1+\frac{3\sqrt{3}}{2}\cos\frac{\theta_1}{2}-\frac{3}{2}\sin\frac{\theta_1}{2}\right).$$

Solving these equations we obtain

$$(2)\quad R\cos\phi=\frac{a}{2}\left(1+\frac{3}{4}\cos 2\theta_1+\frac{9}{4}\cos\theta_1-\frac{3\sqrt{3}}{4}\sin 2\theta_1-\frac{3\sqrt{3}}{4}\sin\theta_1\right)$$

$$(3)\quad R\sin\phi=\frac{a}{2}\left(\frac{3}{4}\sin 2\theta_1+\frac{9}{4}\sin\theta_1+\frac{3\sqrt{3}}{4}\cos 2\theta_1+\frac{3\sqrt{3}}{4}\cos\theta_1\right).$$

We now make the transformation

$$r\cos\psi=R\cos\phi+a/4,\qquad r\sin\psi=R\sin\phi;$$

and equations (2) and (3) become

$$(4)\qquad r\cos\psi=\frac{3a}{8}(2+\cos 2\theta_1-\sqrt{3}\sin 2\theta_1+3\cos\theta_1-\sqrt{3}\sin\theta_1),$$

$$(5)\qquad r\sin\psi=\frac{3a}{8}(\sin 2\theta_1+\sqrt{3}\cos 2\theta_1+3\sin\theta_1+\sqrt{3}\cos\theta_1).$$

Square both sides of (4) and (5) and add, and we obtain

$$r^2=9a^2\,[20+4(\cos 2\theta_1-\sqrt{3}\sin 2\theta_1)+8(3\cos\theta_1-\sqrt{3}\sin\theta_1)]/64.$$

Since

$$(\sqrt{3}\cos\theta_1-\sin\theta_1)^2=\cos 2\theta_1-\sqrt{3}\sin 2\theta_1+2,$$

the last equation may be written

$$(6) \qquad r^2 = 9a^2\left[(\sqrt{3}\cos\theta_1 - \sin\theta_1)^2 + 2\sqrt{3}(\sqrt{3}\cos\theta_1 - \sin\theta_1) + 3\right]/16.$$

Taking the square root and using the positive sign, we have in turn

$$(7) \qquad r = 3a\,(\sqrt{3}\cos\theta_1 - \sin\theta_1 + \sqrt{3})/4,$$

$$(8) \qquad r\cos\psi = 3a\left[(\sqrt{3}\cos\theta_1 - \sin\theta_1)^2 + \sqrt{3}(\sqrt{3}\cos\theta_1 - \sin\theta_1)\right]/8,$$

$$(9) \qquad \cos\psi = \tfrac{1}{2}(\sqrt{3}\cos\theta_1 - \sin\theta_1) = \cos(\theta_1 + \pi/6).$$

Hence we have the equation

$$r = (3\sqrt{3}a + 6a\cos\psi)/4$$

of the locus of the intersections of the tangents such that $\theta_{i+1} = \theta_i + \pi/3$. Six of our intersections, t_1t_2, t_2t_3, t_3t_4, t_4t_5, t_5t_6, t_6t_1, are on this curve, a limaçon. From (9) we see that $\pm\psi = \theta_i + \pi/6$.

We now consider the intersection of t_1 and t_4. We have in turn

$$(t_1): \qquad R\cos\phi\cos\tfrac{3}{2}\theta_1 + R\sin\phi\sin\tfrac{3}{2}\theta_1 = \frac{a}{2}\left(\cos\tfrac{3}{2}\theta_1 + 3\cos\tfrac{1}{2}\theta_1\right),$$

$$(t_4): \qquad R\cos\phi\sin\tfrac{3}{2}\theta_1 - R\sin\phi\cos\tfrac{3}{2}\theta_1 = \frac{a}{2}\left(\sin\tfrac{3}{2}\theta_1 - 3\sin\tfrac{1}{2}\theta_1\right),$$

$$(10) \qquad R\cos\phi = a\,(1 + 3\cos 2\theta_1)/2,$$

$$(11) \qquad R\sin\phi = (3a\sin 2\theta_1)/2.$$

From (10) and (11) we find that

$$(R\cos\phi - \tfrac{1}{2}a)^2 + R^2\sin^2\phi = 9a^2/4, \qquad R^2 - aR\cos\phi = 2a^2.$$

The last equation is that of a circle with center $(a/2, 0)$ and radius $3a/2$. This circle is the locus of the intersections of the tangents such that $\theta_{i+3} = \theta_i + \pi$, or the locus of the intersections t_1t_4, t_2t_5, t_3t_6.

Solved also by C. E. Springer, and the proposer.

3745 [1935, 453]. *Proposed by R. E. Gaines, University of Richmond.*

The same problem as 3744 substituting normals for tangents.

Solution by C. E. Springer, University of Oklahoma.

The normal to the cardioid $\rho = a(1 + \cos\theta)$ at $\theta = \alpha$ is

$$(1) \qquad \frac{2a}{\rho}\sin\frac{\alpha}{2}\cos^2\frac{\alpha}{2} = \sin\left(\frac{3}{2}\alpha - \theta\right)$$

and the six normals are obtained by writing $\alpha = (2\phi + m\pi)/3$ with $m = 0, 1, 2, 3, 4, 5$. The point of intersection of any two of the normals is given by

$$
(2)\qquad
\begin{aligned}
\frac{2a}{\rho}\sin\frac{1}{3}\left(\phi+n\frac{\pi}{2}\right)\cos^2\frac{1}{3}\left(\phi+n\frac{\pi}{2}\right)&=\sin\left(\phi+n\frac{\pi}{2}-\theta\right),\\
\frac{2a}{\rho}\sin\frac{1}{3}\left(\phi+k\frac{\pi}{2}\right)\cos^2\frac{1}{3}\left(\phi+k\frac{\pi}{2}\right)&=\sin\left(\phi+k\frac{\pi}{2}-\theta\right),
\end{aligned}
$$

where $n=0, 2, 4$ and $k=1, 3, 5$. Using rectangular coördinates in (2) and adding and subtracting the equations, we obtain

$$
(3)\qquad
\begin{aligned}
&(2x-a)\sin\left(\phi+\frac{n+k}{2}\frac{\pi}{2}\right)-2y\cos\left(\phi+\frac{n+k}{2}\frac{\pi}{2}\right)\\
&\qquad=a\sin\frac{1}{3}\left(\phi+\frac{n+k}{2}\frac{\pi}{2}\right)\cos\left(\frac{1}{3}\frac{n-k}{2}\frac{\pi}{2}\right)\sec\left(\frac{n-k}{2}\frac{\pi}{2}\right)\\
&(2x-a)\cos\left(\phi+\frac{n+k}{2}\frac{\pi}{2}\right)+2y\sin\left(\phi+\frac{n+k}{2}\frac{\pi}{2}\right)\\
&\qquad=a\cos\frac{1}{3}\left(\phi+\frac{n+k}{2}\frac{\pi}{2}\right)\sin\frac{1}{3}\left(\frac{n-k}{2}\frac{\pi}{2}\right)\csc\left(\frac{n-k}{2}\frac{\pi}{2}\right).
\end{aligned}
$$

Squaring and adding these equations gives (after some trigonometric reduction)

$$
(4)\qquad \frac{(2x-a)^2+4y^2}{a^2}-1=-\cos\frac{1}{3}\frac{(n-k)\pi}{2}\cos\frac{1}{3}\left(2\phi+\frac{(n+k)\pi}{2}\right).
$$

If (n, k) has one of the sets of values (0, 3), (2, 5), or (4, 1), the value of the first factor on the right of (4) is zero and the locus of the intersections of the normals at points of the cardioid corresponding to these numbers is the circle

$$
(5)\qquad (2x-a)^2+4y^2=a^2.
$$

To eliminate the parameter ϕ between equations (3), square and add equations (2) to find

$$
(6)\qquad 4(x^2+y^2)=a^2[4\cos^2\beta+\cos\beta\cos\lambda-4\cos 2\beta\cos^2\lambda],
$$

$$
\beta\equiv\frac{1}{3}\frac{(n-k)\pi}{2},\qquad \lambda\equiv\frac{1}{3}\left(2\phi+\frac{(n+k)\pi}{2}\right).
$$

Substituting the value of $\cos\lambda$ from (6) into (4) we find

$$
(7)\qquad
\begin{aligned}
&4\cos 2\beta[(2x-a)^2+4y^2]^2+a^2(\cos^2\beta-8\cos 2\beta)[(2x-a)^2+4y^2]\\
&\qquad+4a^2\cos^2\beta(x^2+y^2)=4a^4\cos^4\beta-7a^4\cos^2\beta+4a^4.
\end{aligned}
$$

If (n, k) has any one of the sets of values (0, 3), (2, 5), (4, 1), then $\cos\beta=0$ and equation (7) reduces to the circle (5) as noticed above. If (n, k) has any one of

the sets of values (0, 1), (2, 3), (4, 5), (0, 5), (2, 1), (4, 3), then $\cos^2\beta=3/4$ and (7) reduces to

$$32(x^2+y^2)^2-10a^2(x^2+y^2)+3a^2\left(x-\frac{a}{12}\right)=0, \tag{8}$$

where x has been replaced by $x+a/2$. Now, on replacing x by $x+a/4$, equation (8) reduces to

$$\left(x^2+y^2+\frac{ax}{2}\right)^2=\frac{3a^2}{16}(x^2+y^2). \tag{9}$$

On changing to polar coördinates this becomes

$$\rho=\frac{a\sqrt{3}}{4}-\frac{a}{2}\cos\theta,$$

which is a limaçon of the form desired.

Editorial Note. The construction of the required tangents and their intersections is simplified by the following theorem:

Given the circle with center O and diameter $AD=2R$, draw any chord AM_1 through A. Then the chord M_1R_1 which is the reflection of AM_1 in the diameter M_1M_1' is a tangent to the cardioid with its cusp at C on OD so that $OC=R/3$. The point of tangency is the intersection of M_1R_1 with the parallel to M_1M_1' through C. See the solution of 3642 [1934, 638].

The reflection of the chord AM_1' in $M_1'M_1$ is $M_1'R_1$. Hence M_1R_1 and $M_1'R_1$ are two perpendicular tangents meeting in R_1 on the circle. If we denote by α the angle DAM_1, considering AD as the positive direction, then the inclination of OM_1 to this direction is 2α and that of R_1M_1 is 3α. Hence, if we draw from A chords with inclinations $\alpha+\pi/3$ and $\alpha+2\pi/3$, we shall have two tangents M_2R_2, M_3R_3 parallel to M_1R_1. Since the angle AOR_1 is 4α, the three intersections R_1, R_2, R_3 are the vertices of an equilateral triangle. This completes the proof of the first part.

The second part is not so easy when we come to the determination of the locus. Having constructed the tangent M_1R_1, we now draw the chord AM_2 with the inclination $\alpha+\pi/6$; then the tangents M_1R_1 and M_2R_2 are perpendicular at their intersection P. Moreover, OM_2M_1 is equilateral as also $OM_2'M_1'$. The tangents $M_1'R_1$ and $M_2'R_2$ are perpendicular at P'. Since the triangles M_2PM_1 and $M_1'P'M_2'$ have their sides parallel, $P'P$ is parallel and equal to $M_2'M_1$. Hence $P'P=\sqrt{3}R$. Let E be the intersection of $P'P$ with AD. It will be shown later that $OE=R/2$. The inclination of $P'P$ is $2\alpha+\pi/6$. Then by replacing as before α by $\alpha+\pi/3$, $\alpha+2\pi/3$, we shall have six of the required intersections paired so that the three straight lines joining pairs meet in E and enclose in succession the angle $2\pi/3$. Let M_{12} be the mid-point of $M_1M_2(=R)$; then, since angle M_2PM_1 is right, $M_{12}P=R/2$. Also from the angles of the figure it is ob-

vious that $OEPM_{12}$ is an isosceles trapezoid. This determines the locus of P completely. Consider a fixed circle with center O and radius equal to $OM_{12}/2=\sqrt{3}R/4$, and an equal circle with center M_{12} which carries a point P fixed on a certain radius extended so that $M_{12}P=R/2$. As the second circle rolls on the first, P traces the locus. Or we may use a base circle with center O and radius $OE=R/2$. Draw chords UE and on each locate a point P so that $UP=\sqrt{3}R/2$ measured in the sense of UE. If we set $\phi=2\alpha+\pi/6$ and $EP=\rho$, we have at once the polar equation of the locus

$$\rho = \frac{R}{2}\,[\sqrt{3} - 2\cos\phi], \qquad R = 3a/2.$$

The problem regarding the normals easily reduces to the one just solved. Consider the circle with center O and radius $OC=R/3$, cutting OM_1 in N_1. The circle on N_1M_1 as a diameter cuts M_1R_1 in T_1 the point of tangency of the reflected ray M_1R_1. Here we may regard the given cardioid as the path of the point T_1, fixed on the circle (N_1M_1), as this circle rolls on the equal fixed circle (BC), where B is the other extremity of the diameter at C. The normal at T_1 is then T_1N_1 cutting (BC) again in S_1. Hence N_1S_1 is the reflection of the ray CN_1 in ON_1. Thus the envelope of the normal to the given cardioid is another cardioid with its cusp at C' on OB so that $OC'=OB/3=R/9$, and the original reflecting circle (AOD) is now replaced by the reflecting circle (COB). The solution of this problem is then an immediate consequence of that of the previous one.

Note on "The Queen of the Sciences"

In the January number of this Monthly the editor asked if someone would supply a reference for the statement that Gauss called mathematics "The Queen of the Sciences." Professor E. T. Bell writes that the statement is not to be found in Gauss' Collected Works or in any of his letters, but that the saying is ascribed to Gauss by W. Sartorius von Waltershausen in his book, "Gauss zum Gedächtnis" (Leipzig, 1856, page 79). The pertinent quotation reads, "Die Mathematik hielt Gauss, um seine eigenen Worte zu gebrauchen, für die Königin der Wissenschaften und die Arithmetik für die Königin der Mathematik. Diese lasse sich dann öfter herab der Astronomie und andern Naturwissenschaften einen Dienst zu erweisen, doch gebühre ihr unter allen Verhältnissen der erste Rang." This is the basis of Professor Bell's quotation:

"Mathematics is queen of the sciences and arithmetic the queen of mathematics. She often condescends to render service to astronomy and other natural sciences, but under all circumstances the first place is her due." (E. T. Bell: The Queen of the Sciences, The Williams and Wilkins Company, Baltimore, 1931, page 1.)

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items of interest to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

The General Education Board has awarded a grant of $5000 to the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics on "The place of mathematics in secondary education." As a result of this grant the Commission was able to hold meetings in Chicago on February 21 and 22, following the meetings of the National Council. A program of meetings of the entire Commission and of subcommittees to be held during the year was adopted. The chairman of the commission is Professor K. P. Williams of Indiana University.

At a meeting of the National Council of Teachers of Mathematics held in Chicago on February 19–20, a number of papers of interest to college teachers of mathematics were given, including the following: Mathematics and life, by Professor A. A. Bennett, Brown University; Off the beaten path, by Professor Mayme I. Logsdon, University of Chicago; Some problems of Junior College mathematics, by Professor H. W. Bailey, University of Illinois; Business and finance mathematics in the Junior College curriculum, by Professor W. S. Schlauch, New York University; Bridging the gap between High School and College mathematics, by Professor J. O. Hassler, University of Oklahoma; A new deal from old cards, by Professor H. E. Buchanan, The Tulane University of Louisiana. Miss Martha Hildebrandt, President of the National Council, presided at the general meetings. Most of the program was devoted to questions of interest primarily to High School teachers, but one session was devoted to Junior College mathematics with Professor E. J. Moulton presiding. A special feature of the meeting was the presentation of a phonographic record of an address by Dr. H. E. Slaught, and a testimonial in recognition of his splendid work in the promotion of the teaching of mathematics, both in colleges and in secondary schools.

The mathematics division of the Society for the Promotion of Engineering Education is planning a series of conferences with the division of Electrical Engineering to be held in connection with the annual meeting of the Society for the Promotion of Engineering Education, June 28 to July 2, at the Massachusetts Institute of Technology. The program committee consists of Professor J. L. Barnes, Tufts College; Professor A. A. Bennett, Brown University; Professor R. S. Burington, Case School of Applied Science; Professor Joseph Spear, Northeastern University; and Professor H. B. Phillips, Massachusetts Institute of Technology, Chairman. This committee is working in cooperation with the electrical engineers and has chosen for discussion the topic "Basic mathematics for electric engineering." The discussion of this topic will be carried on by representatives from Massachusetts Institute of Technology, Ohio State University, General Electric Company, and Bell Telephone Laboratories.

A mathematics club, meeting every two weeks, is conducted jointly by the departments of mathematics of Bryn Mawr College, Haverford College, Swarthmore College and the University of Pennsylvania. Up to the end of December 1936, papers were given before the Club by Professor W. L. Ayres of the University of Michigan, Professor F. D. Murnaghan of Johns Hopkins University, Professor Tullio Levi-Civita of the University of Rome, and Professor Hans Rademacher.

During the Summer Quarter of 1937 at the University of Chicago a seminar in the calculus of variations will be conducted by Professor G. A. Bliss, Professor E. J. McShane, Dr. W. T. Reid and Dr. M. R. Hestenes. Others who may be interested are invited to participate. It is planned to select topics of discussion in accord with the special interests of those attending.

Informal notes of some of the mathematical courses given this year at Princeton University and the Institute for Advanced Study are being photo-lithoprinted: Bochner, *Principles of Harmonic Analysis;* Bohnenblust, *Functions of Real Variables;* Hardy, *Mathematical Work of Ramanujan;* Lefschetz, *Algebraic Geometry I;* Morse, *Analysis in the Large;* von Neumann, *Continuous Geometry;* and Wilks, *Theory of Statistical Inference.* Notes of Wigner, *Nuclear Physics*, are being mimeographed; and the 1935–36 notes of Church, *Mathematical Logic*, have been re-run. The above list is tentative; a final list with prices (which will be at cost) should be available in May, but some of the notes may not be ready for distribution until later in the summer. Inquiries and orders may be addressed to "Mathematical Lecture Notes," Fine Hall, Princeton, N. J., in care of Professor A. W. Tucker.

A quarterly devoted to an integration of the scientific disciplines and to a study of the interdependence of science and society has recently commenced publication under the title *Science and Society: A Marxian Quarterly.* The editors are A. E. Blumberg, E. B. Burgum, V. J. McGill, Margaret Schlauch, and B. J. Stern. Among the foreign editors are J. D. Bernol of Cambridge University, Paul Langevin of the Collège de France, and H. Levy of the Imperial College of Science, London. Of the mathematical articles that have appeared, one is a historical critique of mathematics by D. J. Struik, and the second is a methodological and historical survey of the laws of probability by H. Levy. Editorial communications and manuscripts may be addressed to the managing editor, W. T. Parry, $6\frac{1}{2}$ Holyoke St., Cambridge, Massachusetts, while subscriptions (at $1.00 per year) may be sent to the business manager, H. F. Mins, Jr., 310 E. 75th St., New York, N.Y.

The College Entrance Examination Board has issued a "Report on the Mathematics Attainment Test of June, 1936," prepared by J. M. Stalnaker. It is believed that this test shows marked improvement in the technique of examination procedure. This report sets forth the principles underlying the test, the methods of assembling and pre-testing the individual items, and the scoring

procedure; in fact, it describes each step in the development of the test from the preparation of items to the reporting of final grades. The publication of every third item of one part of the test, and of all the items in the other part, will indicate to teachers the form and character of the test. The Board realizes that unless teachers understand fully the nature of the test they cannot be expected to appreciate its full values and to interpret effectively the results obtained from its use. This test was devised primarily for three types of students: (α) those who have had some mathematical training in the secondary schools, but who are not qualified for collegiate work in mathematics or natural science; (β) those who are prepared to fulfill the minimum college requirements in mathematics and the natural sciences; (γ) those who expect to continue in mathematics and science. The examination was in two parts, of which Part I contained 160 questions which could be answered by a short response, while Part II contained 10 longer problems for which the student had to supply a detailed solution. Requests for copies of this report should be addressed to the Executive Secretary of the College Entrance Examination Board, 431 West 117th Street, New York, N. Y., and should be accompanied by a remittance of twenty cents to defray the essential cost.

In the December 1936 election, the Association for Symbolic Logic elected the following new officers: Professor E. V. Huntington of Harvard University and Professor E. E. Nagle of Columbia University as members of the Executive Committee, and Professor Rudolf Carnap of the University of Chicago and Professor J. Jørgensen of Copenhagen as additional members of the Council.

N. B. Allison, formerly of the University of Kentucky, is in charge of mathematics at Kentucky Wesleyan College, Winchester, Kentucky.

The University of London has conferred an honorary doctorate on Professor Albert Einstein of the Institute for Advanced Study at Princeton.

Dean G. D. Birkhoff and Professor J. L. Coolidge of Harvard University have been made officers of the Legion of Honor.

Professor Constantin Carathéodory of the University of Munich was the Carl Schurz Memorial Professor at the University of Wisconsin for the first semester of this year, 1936–37.

Assistant Professor L. L. Garner of the University of North Carolina has been granted leave of absence for the coming session. He will continue his studies at the University of Michigan.

Associate Professor V. G. Grove of Michigan State College has been promoted to a professorship in mathematics.

At Indiana University, Professor U. S. Hanna has leave of absence for the year 1936–37, and Professor H. T. Davis has leave of absence for the second semester.

Dr. E. D. Jenkins, formerly of the University of Kentucky, has been at Eastern Kentucky State Teachers College, Richmond, Kentucky since February 1, 1937.

Dr. Cornelius Lanczos, of Purdue University, has been appointed acting assistant professor at Indiana University for the second semester.

Dr. V. S. Lawrence, of Cornell University has been promoted to an assistant professorship.

Dr. Karl Menger, professor of mathematics at the University of Vienna, has joined the department of mathematics at the University of Notre Dame for the second semester of 1936–37.

Dr. F. R. Moulton, formerly professor of astronomy at the University of Chicago, has been elected permanent secretary of the American Association for the Advancement of Science.

Dr. Emma J. Olson has been appointed instructor at Kent State College, Kent, Ohio, for the second semester 1936–37.

Miss Sallie Pence of the University of Kentucky is on leave of absence the second semester of 1936–37 and is studying at the University of Illinois.

Dr. H. R. Pyle has been appointed assistant professor and head of the department of mathematics at Earlham College, Earlham, Indiana.

Professor E. L. Rees of the University of Kentucky is on leave of absence, and is traveling in Australia and South Africa.

J. T. Rule has been appointed to an assistant professorship at the Massachusetts Institute of Technology.

Assistant Professor D. E. South of the University of Kentucky is on leave of absence, and is studying at the University of Michigan.

Professor J. H. Van Vleck of Harvard University has been given the honorary degree of Sc.D. by Wesleyan University.

Professor Mary E. Wells of Vassar has been on leave of absence the first semester of 1936–37 to teach and to help reorganize the departments of mathematics and physics in the Women's Christian College, Madras, India.

Dr. S. S. Wilks of Princeton University has been promoted to an assistant professorship.

The following appointments to instructorships in mathematics have been announced:

Princeton University, Dr. A. H. Taub;
Purdue University, Dr. A. C. Schaeffer;
University of Kentucky, C. W. Williams;
University of Wisconsin, Dr. M. T. Bird.

The following appointments have been made in the Harvard University summer school: Dr. J. J. Gergen, Dr. A. E. Pitcher, Dr. G. B. Price, and Dr. Wladimir Seidel.

The following courses in mathematics are announced for the summer of 1937.

Catholic University of America. In addition to the usual elementary courses the following advanced courses will be offered: By Dr. E. J. Finan: Theory of Equations; Modern algebraic theories. By Professor Rice: Advanced calculus. By Professor Ramler: Differential equations; Analytic projective geometry.

University of Chicago. First term, June 16 to July 21; second term, July 22 to August 27. In addition to Advanced calculus, Theory of equations, and Differential equations, the following courses will be offered: By Professor G. A. Bliss: Seminar in the calculus of variations. By Professor A. C. Lunn: Statistics and probability; Analytic mechanics. By Professor E. J. McShane: Calculus of variations. By Professor Mayme I. Logsdon: Algebraic geometry; Analytic projective geometry. By Professor R. W. Barnard: Continuous groups; Higher algebra. By Professor A. A. Albert: Galois Theory. By Dr. W. T. Reid: Fourier series. By Dr. M. R. Hestenes: Theory of functions of a complex variable. By Dr. L. R. Wilcox: Postulational methods.

University of Colorado. First term, June 21 to July 23; second term, July 26 to August 27. In addition to the usual elementary courses the following advanced courses will be offered: By Professor C. A. Hutchinson: Functions of a complex variable (both terms). By Professor A. J. Kempner: Modern algebra (both terms); Teachers' course (elementary mathematics from an advanced standpoint, first term).

Columbia University. July 12 to August 20. In addition to the usual elementary courses the following advanced courses will be offered: By Professor E. Kasner: General introduction to mathematics; Geometric transformations and groups. By Professor A. C. Berry: Differential equations. By Professor J. F. Ritt: Theory of functions of a real variable. By Professor A. B. Brown: Theory of finite groups.

Cornell University. July 5 to August 14. In addition to the usual elementary courses the following advanced courses will be offered: By Professor Jones: Theory of equations. By Professor Agnew: Elementary differential equations. By Professor Hurwitz: Advanced calculus. By Professor Carver: Projective geometry. By Professor Lawrence: Analytic projective geometry.

Duke University. First term, June 9 to July 20. By Professor Carlitz: Thesis seminar; Elliptic functions, or Theory of algebraic numbers. By Professor Elliott: Advanced calculus; Theory of equations. By Dr. Dressel: Modern geometry. Second term, July 1 to August 11. By Professor Rankin: The teaching of

mathematics; Fundamental concepts of algebra and geometry. Third term, July 21 to August 31: By Professor Miles: Advanced calculus; Modern higher algebra. By Professor Roberts: Projective geometry; Topology, or Real variables.

University of Illinois. In addition to the usual elementary courses the following courses will be offered: By Professor A. B. Coble: Theory of probability; Geometry. By Dr. Echo D. Pepper: Teachers' course; Fundamental concepts. By Professor H. R. Brahana: Introduction to higher algebra; Algebra. By Professor Trjitzinsky: Advanced calculus; Analysis. By Dr. P. W. Ketchum: Introduction to higher geometry. By Professor R. D. Carmichael: The theory of numbers.

University of Iowa. The Summer Session June 14 to August 6. The eleven weeks' session has been reorganized into an eight weeks' teaching period followed by a three weeks' independent study unit. The study unit is designed for students who are recommended by the heads of their departments as qualified to pursue work on theses or assigned problems. To meet requirements for admission to this unit, students must have been registered in the University during the year or summer session. In addition to courses in College algebra, Trigonometry, Analytic geometry, and Calculus, the following advanced courses are offered: By Miss Ruth Lane: Methods of teaching mathematics. By Professor Craig: Higher algebra-matrices and determinants, Analytical methods of mathematical statistics. By Professor Ward: Higher geometry—analytic, Calculus of variations. By Professor Wylie: Mathematics of finance, Astronomy, Current astronomical problems. By Professor Reilly: Differential equations, Spherical harmonics. By Professor Chittenden: Topics for teachers, Functions of real variables. By the staff: Seminar in analysis (Chittenden and Ward), Reading and research.

Indiana University. In addition to the usual elementary courses the following courses will be offered: By Professor Davisson: Advanced geometry. By Professor Rothrock: Advanced calculus; Fourier series. By Professor Hacker: Differential equations; Mathematical physics.

University of Kansas. In addition to the usual elementary courses the following courses will be offered: By Professor Smith: Modern synthetic geometry; Higher plane curves. By Professor Stouffer: Advanced algebra. By Professor Wheeler: Series.

University of Kentucky. In addition to the usual elementary courses the following courses will be offered: First term, By Professor Boyd: Curve tracing. By Professor C. G. Latimer: Advanced analytics. Second term, By Professor H. H. Downing: Solid analytic geometry; Complex variable theory.

Massachusetts Institute of Technology. First term, June 14 to July 24; second term July 26 to September 4. In addition to the usual elementary courses the

following advanced courses are offered. First term: By Professor Franklin: Advanced calculus; Functions of a complex variable. Second term: By Professor Struik: Advanced calculus. By Professor Zeldin: Vector analysis.

University of Michigan. June 28 to August 20. In addition to elementary courses and the standard courses in Differential equations, Theory of equations, Solid analytic geometry, and Advanced calculus, the following courses will be offered: By Professor Anning: Modern geometry; Teaching of geometry; History of arithmetic and algebra. By Professor Ayres: Theory of functions of a real variable. By Professor Carver: Finite differences; Mathematical theory of statistics. By Professor Churchill: Fourier series; Methods in partial differential equations. By Professor Coe: Analytic mechanics. By Professor Copeland: Theory of probability. By Professor Craig: Advanced theory of statistics. By Professor Dushnik: Graphical methods. By Professor Field: Analytic projective geometry. By Dr. Hull: Modern algebra. By Professor Menge: Advanced theory of interest and life contingencies. By Professor Poor: Vector analysis. By Professor Rainich: Continuous groups; Tensor analysis. By Professor Wilder: Introduction to the foundations of mathematics; Combinatorial topology. In addition there will be a seminar in pure mathematics conducted by Professors Rainich and Wilder, and one in statistics by Professor Craig.

University of Minnesota. First term, June 15 to July 24. In addition to all the usual undergraduate subjects, the following graduate courses will be offered: By Professor W. L. Hart: Differential equations. By Professor Dunham Jackson: Vector analysis; Limits and series. By Professors Hart and Jackson: Selected topics in advanced mathematics. No advanced courses will be offered during the second term.

University of Missouri. The courses in advanced mathematics to be offered next summer are: Advanced algebra; Advanced calculus; Advanced geometry; Dimension theory; Theory of fields; Topics from analysis.

University of North Carolina. First term: June 10 to July 21. In addition to the usual elementary work in Algebra, Trigonometry, Analytic geometry and the Calculus, the following courses will be offered: By Professor Henderson: Differential equations; Complex variable; Finite groups. By Professor Lasley: Differential geometry. By Professor Hill: History of mathematics. By Professor Garner: Theory of equations. Second term: July 22 to August 28. By Professor Mackie: Differential equations (continued); Complex variable (continued); Theory of equations (continued). By Professor Winsor: College geometry. By Professor Browne: Theory of numbers.

Northwestern University. Eight weeks session, June 21 to August 13. In addition to courses in Analytic geometry and Elementary calculus, the following courses will be offered: By Professor E. J. Moulton: Vector analysis. By Professor N. J. Lennes: Reorganization of secondary school mathematics; Intro-

duction to functions of real variables. By Professor H. S. Wall: Functions of several variables. By Professor H. L. Garabedian: Higher geometry.

Ohio State University. June 21 to September 3. In addition to the usual elementary courses the following courses will be offered: By Professor Bamforth: Advanced calculus; Integral equations. By Professor Bareis: Finite groups. By Professor Beatty: Advanced euclidean geometry. By Professor LaPaz: Introduction to modern mathematics; Partial differential equations.

University of Pennsylvania. In addition to the usual elementary courses up to and including the Calculus, the following advanced courses are offered: By Professor G. H. Hallett: Advanced calculus; Theory of finite groups. By Professor M. J. Babb: Differential equations. By Professor P. A. Caris: Modern analytic geometry. By Dr. J. A. Clarkson: Theory of abstract spaces.

Syracuse University. July 5 to August 13. In addition to the usual elementary courses the following advanced courses are offered: By Professor Campbell: Introduction to projective geometry; College solid geometry. By Professor Decker: Theory of numbers; Fundamental mathematical concepts.

Teachers College, Columbia University. By Professors Reeve, J. R. Clark and Shuster, Dr. Sanford, Dr. Swenson, Dr. Wolff and Miss Sutherland: Conferences and discussions on current questions in the teaching of mathematics. By Miss Sutherland: Teaching arithmetic in primary grades; Teaching arithmetic in intermediate grades; Professionalized subject matter in junior high school mathematics. By Professor C. N. Shuster: Modern business arithmetic; Methods of teaching in junior and senior high schools; Field work in mathematics. By Dr. Vera Sanford: Demonstration class in social and economic arithmetic; History of mathematics. By Professor W. D. Reeve: Teaching algebra in secondary schools; Teaching and supervision of mathematics in junior high schools. By Professor J. R. Clark: Teaching geometry in secondary schools; Teaching intuitive geometry in junior high schools. By Dr. J. A. Swenson: Demonstration class in integrated mathematics for the tenth year; Professionalized subject matter in senior high school mathematics. By Dr. Georg Wolff: Teaching mathematics in the secondary schools of Germany; The correlation of secondary school mathematics with science and art.

University of Washington. Professor Oswald Veblen of the Institute for Advanced Study has been appointed to a Walker Ames Professorship in Mathematics for the first term of the 1937 Summer Session. He will conduct a seminar in the theory of Spinors, assisted by Dr. A. H. Taub, formerly of Princeton University. In addition the following regular courses will be offered. By Professor J. P. Ballantine: Advanced calculus. By Dr. A. H. Taub: Modern algebra; Introduction to complex variables. By Professor R. M. Winger: Teachers course in mathematics; Reorganization of secondary mathematics; Finite groups.

University of Wisconsin. In addition to the usual elementary courses the following courses will be offered: By Professor MacDuffee: Advanced calculus; Continuous groups; Projective geometry. By Professor Evans: Topics in the theory of probability; Differential geometry (Six or nine weeks). By Professor Ingraham: Mathematics of educational statistics (Six or nine weeks); Higher algebra, (Six or nine weeks). By Professor Sokolnikoff: Topics in applied mathematics. By Mr. Trump: College geometry. By Miss Wolf: Theory of equations.

HEDRICK MADE PROVOST AT U.C.L.A.

Professor E. R. Hedrick has been appointed Vice-president of the University of California and Provost of the University of California at Los Angeles, his new duties commencing on March 19th. From the nature of the organization of the university he has a large administrative responsibility, especially for the division of the university which is located at Los Angeles.

Since Hedrick was the first president (1916) of the Mathematical Association of America, of which this MONTHLY is the official journal, it seems appropriate to give more than a passing notice of his new honor and responsibilities. Those familiar with the history of the Association unanimously give a lion's share of credit to Hedrick for its inception and for the courageous and effective efforts which he has put forth in its behalf.

Throughout his career Hedrick has shown remarkable organizing and administrative ability. His work on scientific and educational committees and commissions has been arduous and efficient, as is indicated by the fact that during the last year he has been a member of over forty such groups. In 1929–1930 he was president of the American Mathematical Society, and he has been editor-in-chief of the *Bulletin* of that organization since 1921. He has been a vice-president of the American Association for the Advancement of Science and is now secretary of Section A of that association. He has served as an editor of mathematical and engineering books for the Macmillan Company, forty-two mathematics books and twenty-eight engineering books appearing in the series. He has also found time to write two mathematical text-books, translate a couple of mathematical books from French and German, write about twenty research articles, give a great many addresses on mathematical and engineering education, teach in summer sessions as well as during the academic year,—and play an occasional rubber of bridge.

Provost Hedrick was born in Union City, Indiana, September 27, 1876. He studied at Michigan, Harvard, Göttingen, and Paris, and has taught at Yale, Missouri, and California. He married Helen Seidensticker at Cambridge in 1901, and his home has been enlivened by seven daughters and three sons.

A subscriber wishing to have his copies of the MONTHLY forwarded during the summer months should leave for each copy four cents postage with the local post office or carrier with instructions for forwarding.

Legal Form for Gifts and Bequests

I hereby give[1] to the Board of Trustees of the Mathematical Association of America the sum of.................................... Dollars, to be known as the............................... Fund, and to be used

for[2] { Endowment—the income only of which may be expended.
Special Projects—for which both principal and income may be expended.

Witness:...................... Signature......................

[1] In case of a bequest, the first line should read "I hereby give and bequeath," etc.
[2] Indicate which one of the two purposes is desired, and omit the other.

The Association needs funds for scientific publications and for the promotion of scientific activities.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, E. J. MOULTON, Northwestern University, Evanston, Ill.

BOOKS FOR REVIEW should be addressed to REVIEW EDITOR, American Mathematical Monthly, 531 West 116th Street, New York, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of addresss to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-first Summer Meeting, Pennsylvania State College, Sept. 6–7, 1937.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1937 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Waynesburg, Pa., May 1.
ILLINOIS, DeKalb, May 14–15.
INDIANA, Greencastle, April 30–May 1.
IOWA, Dubuque, April 16–17.
KANSAS, Wichita, April 3.
KENTUCKY, Louisville, May 1.
LOUISIANA-MISSISSIPPI, Hammond, La., March 5–6.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Lynchburg, Va., May 8.
MICHIGAN, Ann Arbor, March 20.
MINNESOTA, St. Paul, May 15.
MISSOURI.
NEBRASKA, Lincoln, May 7.
OHIO, Columbus, April 1.
OKLAHOMA, Tulsa, February 5.
PHILADELPHIA, Haverford, Nov. 27.
ROCKY MOUNTAIN, Greeley, Colo., April 16–17.
SOUTHEASTERN, Nashville, Tenn., April 16–17.
SOUTHERN CALIFORNIA, Los Angeles, March 6.
SOUTHWESTERN, State College, N.M., April 2–3.
TEXAS, Houston, April 23–24.
WISCONSIN, Milwaukee, May 8.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

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(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

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PRELIMINARY ANNOUNCEMENT OF THE TWENTY-FIRST SUMMER MEETING OF THE ASSOCIATION

The twenty-first summer meeting of the Mathematical Association of America will be held at the Pennsylvania State College, September 6–7, 1937, in conjunction with the meeting of the American Mathematical Society. Sessions will be held on Monday afternoon and Tuesday morning, comprising addresses by invited speakers including the retiring presidential address by Professor D. R. Curtiss. The meetings of the Society will continue from Tuesday afternoon until Friday of that week, and will include Colloquium Lectures on "Continuous geometry" by Professor John von Neumann, and an address on "Topological properties of differentiable manifolds" by Professor Hassler Whitney.

The dormitories of State College will be available to the visiting mathematicians and their families at the rate of $1.00 a day for single rooms and $1.50 a day for double rooms. The Nittany Lion Inn on the campus is also available. Meals will be served in the campus restaurant at approximately $1.50 a day. The joint dinner will be held at the Nittany Lion Inn Thursday evening. A picnic excursion into the mountains is planned for Wednesday afternoon. Numerous facilities for recreation will be available.

The detailed program will be mailed to members of the Mathematical Association early in July as usual, and reservations can be made at that time.

THE FALL MEETING OF THE KENTUCKY SECTION AND THE TENNESSEE GROUP

The second fall meeting of the Kentucky Section was held jointly with the Tennessee group in Nashville, Tennessee, on Saturday, November 21, 1936. The chairman, Professor W. L. Moore of the University of Louisville, presided at the morning session which was held at Peabody College, and Professor W. L. Miser of Vanderbilt University presided at a cafeteria luncheon and at the afternoon session which were held at Vanderbilt University.

The attendance was sixty-one, including the following twenty members of the Association: H. G. Ayre, R. V. Blair, P. P. Boyd, M. C. Brown, H. H. Downing, Tryphena Howard, R. O. Hutchinson, J. A. Hyden, H. T. Karnes, J. F. Locke, A. N. McPherson, W. L. Miser, W. L. Moore, J. S. Morrel, Mabel I. Nowlan, J. K. Peterson, Tibor Radó, Augustus Sisk, F. L. Wren, H. M. Yarbrough.

Dean Boyd acted for the secretary at this meeting. Professor Tibor Radó of Ohio State University was the guest speaker. His presence was appreciated and enjoyed by the group. At the luncheon Professor Miser introduced Dr. F. C. Paschal, dean of the College of Arts and Sciences at Vanderbilt University; Dr. P. P. Boyd, dean of the College of Arts and Sciences at the University of Kentucky; and C. M. Sarratt, dean of men and head of the department of mathematics at Vanderbilt University. Each gave a short talk.

The following ten papers were read:

1. "On representation of functions of a complex variable by line complexes" by Mary Dean Clement, Ward-Belmont School, introduced by the chairman.

2. "The mathematical preparation of freshmen" by Professor G. S. Burton, University of the South, introduced by the chairman.

3. "Mathematizing the 1930 U. S. life tables" by Professor M. C. Brown, University of Kentucky.

4. "Formulas for sines and cosines of multiple angles" by Professor H. H. Downing, University of Kentucky.

5. "Some notes on contact curves" by Dean P. P. Boyd, University of Kentucky.

6. "On maximum and minimum" by Professor Tibor Radó, Ohio State University.

7. "The problem of Lagrange with differential inequalities as added side conditions" by F. A. Valentine, University of Tennessee, introduced by Professor Miser.

8. "Mathematical results of sampling applied to the distribution of college grades" by Professor R. O. Hutchinson, Tennessee Polytechnic Institute.

9. "A characteristic property of integral valued polynomials" by Professor Fritz John, University of Kentucky, introduced by Professor Miser.

10. "Plane sections of a surface" by Professor M. L. MacQueen, Southwestern College, introduced by Professor Miser.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Miss Clement set up a geometric representation of functions of a single complex variable by means of the totality of lines in three-dimensional space, and studied such functions analytically by means of three mutually independent equations in Plücker coördinates. She investigated the behavior of this system of representation for a singular point of a function.

2. The results of a study of the preparation of college freshmen were presented by Professor Burton, showing why many secondary school graduates are poorly prepared to do work in mathematics at the college level. The four factors that enter into the preparation of a student during his school period were treated, these factors being the school, its systems and standards; the teacher; the textbook; and the subject matter.

3. In this paper Professor Brown pointed out that the American Experience Mortality Table was first published, 1868, in the schedule of an act prescribing it as a basis of valuation in the State of New York. Since the 1930 U. S. Life Tables more nearly represent actual conditions in the U. S. at the present time, and since these tables give facts about persons from ages 0 up to 105, the State of Kentucky has recently passed an act prescribing this table as a basis for inheritance tax computations. At the death of a person, several heirs of different ages may receive periodically for their lifetime parts of the future income from the estate. This brings about the need for the present value of a life annuity on two or more lives. To obtain such value, the graduation of the 1930 U. S. Life

Tables (White Males), the calculation of the necessary Makeham constants, and some comparisons with other tables have been considered.

4. In this paper Professor Downing made reference to various formulas expressing sin $n\theta$ and cos $n\theta$ as polynomials in sines and cosines of the simple angle θ, n being a positive integer.

5. Professor Boyd outlined the Jonquières paper in *Liouville*, 1861, on *General Theorems* etc., in which the index N of a non-linear system of curves was introduced and theorems were proved that had to do with the number of curves of the system that touch one or more fixed curves. Jonquières's formulas failed for conics, and Chasles in *Comptes Rendus*, 1864, supplied the deficiency by the introduction of a second index or characteristic, N^1. By means of Chasles's theorem the enumeration of conics determined by the various possible combinations of points, tangent lines and tangent conics was effected and set down as in the table by Sporer.

6. The problem of determining the maximum or the minimum of a function over a given range is treated in calculus in terms of differential coefficients. The rules developed in calculus are concerned with conditions for a maximum or a minimum being reached at an *interior point*, while the *boundary points* must be treated individually in each case. Professor Radó gave a survey of certain important classes of functions for which the interior points can be ruled out altogether. The classes discussed included monotonic functions and convex functions of a single variable and saddle-functions and subharmonic functions of two variables. Applications in recent research work were sketched briefly in conclusion.

7. The problem presented by Mr. Valentine consisted in finding in the class of admissible arcs $y_i = y_i(x)$ joining two fixed points and satisfying a set of differential equations and a set of differential inequalities that one which minimizes an integral of the form $J = \int_{x_1}^{x_2} f(x, y, y')dx$. He indicated how an equivalent problem yields a number of necessary conditions. Furthermore, it is possible to obtain enough necessary conditions which when properly strengthened enable one to establish a sufficiency proof without the assumption of normality. Further major results were briefly summarized.

8. Professor Hutchinson presented equations for calculating the probability that the distribution of abilities in a medium sized class would approximate the distribution in a very large group from which the class had been selected at random. By tables calculated from these equations he showed that, if the original distribution were that of the normal probability curve, only 14 per cent of all classes of twenty-five students each could be expected to have a distribution deviating from the normal curve by less than 25 per cent.

9. Let $f(n)$ denote a function defined for every integer $n > 0$, the values of which are integers. Professor John proved the theorem: $f(n)$ is a polynomial if it satisfies the following two conditions: (1) $f(n_1) \equiv f(n_2)$ (mod p) for every prime p for which $n_1 \equiv n_2$ (mod p); (2) $\lim_{n\to\infty} \sup (1/n) \log f(n) < \log (e-1)$. It may be possible to replace $\log (e-1)$ by a better constant in this connection.

10. A line l_1 at a point O of an analytic surface S in ordinary space is defined to be a line that passes through the point but does not lie in the tangent plane of the surface at the point. In this paper Professor MacQueen supplements the investigations of Wilczynski by extending the study of the loci of geometrical elements projectively associated with the plane curves of section of a surface made by a variable plane through a line l_1 at an ordinary non-parabolic point O of a surface S in projective space of three dimensions.

A. R. FEHN, *Secretary*

THE NOVEMBER MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The seventh regular meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at Westminster College, New Wilmington, Pennsylvania, on Saturday, November 7, 1936. Professor L. L. Dines, Chairman of the Section, presided at both the morning and afternoon sessions.

Fifty-nine representatives from eleven colleges, three research laboratories, and three high schools attended the meetings, including the following eighteen members of the Association: C. S. Atchison, O. F. H. Bert, H. L. Black, Helen Calkins, W. E. Cleland, L. L. Dines, C. W. Foard, F. A. Foraker, N. C. Grimes, H. C. Hicks, B. P. Hoover, L. T. Moston, E. G. Olds, J. B. Rosenbach, E. Saibel, H. C. Shaub, J. S. Taylor, E. A. Whitman.

At the annual business meeting the following officers of the Section were elected: Chairman, L. L. Dines, Carnegie Institute of Technology; Secretary-Treasurer, J. S. Taylor, University of Pittsburgh; Member of Executive Committee, F. W. Owens, Pennsylvania State College. Professor H. L. Black, Westminster College, continues in office for the second year of his term as the adtional member of the executive committee. The spring meeting was set for Saturday, May 1, at Waynesburg College, Waynesburg, Pennsylvania. Following the afternoon session those in attendance were entertained by the members of the Westminster College chapter of Delta Nabla at a very delightful tea.

Following a welcoming address by President Robert Ferguson Galbreath of Westminster College the following five papers were read:

1. "A family of plane non-involutorial Cremona transformations which leaves curvewise invariant a pencil of cuspidal cubics" by Professor H. C. Shaub, Washington and Jefferson College.

2. "The use of foreign language mathematics texts" by Professor H. L. Black, Westminster College.

3. "Some applications of vector methods" by Professor L. T. Smith, Carnegie Institute of Technology, introduced by Professor Dines.

4. "Group theory in combinatorial topology" by H. Serbin, University of Pittsburgh, introduced by the Secretary.

5. "Recent progress in lightning studies" by P. L. Bellaschi, Westinghouse Electric and Manufacturing Co., introduced by the Secretary.

Abstracts of these papers follow, with the numbers corresponding to the numbers in the list of titles:

1. A one-to-one non-involutorial correspondence K which leaves each cubic of a pencil of cuspidal cubics invariant as a whole was set up by Professor Shaub. The algebraic formulation of K led to a non-involutorial Cremona transformation of order seven in which the fundamental elements have interesting coincidences. The relations between the fundamental configurations were discussed. This problem, unnoted in Miss Hudson's *Cremona Transformations*, has been treated, by different methods, by Godeaux, *Nouvelles Annales* vol. 3, 1924.

2. In Professor Black's paper three principal reasons for the use of foreign language texts in advanced undergraduate mathematics classes were stated and discussed: first, the desirability of applying the acquired language skills as soon as possible after the basic courses have been completed; second, the psychological advantage of a feeling of language mastery before graduate work is undertaken: and third, the value of early acquaintance with the mathematical points of view of foreign schools. The use of foreign language texts in junior and senior classes of selected students at Westminister College has produced favorable results.

3. Professor Smith pointed out that the gradient of a function and the divergence and the curl of a vector field may be defined as the limit of the ratio of certain surface integrals over closed surfaces to the volumes contained within the closed surfaces when the volume is allowed to shrink to a point. From these definitions the physical meanings of these operators are made evident rather easily. It is also rather easy to obtain corresponding integral identities such as the divergence theorem. This method of derivation makes it evident that the usual expressions for these operators in a rectangular coördinate system are invariant under changes of axes.

4. In presenting an expository paper on combinatorial topology Mr. Serbin began by defining simplex, complex, and K-chains. It was shown that the set T^k of K-chains is an additive group and that the boundary operator is a homomorphism of T^k onto a subgroup of N^{k-1} (bounding chains) of T^{k-1}. From a fundamental theorem on groups one infers the existence of a subgroup G^k (closed chains) of T^k such that $T^k/G^k \cong N^{k-1}$. After proving that the boundary operator when applied twice to a chain gives a zero chain, it is possible to derive the standard numerical relations on the ranks of the subgroups involved. In conclusion, the homology group was defined as G^k/N^k.

5. After briefly discussing the generation of the thunderstorm, Mr. Bellaschi directed attention to recent progress in the study of the mechanism of the lightning stroke flash. This may consist of single or multiple discharges. In the latter case the first stroke generally is the most severe. The development of the lightning stroke generator at the Sharon Laboratory of the Westinghouse Company has furthered progress in our knowledge of the lightning stroke discharge. The author has established that the core of the lighting stroke channel is about 1/2 to 3/4 inches in diameter.

J. S. TAYLOR, *Secretary*

MARY HEGELER CARUS, 1861–1936

By DAVID EUGENE SMITH, Columbia University

In a small two-story house in La Salle, Illinois, on January 10, 1861, there was born a daughter to Mr. and Mrs. Edward C. Hegeler. It was an event in the village, a greater event in the household but, as it turned out, a far greater one in the country at large.

Mr. Hegeler had come from Bremen, an ancient city of religious and political strife for a thousand years and more, and for a time a Free City State of the Hanseatic League. It had been the desire of his father that one of his sons should settle in the United States, in the development of which he was greatly interested. With this purpose in mind he planned the course of education of his youngest son, Edward, at the same time instilling into the latter's mind a feeling of the American spirit and almost a patriotic desire to carry to the new world the German spirit and culture and the thoroughness of education which characterized his home land. With this in view Edward was sent to the Academy at Schnepfenthal for a time, and later to the Polytechnic Institute at Hanover where he devoted his attention to mechanical engineering. This was followed by a course in the School of Mines at Freiberg, Saxony, where he studied under Professor Julius Weisbach, then a well-known scientist, and where he met his future wife, Camilla, the professor's daughter.

In 1857 he came to America with his friend and fellow-student, F. W. Matthiessen, each with the idea of building up a zinc plant such as had not as yet been established in the United States. After investigating the local conditions in Bethlehem, Pennsylvania, where one attempt had already been made, and southwestern Missouri, they selected La Salle, Illinois, as being the best location on account of its proximity both to coal fields and to the zinc mines of Wisconsin.

Three years later, in 1860, business had prospered sufficiently to enable Edward to make a visit to Germany and marry the Camilla Weisbach whom he had met in Freiberg. The young couple returned to La Salle, where Mary (the first of their ten children) was born in 1861; and with this began the interesting story of Mrs. Carus, a brief summary of which this MONTHLY gladly welcomes.

Mary Hegeler attended school at La Salle, and at the age of sixteen we find her working in the weigh-office of the zinc plant, and assisting her father in various other departments. Soon thereafter she entered the University of Michigan where she devoted her attention chiefly to mathematics and chemistry, graduating with the class of 1882. She then went over to her father's old home at Freiberg, attending lectures on metallurgy in the Mining Academy, and working in the laboratory of her uncle, the famous chemist Clemens Winkler. It was while she was assisting in this laboratory in 1886 that Dr. Winkler discovered in the argyrodite of Freiberg the chemical element Germanium (Ge), isolating and purifying it, the properties having been predicted in 1871 by Mendeléef (Mendeleyev). The metal is still very rare and has been the object of much study in recent years.

The influence of all these studies upon the training of a young woman of twenty-five may well be imagined.

Upon her return from Germany she again entered the plant of her father, who soon found himself depending upon her judgment and her ability.

Meanwhile Mr. Hegeler had made the acquaintance of a young German scholar, Dr. Paul Carus, and had engaged him to tutor his children and to do some translating and writing for the *Open Court*, a magazine founded by the former early in 1887. Such was the ability shown by Dr. Carus that in December he took over the editorship of the magazine. In the following year he and Mary Hegeler were married, and now began the relation of Mrs. Carus to the notable work initiated by the founding of the *Open Court* and the *Monist*, and by the publication of the *Carus Mathematical Monographs* under the auspices of the Mathematical Association of America.

As a matter of history it should be recorded that when the World War broke out it became evident that the publication of certain European periodicals must cease. Among these was the *Bibliotheca Mathematica* (published in Leipzig) which had for many years been edited by Gustav Eneström, who asked this writer if it would be possible to continue its publication in America. Professor H. E. Slaught and the writer considered the possibility of connecting the *Bibliotheca* with the *American Mathematical Monthly*, perhaps in the form of a supplement. With this in view we called upon Mrs. Carus at her home in La Salle to see if she would be interested in financing such a project. We went over the plan carefully and she was very sympathetic with the idea of assisting in this or some similar project, and promised that she would take the matter up with her son, Dr. Edward Carus. It turned out that they felt that it would be better to sponsor some such project as the *Carus Mathematical Monographs*. This was later "made possible by a notable gift to the Mathematical Association of America by Mrs. Carus as sole trustee of the Edward C. Hegeler Trust Fund. . . . The scope of this series includes also historical and biographical monographs"—as the introduction to each volume states.

It was originally intended by Mrs. Carus eventually to endow the series. Circumstances led, however, to a modified plan by which the Open Court Publishing Company, as representing Mrs. Carus, financed the first four volumes directly, and also enabled the Mathematical Association to accumulate a Revolving Publication Fund by means of which, together with such further support as may be necessary from the Open Court Publishing Company, the *Carus Mathematical Monographs* could continue indefinitely. This plan was inaugurated with number five and is expected to continue with all future numbers.

It should be added as a further matter of history that not only is the organization of the Mathematical Association of America due to the vision and the persistent efforts of Professor H. E. Slaught of the University of Chicago, but that the initial suggestion for the Monographs was probably made by him to Mrs. Carus who so generously made the suggestion bear fruit.

It is impossible to know or to appreciate Mrs. Carus without understanding

to some degree the ideals and purposes of the two men to whom her life was devoted—to her father and her husband. The father, Mr. Hegeler, was a man of deeply religious nature, his ancestors having belonged to the Reformed Church, and the atmosphere surrounding him having been liberal in the best sense of the term. At Schnepfenthal, in his youth, he came under the pietistic influence of that school and was deeply impressed by the devotional spirit. As later experience broadened his view he naturally surrendered his belief in Christian dogmatism, but preserved that seriousness of purpose, that moral endeavor, and that profound faithfulness which characterizes all true religion. He found the necessary correctives in the Monistic conception of science. His idea of the theological God changed, but his religion of science could not dispense with a God of some kind. With Goethe he saw God in nature, recognizing Him as that power which enforces a definite kind of conduct. To him, morality was not what we think is good, but that which can stand the test of continued and thorough experience.

To present his solution of the religious problem and to place both religion and ethics on a scientific basis he founded the *Open Court* as a fortnightly magazine in February 1887. It was through an article by Dr. Carus, published in this periodical, that Mr. Hegeler became acquainted with the man himself—a Monist whose philosophy was essentially the same as his own.

Dr. Carus was born in Ilsenburg, Germany, where his father was a pastor. He received a thorough training in mathematics and the classics in the gymnasia at Posen and Stettin. In the latter school he studied under Hermann Günther Grassmann, author of the celebrated *Ausdehnungslehre* (a theory akin to Hamilton's *Quaternions* and Gibbs's *Vector Analysis*.) Dr. Carus later studied philosophy, philology, and the natural sciences in the universities of Greifswald, Strassburg (Strasbourg), and Tübingen.

In 1890 the *Open Court* became a more popular magazine, the more abstract and specifically scientific articles being published in a new quarterly, the *Monist*. This was the beginning of many years of even closer coöperation between Dr. Carus as editor, and Mrs. Carus and Mr. Hegeler as associate editors of the magazines and books issued by the Open Court Publishing Company.

Mrs. Carus joined heartily with her husband in emphasizing that the sources of knowledge, so claimed, must be critically examined, and this led them to seek for fragments of truth in poetry and art, and in the myths and symbols of religion. Dr. Carus's conception of form constituted the central idea of his philosophy. By a new and more precise formulation of this conception he bridged the chasm between object and subject and arrived at his monistic conception of the world. In his opinion the laws of nature and the formal laws of thought were identical inasmuch as consistency as the primary attribute of form applies to both. In fact he felt that it is form in which the spiritual and material unite.

After the death of Dr. Carus, in 1919, Mrs. Carus, with able assistants, carried on the work of her husband, always keeping in close touch with the organization and directing its policies. Because of the connection of Monism with

mathematics, and the sympathic relation of Mrs. Carus to each, a considerable number of the *Open Court* publications were on mathematics.

Mrs. Carus arranged for financing the *Carus Mathematical Monographs*, initiated in 1925 and published by the Mathematical Association of America. The purpose was to make accessible "mathematical and formal thought as contributory to exact knowledge and clear thinking, not only for mathematicians and teachers of mathematics but also for other scientists and the public at large."

After only a brief illness Mrs. Carus died on June 27, 1936, at her home in La Salle. Throughout her life she bore bravely and successfully many important responsibilities, from 1903 until her death holding the office of either president or vice-president of the Matthiessen and Hegeler Zinc Company, and serving as associate editor and adviser to the Open Court Publishing Company. She was also actively interested in the management of the latter Company as well as in the books and magazines which it published. In addition to all these activities she was the mother of six children, her home duties, however difficult, demanding and receiving her first attention.

Probably the most outstanding qualities of Mrs. Carus were her fairness and unselfishness. Never in any of her business or personal dealings did she take unfair advantage of any person,—always ready to hear any case that was presented, and giving not only advice but frequently financial assistance, even at a personal sacrifice. She lived modestly and simply, welcoming to her home many friends and all who needed her wise counsel. Although quiet and unassuming in manner, she had a broad foundation of knowledge and a great interest in life as a whole. Her keen insight and far-sighted judgment in all matters that came to her attention were so unusual as to seem to all her acquaintances irreplaceable.

As we think of her and of what she did for the world we may with perfect justice recall the lines of Wordsworth:

> "A perfect woman, nobly planned
> To warn, to comfort, and command."

As one who for more than forty years had occasion to know Dr. and Mrs. Carus and to have had several books published by the Company, this writer can whole-heartedly attest to the generous assistance rendered by each in the preparation and publication of the various works and to the friendships which thus grew up in the course of the many years.

It is a pleasure to add a few words of thanks especially to the members of the Carus family whose assistance in preparing this article and in securing certain data is deeply appreciated.

"The moving power of mathematical invention is not reasoning but imagination." A. De Morgan, Quoted in Graves' *Life of Sir W. R. Hamilton*, vol. 3, 1889, p. 219.

UNDERGRADUATE INSTRUCTION IN MATHEMATICS*

By J. I. TRACEY, Yale University

During the past quarter of a century there has been in this country a phenomenal increase in productive scholarship in the field of mathematics. The volume of research which is being produced and which is seeking an outlet has caused our societies and universities to enlarge existing journals of mathematics and to establish new ones. The effect of this has been to enhance the prestige of American mathematics until today it compares favorably with that of any other nation.

At the same time this development has taken place if we examine the other side of the picture we see that the general level of instruction in mathematics has fallen correspondingly. The requirement of mathematics has gradually been eliminated from our colleges, and in a number of states in recent years the study of algebra and geometry is no longer required in the high schools. Between these opposing trends of mathematical research and secondary instruction comes the undergraduate instruction in mathematics in the colleges, and it is inevitable that the first two years of college mathematics should be adversely affected by the evils which exist in the earlier instruction rather than stimulated by the achievements of productive mathematicians.

Before going into a discussion of undergraduate instruction let us trace some of the factors which have had a bearing on the secondary instruction. In that excellent book by Felix Klein, *Elementary Mathematics from an Advanced Standpoint*, of which there is a translation by Hedrick and Noble, there is a brief and interesting account of the historical development of mathematics. Therein Klein distinguishes different types of progress, or trends. The first one is the formal development of the subject by subdivisions within rather narrow limits. This he calls plan A. The second type, or plan B, is the development of a combination of partial fields under a uniform point of view so as to give a perspective to the subject as a connected whole. Thus the formal development of algebra, or euclidean geometry, as a separate subject typifies plan A; while analytic geometry, in which the central idea is to fuse number perception and space perception, is an illustration of plan B. In a general way Klein labels the important discoveries in modern mathematics under one or the other of these two plans, and states that both of them have played important roles in extending mathematical knowledge, whether used separately or in combination. In his conclusion he says that mathematical instruction in the secondary schools has been under the one-sided control of plan A, and any movement toward reform must be in the direction of plan B, especially in the direction of giving prominence to the notion of function under the fusion of number perception and space perception.

This book by Klein was the result of a series of lectures which he delivered

* Presented to the Mathematical Association of America at the meeting at the University of North Carolina, December 31, 1936.

during the closing years of the past, and the first years of the present century. During this same period there was a demand among the leaders in mathematics in this country for curricula reform in the schools which would revitalize the subject and make it conform to the needs of our modern civilization. There was considerable agitation to do away with the traditional subdivisions of mathematics and teach courses in general mathematics. However, at this time the enrollment in our secondary schools was increasing at an unprecedented rate. Whereas formerly it was only the exceptional pupil in the rural or village school who was urged to attend the county or town high school, the principle of mass education was extending so rapidly that from 1890 to 1930 while the population doubled, the secondary school enrollment increased 1300%.

The immediate effect of this was to create such a demand for teachers that great numbers of inexperienced persons without adequate training in mathematics were drawn into the profession. Many of the old drill-masters were less effective in attempting to teach the revised courses, and the younger teachers lacked the background and the mathematical maturity to teach the subject as a living, growing, and unified whole. Before we realized it we were in a jazz age of mathematics as well as of music. Having burst the shackles of Euclid the number of new texts in plane geometry seemed to correspond to the number of permutations of certain groups of theorems. When one considers the combined effect of intuitional geometry, of the theorems postulated, and of the theorems accepted without proof, how could even a good student be conscious of having built up a body of doctrine by logical deductions from a given set of axioms? The combination of poor teaching, poor texts, and an excessive number of failures of improperly qualified students has produced a mighty volume of protest against mathematics.

How can this situation be improved? Those teachers who lack a proper training in mathematics cannot compensate for this deficiency by taking courses in education, or even courses in "how to teach mathematics," however much they may be required or advised to do so. Moreover we college teachers are not immune from the shafts which are leveled at the secondary school teachers. For the most part they were taught by college teachers, and we are now reaping from some of the seed we have sown in former years. The successful teaching of elementary mathematics requires considerable drill work and mental discipline similar to that of teaching the classics, and these subjects are not especially popular in a system of mass education. Under the influence of the modern educators much of the essential discipline of early mathematical training has been eliminated with the result that many students cannot perform the simple operations of arithmetic which are required in other courses.

The critical problems of undergraduate instruction are those which center about the freshman and sophomore courses. The freshman course is attended by those students who elect mathematics *per se*, and others who may need it in some future course in science or engineering. The upper class courses are practically restricted to the few who aim to specialize in mathematics or the physi-

cal sciences. Since the majority of students take not more than one year of college mathematics it is evident that the attitude of the average college graduate toward mathematics will be influenced largely by the instructor in his freshman course.

Without doubt the recitation is the most effective method of teaching courses in freshman mathematics, and to a certain extent the same is true for sophomore courses. Is there a better way to help a student crystallize his ideas, and to express himself, than to have him discuss both topics and problems before the other members of his class as friendly critics? To acquire a proper mathematical vocabulary and to understand its terminology is a vital asset to the student. Too often he can imitate mechanical manipulation but has not the command of language to explain the operations he has performed. A lucid explanation of a topic is the best test of one's comprehension of it.

The content of the freshman and sophomore courses in practically every college will include some trigonometry, analytic geometry, and calculus, or at least two of these subjects. Since the trigonometric functions are usually defined by means of rectangular coördinates this is a good opportunity to teach the student the proper use of directed line segments. The student will grasp the idea more quickly if he is required to read the segment as directed and not refer to it by a single symbol, for in the latter case he is oblivious to its directed sense. Trigonometric identities and inverse trigonometric functions should be given proper emphasis. Although the latter topic is frequently neglected, the best argument in its favor is summed up in the quotation: "One does not fully understand the significance of an operation until one has considered its inverse."

Many topics arise in any subject which give the teacher opportunities to present supplementary information and to unfold the subject in its true perspective, but the richest field in which to interest even the average normal student is the calculus. Virtually every topic both in its development and in its applications has some features which help to give the student a conception of its value. Without becoming pedantic a teacher can present the subject logically, and with but few exceptions, rigorously. Common sense should dictate these exceptions depending on the maturity of the students. Differentiation means much more to the student if he has grasped the essential idea of limits and the nature of the derivative. Above all, the teacher should be alert for material and problems which might be classed under "training for transfer," and frequently he can guide the student to discover for himself important properties of the subject. Such objectives are far more stimulating to the student than the mere repetition of mechanical operations.

Many teachers are enthusiastic for the so-called combination courses and object to having mathematics taught in separate sub-divisions, or as they frequently express it, in "water-tight compartments." Being an abstract subject, the average student must exert real effort to gain a comprehensive knowledge of almost any course in mathematics. Hence it would seem advisable to direct the student's efforts along proper channels, rather than spread them out in every

direction simultaneously under the magic name of function. *The essential thing is whether or not the teacher is in a "water-tight compartment."* The teacher who cannot interest a class of students in trigonometry, in analytic geometry, or in the calculus will be much less of a success in a course which combines all three subjects. On the other hand a good combination text is preferable to poor texts in the separate subjects, especially if the teacher is limited to the material between the covers of the text. The choice in general should depend on the individual teacher and on the objectives of the course.

The development of the combination course is distinctly of the type of plan B in Klein's classification, and one danger in such courses, particularly in courses for secondary schools, is that the logical and rigorous presentation of the topics will give way to an intuitional description. It seems that Klein sensed a danger in carrying his proposed changes too far, for in the preface to his book mentioned previously he says: "I have endeavored here as always to combine geometric intuition with the precision of arithmetic formulas." That is, geometric intuition should supplement, not supplant a logical and exact development of the subject. How stimulating to have a precise demonstration enriched with geometric intuitions, physical applications, and philosophical implications! How ineffective and shallow the embellishments become when given superficially without the supporting foundation!

Our most intelligent efforts should be directed toward stimulating our best students, but on the other hand the average student should not be neglected on the plea that it is giving too much attention to the poor student. Although many who enter college are not capable of being educated, nevertheless the great majority of college students, perhaps 80% of the whole, are capable of being instructed and usefully trained in one or even two years of college mathematics. At the present time more and more mathematics at the higher levels is being demanded by science for our modern civilization, simultaneously less and less of it is being offered at the lower levels. Although our students as a rule are poorly prepared, it is quite possible that the solution of the difficulty rests in part with us. An improvement in undergraduate instruction, including more real teaching, and perhaps less talking on the part of the instructor, will turn better teachers of mathematics into the secondary schools in a few years; and if these in turn emphasize the same quality and thoroughness of instruction we may eventually succeed in getting arithmetic taught properly in the elementary schools.

In closing let me summarize by saying that the movement to modernize and improve the instruction in mathematics in the secondary schools in this country was engulfed by the unprecedented wave of mass education since the beginning of the present century. It is idle to argue at this time whether or not the general situation would have been improved if there had been no attempt at revision. Certainly every college teacher wishes that his freshmen had a more thorough understanding, and a greater facility in performing the fundamental operations of arithmetic and introductory algebra. The problem of secondary instruction

is a serious one, and who is wise enough to suggest how educators, school superintendents, and politicians can be made to realize the intrinsic values in the study of mathematics? The new type of examination of the College Entrance Examination Board in mathematics may help in a selected group of schools which are necessarily sensitive to the Board's requirements. Nor must we overlook the fact that some students enter with excellent preparation, and a selected few are given advanced standing.

The critical problems of undergraduate instruction are in the freshman and sophomore years, and these should receive more serious consideration of our departments. We take pride in the brilliant record which our younger mathematicians are making in research, but to support this superstructure we must create among the intelligent public a broader interest in mathematics and a better appreciation of its values. An important factor here will be the attitude of the college trained man in the community, and this will become more tolerant, or even enthusiastic, if we can improve the quality of the undergraduate instruction. Never before has there been available for all teachers such an abundance of excellent material to supplement and broaden our knowledge of mathematics. Without introducing superficial or special courses, and without lowering standards, the problems of undergraduate instruction can be met with the effective teaching of real mathematics.

ON THE OSGOOD-CARATHÉODORY THEOREM

By E. J. McSHANE, University of Virginia

One of the most interesting and important theorems in the theory of functions of a complex variable is the Riemann mapping theorem. For this theorem there are several proofs suitable for class use. However, for some applications (in particular in constructing the modular function so useful in proving the theorems of Picard and Schottky) it is necessary to know that the function which maps the interior of a Jordan region conformally on the interior of a circle can be extended continuously to the boundary. For class-room use, the proof here given of this theorem (the Osgood-Carathéodory theorem) seems more suitable than any which I have found in the literature.*

THEOREM: *Let the function $w=f(z)$ map the unit circle $|z|<1$ in a one-to-one conformal way on a region R of the w-plane bounded by a Jordan curve J. Then the function $f(z)$ can be assigned values on $|z|=1$ in such a way that $f(z)$ is continuous on the closed circle $|z|\leqq 1$, and maps $|z|\leqq 1$ in a one-to-one way on the closed region $R+J$, while mapping the circumference $|z|=1$ in a one-to-one way on J.*

* Other proofs of this theorem can be found in various books and articles; for example: C. Carathéodory, Theory of Conformal Mapping (No. 28 of the Cambridge Tracts), pp. 81–86; L. Bieberbach, Funktionentheorie, vol. II, pp. 17–30; Hurwitz-Courant, Funktionentheorie, pp. 400–405. Two especially interesting proofs are obtained by Jesse Douglas (Transactions of the American Mathematical Society, vol. XXXIII, 1931, p. 263) and R. Courant (paper to be published in Annals of Mathematics) as by-products of their solutions of the problem of Plateau.

Let us first notice that if a region B of the z-plane is mapped conformally and one-to-one on a region R of the w-plane, and $z_1, z_2, \cdots$ is a sequence of points of B tending to a boundary point of B, then all the accumulation points of the sequence $f(z_i)$ lie on the boundary of R. For let w_0 be any point of R. There is a point z_0 of B such that $f(z_0)=w_0$. Since z_i tends to a point different from z_0, there is a neighborhood U of z_0 contained in B and containing only a finite number of the z_i. Hence in the w-plane the image W of U contains only a finite number of the images $f(z_i)$. But W is a neighborhood of w_0, so w_0 is not an accumulation point of the $f(z_i)$. That is, no interior point of R is an accumulation point of the $F(z_i)$. Thus our statement is established.

We now establish a lemma.

LEMMA. *Let C be a Jordan arc lying in $|z|<1$ except for its ends, which are on $|z|=1$. Suppose that as z tends along C to each end of C the function $f(z)$ tends to a limit. Then these two limits are distinct.*

Suppose, on the contrary, that as z tends to either end of C, the function $f(z)$ tends to one and the same limit w_0. As we have noticed, w_0 is on J; no other point of the image of C is on J. Moreover, no two distinct points of C, except the ends, have the same image; so the image of C is a Jordan curve. This curve bounds a Jordan region R_1 contained in R. Then R_1 is the image of one of the two regions into which C cuts the circle $|z|\leqq 1$; we call this portion K_1. In K_1, let z_i be any sequence of points approaching any point of the arc of $|z|=1$ which bounds K_1. The accumulation points of the sequence $f(z_i)$ all lie on the boundary of R_1 and also on the boundary of R. There is only one such point, w_0; hence $f(z_i)\rightarrow w_0$.

Hence as z tends to any point of the arc of $|z|=1$ bounding K_1, $f(z)$ tends to w_0. Thus $f(z)$ is continuous and constant on this arc. By Schwarz's reflection principle, it is analytic on this arc; and being constant on the arc, it is a constant function: $f(z)\equiv w_0$. This is impossible, since $f(z)$ maps $|z|<1$ on all of R.

We now proceed to the proof of the theorem. Let z_0 be any point of $|z|=1$; we wish first to show that for any sequence $z_i\rightarrow z_0$, the values $f(z_i)$ tend to a limit.

The (finite) area of R is given by

$$A = \iint |f'(z)|^2 dxdy,$$

where $z=x+iy$ and the integration is over the region $x^2+y^2<1$. Let us write $z=(z-z_0)+z_0=re^{i\theta}+z_0$; then $f(z)$ is a function of r and θ, which can be written $u(r,\theta)+iv(r,\theta)$. We can then evaluate $f'(z)$ by holding r constant and letting θ vary; we get

$$\frac{df}{d\theta} = u_\theta + iv_\theta, \qquad \frac{dz}{d\theta} = rie^{i\theta}, \qquad \frac{df}{dz} = -\, i(u_\theta + iv_\theta)r^{-1}e^{-i\theta},$$

$$|f'(z)|^2 = (u_\theta^2 + v_\theta^2)r^{-2}.$$

Consequently, changing the integral expression for A to polar coördinates, we obtain

$$A = \iint_{|z|<1} (u_\theta^2 + v_\theta^2) r^{-2} \cdot r dr d\theta.$$

From this we wish to show that it is possible to find a sequence of positive numbers r_n tending monotonically to zero (i.e., $r_{m+1} < r_m$ for every m) such that

$$\lim_{n\to\infty} \int_{C_{r_n}} (u_\theta^2 + v_\theta^2) d\theta = 0, \tag{1}$$

where C_{r_n} means the arc of the circle $|z - z_0| = r_n$ lying in the unit circle $|z| \leqq 1$. Here we consider the integrals to be Lebesgue integrals; the proof can be carried through with Riemann integrals, but is longer.

Expressing A as an iterated integral,

$$A = \int_0^1 \left\{ \int_{C_r} (u_\theta^2 + v_0^2) d\theta \right\} r^{-1} dr.$$

If ϵ be any positive number, then there are values of r less than ϵ such that the integral over C_r of $(u_\theta^2 + v_\theta^2)$ is less than ϵ. Otherwise we would have

$$A \geqq \int_0^\epsilon \{\epsilon\} r^{-1} dr = \infty.$$

Hence, choosing a sequence ϵ_n tending to 0, we can find corresponding r_n such that $r_n < \epsilon_n$ and (1) holds.

Now by Schwarz's inequality,

$$\left\{ \int_{C_{r_n}} \sqrt{u_\theta^2 + v_\theta^2}\, d\theta \right\}^2 \leqq \int_{C_{r_n}} (u_\theta^2 + v_\theta^2) d\theta \int_{C_{r_n}} 1 d\theta.$$

The first factor on the right tends to zero by the choice of r_n, and the second factor does not exceed 2π; hence the left member of the inequality tends to zero as $n \to \infty$. That is, the image of C_{r_n} has a finite length L_n which tends to zero as $n \to \infty$. This implies at once that as z tends to either end of C_{r_n}, the point $f(z)$ tends to a unique limit; for otherwise the image of C_{r_n} would have to travel back and forth infinitely many times between the neighborhoods of different accumulation points, and would run up an infinite length in the process. Let these two limits (distinct by our lemma) be w_n and W_n. The image of C_{r_n} divides R into two Jordan regions; we choose the notation so that the counterclockwise arc (w_n, W_n) bounds that region R_n corresponding to $|z - z_0| < r_n$. Since w_n and W_n are joined by a curve of length L_n, we have

$$| w_n - W_n | \leqq L_n, \qquad \lim_{n\to\infty} L_n = 0. \tag{2}$$

If we fix on any n, then, with possibly a finite number of exceptions, the points z_i are in $|z - z_0| < r_n$, so the $f(z_i)$ are in R_n. All accumulation points of the

$f(z_i)$ are on the boundary of R_n, and also on the boundary of R; therefore all such accumulation points are in the common part of the boundaries, namely the arc (w_n, W_n). This is true for all n; hence all accumulation points of the $f(z_i)$ belong to all intervals (w_n, W_n). If we now show that there is only one point w_0 common to all the intervals, then all accumulation points of the $f(z_i)$ will be w_0, so that $f(z_i)$ will tend to the unique limit w_0.

Consider then any two arcs C_{r_n} and $C_{r_{n+1}}$. Since the latter is in the region $|z-z_0|<r_n$, its image will be in R_n, and the end-points will be on the boundary of R_n as well as on the boundary of R, and therefore in (w_n, W_n). That is, the points w_n, w_{n+1}, W_{n+1}, W_n occur in that (counterclockwise) order. So the intervals (w_n, W_n) form a steadily shrinking sequence of intervals. The common part will be either a single point w_0 or an interval (w, W). In the latter case, each w_n will lie in the arc (w_1, w) and each W_n in the arc (W, W_1). These arcs are disjoint closed sets and have a positive distance δ. Therefore, $|w_n-W_n|\geqq\delta$ for every n. But by (2), $|W_n-w_n|\leqq L_n$, which is less than δ if n is large enough. This contradiction proves that the common part of the intervals (w_n, W_n) is a single point w_0, and the existence of $\lim_{z\to z_0}f(z)$ is established.

Let us now define $f(z_0)=\lim_{z\to z_0}f(z)$. If we use the neighborhood definition of limit, we see that if z_0 is any point of $|z|=1$, then for every $\epsilon>0$ there is a $\delta>0$ such that $|f(z)-f(z_0)|\leqq\epsilon$ if z is in the set U defined by the inequalities $|z-z_0|\leqq\delta$ and $|z|<1$. But for all points z with $|z|=1$ and $|z-z_0|\leqq\delta$, the value of $f(z)$ is defined as the limit of values of $f(z)$ lying in U; so for all such z we have $|f(z)-f(z_0)|\leqq\epsilon$. That is, $f(z)$ is continuous on the closed circle $|z|\leqq 1$.

We have so far shown that to each point of $|z|=1$ corresponds exactly one point of J. Conversely, let w_0 be any point of J. We can find a sequence of points w_i in B tending to w_0. For each w_i there is a z_i such that $w_i=f(z_i)$. These z_i have a point of accumulation z_0 on $|z|=1$, and for a subsequence z_α we have $z_\alpha\to z_0$. Then by the continuity of $f(z)$ we know that $f(z_\alpha)\to f(z_0)$. But $f(z_\alpha)\to w_0$; therefore $w_0=f(z_0)$. So to each point of J there corresponds at least one point of $|z|=1$. Suppose now that there are two points z_1, z_2 of $|z|=1$ such that $f(z_1)=f(z_2)=w_0$. We join z_1 and z_2 by a line segment. As z tends to the end point z_i of this segment, by continuity $f(z)$ tends to $f(z_i)$. By our lemma, $f(z_1)$ and $f(z_2)$ are distinct. This contradiction proves that to each w_0 of J there corresponds exactly one point z of the circumference $|z|=1$. Hence the function $f(z)$ maps $|z|=1$ in a one-to-one continuous way on J, while mapping $|z|\leqq 1$ in a continuous one-to-one way on the closed region $B+J$ and mapping $|z|<1$ conformally on B.

"A mathematical science is any body of propositions which is capable of an abstract formulation and arrangement in such a way that every proposition of the set after a certain one is a formal logical consequence of some or all the preceding propositions. Mathematics consists of all such mathematical sciences." Charles Wesley Young, *Fundamental Concepts of Algebra and Geometry*, New York, 1911, p. 222.

AN ANALYTIC STUDY OF THE NON-PERSPECTIVE PICTURIZATION OF QUADRIC SURFACES

By NEIL LITTLE, Purdue University

1. *Introduction.* The problem of the picturization of any object is the problem of the determination of what lines to draw in a plane in order that the impression conveyed to the eye shall be that of a three dimensional object. One way by which this may be accomplished is by projecting upon a plane certain strategic lines or curves of the object and drawing on the paper those projections. To picture some surface whose equation is known we would picture the projections of the traces of the surface in the coördinate planes and parallels thereto, and the projection of any other curve on the surface which seemed desirable.

Certain definitions are here in order. The projection of a curve in space upon a plane is the intersection of the plane and a cylindrical surface each of whose elements passes through a point on the curve. The plane is called the plane of projection, or the projection plane, and the cylindrical surface is called the projecting cylinder. If the elements of the projecting cylinder are normal to the projection plane the projection is said to be orthographic; if not, the projection is oblique.

2. *Oblique projections.* Oblique projections on a plane parallel to a coördinate plane will be considered. Let us suppose the plane of projection to be the plane whose equation is $y-k=0$, where k is any constant, and that the elements of the projecting cylinder are parallel to the line whose directions cosines are l, m, and n respectively. Such a line would be the line whose equations are

$$x/l = y/m = z/n. \tag{1}$$

2.1. *Projections of the coördinate axes.* The plane containing the x-axis and the line (1) has the equation

$$z = ny/m. \tag{2}$$

The intersection of this plane and the plane of projection is the line into which the x-axis projects. The equations of this line are

$$z = nk/m, \qquad y = k. \tag{3}$$

Similarly, the equations of the projections of the y- and z-axes are found to be, respectively,

$$x = lz/n, \qquad y = k, \tag{4}$$

and

$$x = lk/m, \qquad y = k. \tag{5}$$

From these it is seen that, if the plane of projection is taken as the plane of the paper, the projections of the coördinate axes will appear as shown in Figure 1.

2.2. *Projection of the point* (a, b, c). Consider the line

$$(x - a)/l = (y - b)/m = (z - c)/n. \tag{6}$$

This line passes through the point (a, b, c) and is parallel to the line (1). The point where this line intersects the projection plane will be the projection of the point (a, b, c) on that plane. Setting $y=k$ in (6), we obtain

$$(x - a)/l = (k - b)/m = (z - c)/n.$$

Hence

$$x = a + l(k - b)/m, \qquad y = k, \qquad z = \epsilon + n(k - b)/m.$$

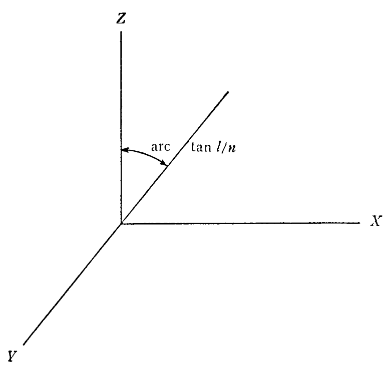

FIG. 1

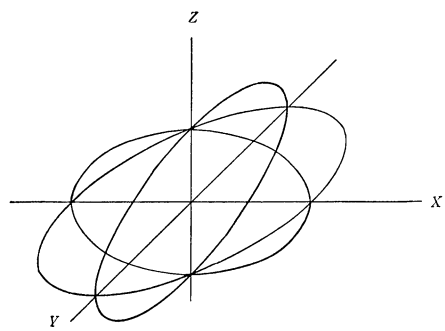

FIG. 2

These are the coördinates in space of the projection of (a, b, c). To find the position of this point relative to the projections of the x- and z-axes, make the transformation

$$x = X + lk/m, \qquad y = Y, \qquad z = Z + nk/m. \tag{A}$$

There results

$$X = (ma - lb)/m, \qquad Y = k, \qquad Z = (mc - nb)/m. \tag{7}$$

The first and third of these are the distances of the projection of the point from the projections of the z- and x-axes respectively, and thus are the coördinates of the projection of the point with respect to the projections of the axes.

We proceed now to find the equations of the projections of various curves in the various coördinate planes and parallels thereto *with respect to the projections of the x- and z-axes.* To distinguish between a point in space and its projection we shall use capital letters to represent the coördinates of the projection.

2.3. *Projection of a curve in the xz-plane.* From the above it is seen that any point $(x, 0, z)$ will project into a point (X, Z) where $x=X$, and $z=Z$. Hence any curve whose equations are

$$f(x, z) = 0, \qquad y = 0$$

will project into a curve identical to itself and situated in the same position relative to the projections of the x- and z-axes as the original is situated relative to the x- and z-axes themselves. This fact constitutes one of the major advantages of oblique projections over orthographic projections.

2.4. *Projection of a curve in the xy-plane.* Consider the surface

$$
\begin{aligned}
a(x - lz/n)^2 + b(x - lz/n)(y - mz/n) + c(y - mz/n)^2 \\
+ d(x - lz/n) + e(y - mz/n) + f = 0.
\end{aligned} \tag{8}
$$

This is the equation of a cylindrical surface whose elements are parallel to the line (1) and whose trace in the xy-plane is the curve

$$ax^2 + bxy + cy^2 + dx + ey + f = 0, \qquad z = 0. \tag{9}$$

The intersection of the surface (8) and the plane of projection is the projection of the curve (9) in the plane. Setting $y = k$ in (8) and making transformation (A), we obtain

$$
\begin{aligned}
aX^2 - (2al + bm)XZ/n + (al^2 + blm + cm^2)Z^2/n^2 \\
+ dX - (dl + em)Z/n + f = 0.
\end{aligned} \tag{10}
$$

The discriminant of (10) is $(b^2 - 4ac)m^2/n^2$. Since this will always have the same sign as the discriminant of (9), it is evident that any conic in the xy-plane will project into a conic of the same kind as itself, except when $lmn = 0$.

2.5. *Projection of a curve in the yz-plane.* Consider the surface

$$
\begin{aligned}
c(y - mx/l)^2 + p(y - mx/l)(z - nx/l) + h(z - nx/l)^2 \\
+ e(y - mx/l) + i(z - nx/l) + f = 0.
\end{aligned} \tag{11}
$$

This is the equation of a cylindrical surface whose elements are parallel to the line (1) and whose trace in the yz-plane is the curve

$$cy^2 + pyz + hz^2 + ey + iz + f = 0, \qquad x = 0. \tag{12}$$

To get the projection of the curve (12) in the plane of projection we set $y = k$ and make the transformation (A). There results

$$
\begin{aligned}
(cm^2 + pmn + hn^2)X^2/l^2 - (pm + 2hn)XZ/l + hZ^2 \\
- (em + ni)X/l + iZ + f = 0.
\end{aligned} \tag{13}
$$

The discriminant of (13) is $(p^2 - 4hc)m^2/l^2$, which will always have the same sign as the discriminant of (12) and a conclusion similar to the one drawn in the last paragraph may be drawn.

2.6. *Projections of curves in planes parallel to the coördinate planes.* The equations of a curve in a plane parallel to a coördinate plane are like those of a curve in a coördinate plane except that one of the equations instead of being x, y, or z equals zero, is x, y, or z equals some constant. If a transformation is made carrying the origin to the point where the parallel plane cuts an axis, and the projection of this point found, the problem reduces to that of one of the cases considered above.

2.7. *The curve of visibility.* Pictured in Figure 2 are the projections of the traces in the coördinate planes of the ellipsoid

$$x^2/25 + y^2/9 + z^2/16 = 1,$$

with l, m, and n all equal to $1/\sqrt{3}$. It is apparent that, as a picture, there is distinctly something lacking. That something is a bounding, or outline curve, or, as it is sometimes called, a curve of visibility. This curve will be the intersection of the plane of projection and a cylindrical surface whose elements are parallel to the line (1) and are tangent to the surface in question. The general equation of this surface is*

$$4F(x, y, z)f(l, m, n) = \left(l\frac{\partial F}{\partial x} + m\frac{\partial F}{\partial y} + n\frac{\partial F}{\partial z}\right)^2, \tag{14}$$

where $F(x, y, z)=0$ is the equation of the quadric surface in question and $f(x, y, z)$ is a function made up of those terms of $F(x, y, z)$ which are of second degree. If we set $y=k$ and make the transformation (A) the equation of the curve of visibility may be found. The general equation is long and cumbersome and will not be worked out here. A few special cases will be considered.

The curve of visibility of the surface

$$ax^2 + cy^2 + hz^2 = 1 \tag{15}$$

is

$$a(cm^2 + hn^2)X^2 - 2ahlnXZ + h(al^2 + cm^2)Z^2 = (al^2 + cm^2 + hn^2). \tag{16}$$

The identification of this curve leads to some interesting results. The discriminant of (16) is

$$-4achm^2(al^2 + cm^2 + hn^2).$$

Several cases may be considered, as follows:

(1) *Ellipsoid.* If a, h, and c are all positive the discriminant is negative, and the curve (16) is an ellipse.

(2) *Hyperboloid of one sheet.* Suppose for definiteness that a and c are positive and h is negative. There are three possible cases.

Case I. If $|al^2+cm^2| > |hn^2|$, the discriminant is positive, and the curve (16) is a hyperbola. See Plate IV.

* Cf. R. J. T. Bell, Coördinate Geometry of Three Dimensions, 1912, Art. 137.

Case II. If $|al^2+cm^2|=|hn^2|$, the discriminant is zero. However, the right hand side of (16) is also zero, and the curve is a degenerate parabola, or a pair of coincident straight lines. See Plate V.

Case III. If $|al^2+cm^2|<|hn^2|$, the discriminant is negative, and the curve (16) is an ellipse. In this case we can "see" down inside the surface. See Plate VI.

Conclusions analogous to those above may be drawn for the cases where any two of the constants a, c, and h are positive and the other one negative.

(3) *Hyperboloid of two sheets.* Suppose a is positive and c and h are negative. There are again three cases.

Case I. If $|al^2|<|cm^2+hn^2|$, the discriminant is positive, and the curve of visibility is a hyperbola.

Case II. If $|al^2|=|cm^2+hn^2|$, the discriminant is zero, but the right hand side is zero, and the curve is again a pair of coincident straight lines.

Case III. If $|al^2|>|cm^2+hn^2|$, the discriminant is negative and the curve is an ellipse. This ellipse may be proved to be imaginary, as follows. If the XZ term is removed from the equation (16) by a suitable transformation, the resulting equation is

$$\begin{aligned}(17)\quad &\frac{2achm^2X'^2}{a(cm^2+hn^2)+h(al^2+cm^2)-\left[4a^2h^2l^2n^2+\left[h(al^2+cm^2)-a(cm^2+hn^2)\right]^2\right]^{1/2}}\\ &+\frac{2achm^2Z'^2}{a(cm^2+hn^2)+h(al^2+cm^2)+\left[4a^2h^2l^2n^2+\left[h(al^2+cm^2)-a(cm^2+hn^2)\right]^2\right]^{1/2}}=1.\end{aligned}$$

By hypothesis $a>0$, $c<0$, $h<0$, and $|al^2|>|cm^2+hn^2|$. From these follow directly that $2achm^2>0$, $a(cm^2+hn^2)<0$, and $|al^2|>|cm^2|$. Hence $(al^2+cm^2)>0$, and so $h(al^2+cm^2)<0$. From these it follows that the coefficient of X'^2 is negative. Since the curve is known to be an ellipse, the coefficient of Z'^2 must also be negative. But the right side is positive. The ellipse is, then, an imaginary one.

Conclusions analogous to those above may be drawn for the cases where any one of the constants a, c, or h is positive and the other two negative.

2.8. *The plates.* In Plate I is shown the same ellipsoid as was used for Figure 2, but with the curve of visibility drawn in and with appropriate parts of the curves dotted to show that they are invisible. In Plates II–VI are shown various other quadric surfaces, with specifications as given.

The curve of visibility of the surface

$$ax^2+cy^2+iz=0,$$

is

$$4acm^2X^2-4alniX+4(al^2+cm^2)iZ-n^2i^2=0.$$

This curve is a parabola.

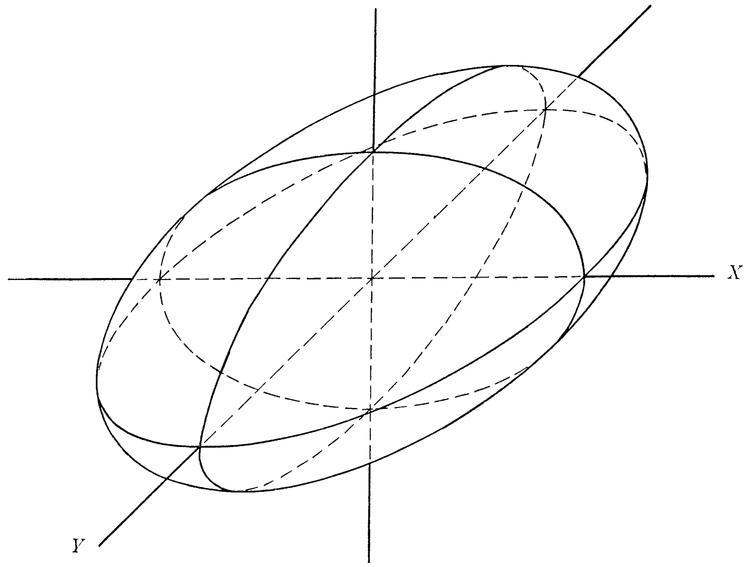

Plate I. Oblique projection of the surface $x^2/25+y^2/9+z^2/16=1$, taking $l=m=n=1/\sqrt{3}$.

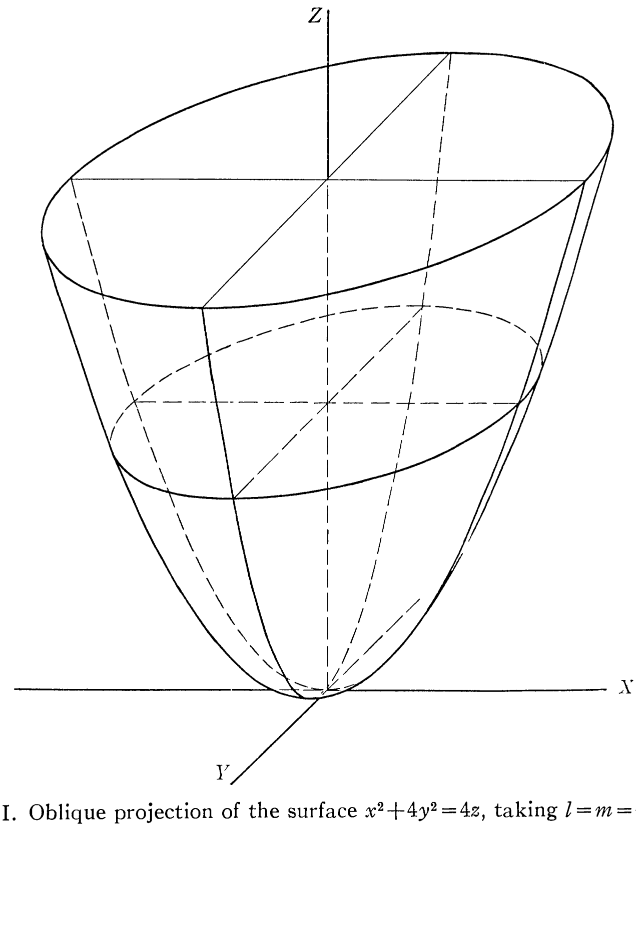

Plate II. Oblique projection of the surface $x^2+4y^2=4z$, taking $l=m=n=1/\sqrt{3}$.

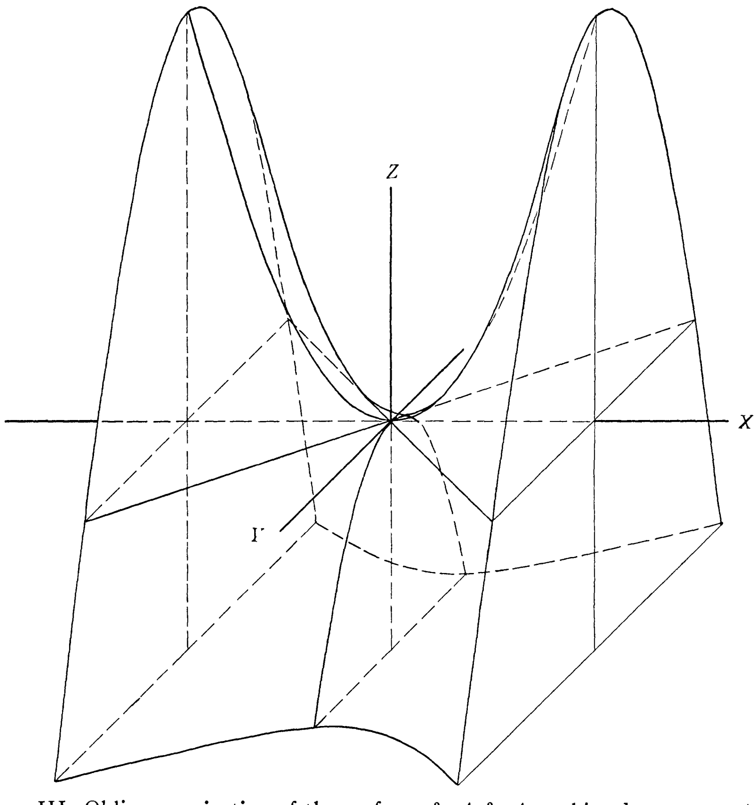

PLATE III. Oblique projection of the surface $x^2-4y^2=4z$, taking $l=m=n=1/\sqrt{3}$.

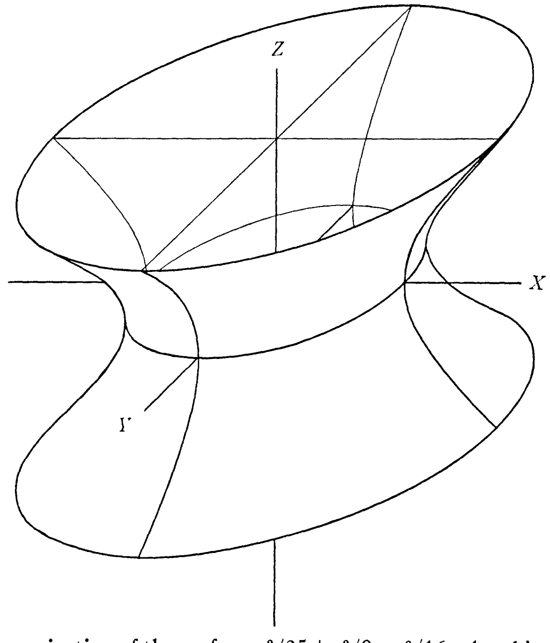

PLATE IV. Oblique projection of the surface $x^2/25+y^2/9-z^2/16=1$, taking $l=m=n=1/\sqrt{3}$.
No hidden lines shown.

3. *Orthographic projections.* The elements of the projecting cylinder will again be taken parallel to the line (1). The plane of projection is then the plane

$$lx + my + nz = k,$$

where k is any constant. The following transformation will be useful:

$$\text{(B)} \qquad \begin{aligned} x &= (Xnl - Ym + Zl\sqrt{m^2 + l^2})/\sqrt{m^2 + l^2}, \\ y &= (Xmn + Yl + Zm\sqrt{m^2 + l^2})/\sqrt{m^2 + l^2}, \\ z &= Zn - X\sqrt{m^2 + l^2}. \end{aligned}$$

This transformation rotates the z-axis into the line (1) and rotates the x- and y-axes into positions so that their projections on the plane of projection will be as shown in Figure 3. The equations of the projections of the various curves with reference to these axes will be found.

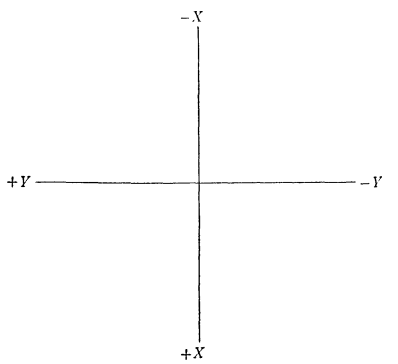

FIG. 3

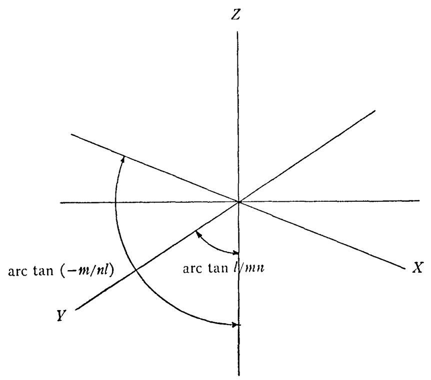

FIG. 4

3.1. *Projections of the coördinate axes.* If the transformation (B) is applied to the equation (2), the result is

$$Y = -mX/nl. \tag{18}$$

This is the equation of the projection of the x-axis. In similar fashion the equations of the projections of the y- and z-axes are found to be, respectively,

$$Y = lX/mn, \tag{19}$$

and

$$Y = 0. \tag{20}$$

Hence the axes will appear as shown in Figure 4.

It will be observed that if l, m, and n are known, the positions of the projections of the axes may be found, and conversely, if the positions of the projections of the axes are known, the values of l, m, and n may be found.

3.2. *Projection of the point* (a, b, c). Applying transformation (B) to the equation (6) and simplifying, we obtain

$$X = \frac{aln + bmn - c(l^2 + m^2)}{\sqrt{l^2 + m^2}}, \qquad Y = \frac{bl - am}{\sqrt{l^2 + m^2}}. \tag{21}$$

These are the coördinates of the point into which (a, b, c) projects.

3.3. *Projection of a curve in the xy-plane.* Applying transformation (B) to the equation (8), we obtain

$$\begin{aligned} &\frac{(al^2+blm+cm^2)X^2+[2cmnl+bn(l^2-m^2)-2amnl]XY+n^2(am^2-blm+cl^2)Y^2}{n^2(m^2+l^2)} \\ &\qquad + \frac{(ld+em)X}{n\sqrt{l^2+m^2}} + \frac{(le-md)Y}{\sqrt{l^2+m^2}} + f = 0. \end{aligned} \tag{22}$$

This is the equation of the curve into which the curve (9) projects. The discriminant of (22) is $n^2(l^2+m^2)^2(b^2-4ac)$. This will always have the same sign as the discriminant of (9), and hence the two curves are of the same type.

3.4. *Projections of curves in the other coördinate planes.* These may be found in a fashion similar to the preceding discussion. The curve

$$ax^2 + gxz + hz^2 + dx + iz + f = 0, \qquad y = 0, \tag{23}$$

projects into the curve

$$\begin{aligned} &\frac{hm^2X^2 + m[2hln + g(m^2+l^2)]XY + [a(m^2+l^2)^2 + gln(m^2+l^2) + hl^2n^2]Y^2}{m^2(l^2+m^2)} \\ &\qquad - \frac{iX}{\sqrt{l^2+m^2}} - \frac{(dm^2+dl^2+iln)Y}{m\sqrt{l^2+m^2}} + f = 0. \end{aligned} \tag{24}$$

Also the curve (12) projects into the curve

$$\begin{aligned} &\frac{hl^2X^2 - l[2hmn + p(m^2+l^2)]XY + [c(m^2+l^2)^2 + pmn(m^2+l^2) + hm^2n^2]Y^2}{l^2(m^2+l^2)} \\ &\qquad - \frac{iX}{\sqrt{m^2+l^2}} + \frac{(em^2+el^2+imn)Y}{l\sqrt{l^2+m^2}} + f = 0. \end{aligned} \tag{25}$$

3.5. *Projections of curves in planes parallel to the coördinate planes.* As in the oblique projections, a transformation can be made carrying the origin to the point where the parallel plane cuts an axis, the projection of this point found, and the results of the previous cases applied.

3.6. *Curve of visibility*. By applying transformation (B) to equation (14) the general equation of the curve of visibility may be found. We shall confine our work here to a couple of special cases. The curve of visibility of surface (15) is

$$
\begin{aligned}
(26)\qquad & h(al^2 + cm^2)X^2 + 2hlmn(c - a)XY + [ac(l^2 + m^2)^2 + hn^2(am^2 + cl^2)]Y^2 \\
& \qquad = (al^2 + cm^2 + hn^2)(l^2 + m^2).
\end{aligned}
$$

The discriminant of (26) is $-4ach(l^2+m^2)^2(al^2+cm^2+hn^2)$. A comparison of this discriminant with that of equation (16) shows that, so far as the type of the curve is concerned, precisely the same cases will arise with precisely the same results here as did previously. Also in the case of the hyperboloid of two sheets, the ellipse arising in Case III can be proved imaginary.

The curve of visibility of the surface

$$ax^2 + cy^2 + iz = 0$$

is

$$(27)\quad 4ac(l^2 + m^2)^{3/2}Y^2 + 4lmni(a - c)Y - 4i(al^2 + cm^2)X - n^2i^2\sqrt{l^2 + m^2} = 0.$$

The curve is evidently a parabola. An interesting property of this curve is that it is tangent to the projection of the trace of the surface in the yz-plane, and the common tangent to the two curves is parallel to the projection of the x-axis. Similarly, the curve of visibility and the projection of the trace of the surface in the xz-plane have a common tangent which is parallel to the projection of the y-axis.

3.7. *Equation of the projection of the curve of intersection of two surfaces*. The equation of the projection of the curve of intersection of two surfaces whose equations are known may be found by subjecting the equation of each surface to the transformation (B) and then eliminating Z between the two resulting equations.

3.8. *Isometric, dimetric, and trimetric projections*. An isometric projection may be defined as an orthographic projection on a plane $x+y+z=k$, where k is any constant. For this case $l=m=n=1/\sqrt{3}$. The name arises from the fact that equal distances on the coördinate axes or parallels thereto project into equal distances.

A dimetric projection is an orthographic projection on a plane $lx+my+nz=k$, where k is any constant and some pair of the three direction cosines, l, m, and n are equal. An infinite number of combinations of l, m, and n can be found for such a condition, a common special one being that in which $l=1/3$, $m=\sqrt{7}/3$, and $n=1/3$. For this case, equal distances on the coördinate axes project into distances having the ratio 2 to 1 to 2.

A trimetric projection is an orthographic projection in which no two of the constants l, m, and n are equal. Such a projection is not commonly used.

3.9. *The figures*. In Plates VII–X some pictures of various quadric surfaces are shown, with specifications as given.

4. *Concluding remarks*. The advantages and disadvantages of the various types of projection will be apparent to the reader. The algebra of the oblique projection is less complicated than that of the orthographic and so also is the

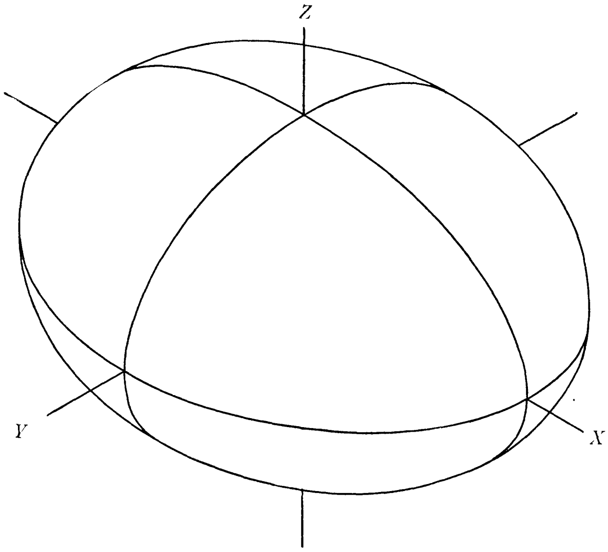

PLATE VII. Isometric projection of the surface $x^2/25+y^2/16+z^2/9=1$.

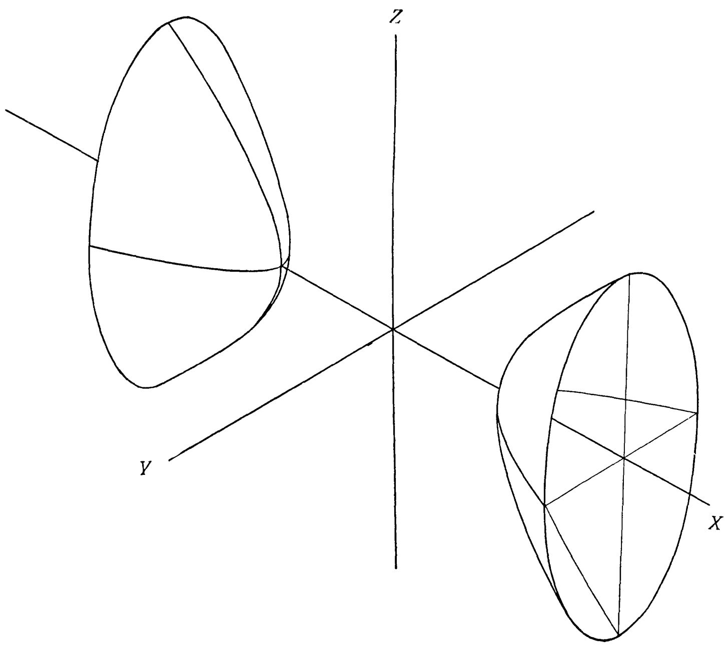

PLATE VIII. Isometric projection of the surface $x^2/25-y^2/4-z^2/16=1$.

final result. The oblique projection gives, however, a more distorted and so less realistic picture than the orthographic. This distortion is particularly impressed on one when it is realized that the curve of visibility of a sphere in oblique projection is an ellipse. Lack of realism is evident also from the fact that it is impossible for the eye to be so situated that two of the axes would appear at right angles, with the other one oblique to each of the first two. It perhaps should be further observed that all of the pictures lack perspective. To achieve that quality, a projecting cone would have to be used instead of a projecting cylinder. This the author hopes to treat in a later paper.

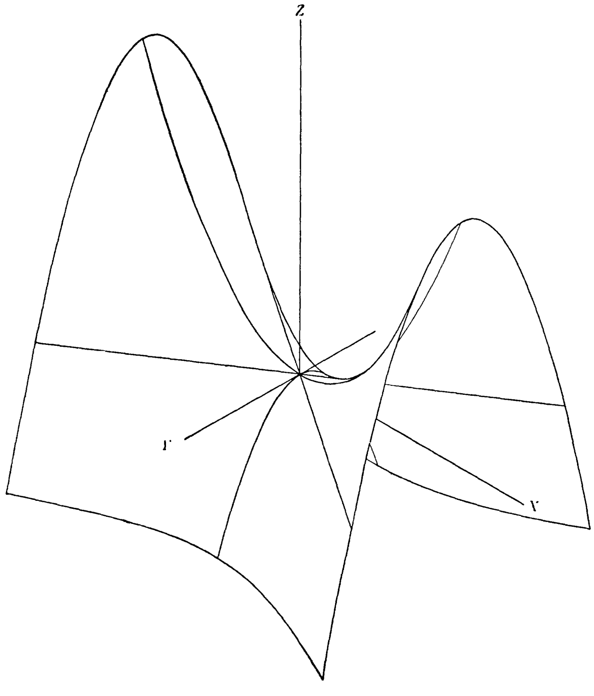

PLATE IX. Isometric projection of the surface $x^2/9-y^2/4=z$.

The reader will observe that the drawings in the various Plates connected herewith differ considerably from those which commonly appear in our analytic geometry and calculus textbooks. The writer feels that these latter constitute a serious breach on the part of our textbook writers, not so much in that the figures are makeshift and inaccurate, but in that the authors do not point out to the student that such is the case. Time and space do not ordinarily permit a textbook treatment of the subject such as is here given, but the student is entitled to know that such a treatment can be made, and if the figures are not correctly drawn, he should at least be told that they are not accurately done and told why such is the case.

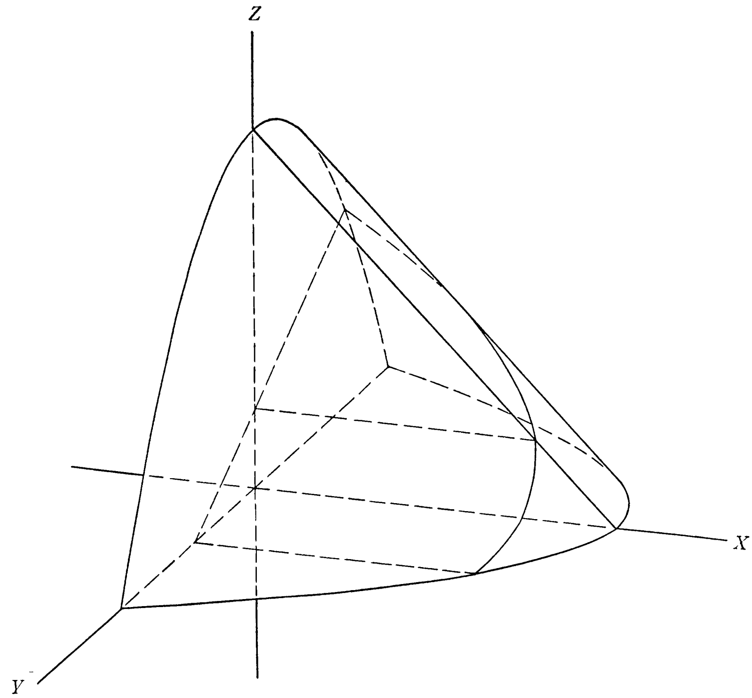

PLATE X. Dimetric projection of the surfaces $y^2+9x+9z-81=0$, and $y+2z-4=0$. $l=n=1/3$, $m=\sqrt{7}/3$.

Note by the Editor. It might be contended by an author of a textbook that the object of his drawings is to convey to the reader as simply as possible an adequate impression of the three-dimensional figure under consideration. Perhaps this can be done by some scheme other than that of using one of the projections discussed by the author of the present paper.—E.J.M.

"If it were always necessary to reduce everything to intuitive knowledge, demonstration would often be insufferably prolix. This is why mathematicians have had the cleverness to divide the difficulties and to demonstrate separately the intervening propositions. And there is art also in this; for as the mediate truths (which are called *lemmas*, since they appear to be a digression) may be assigned in many ways, it is well, in order to aid the understanding and memory, to choose of them those which greatly shorten the process, and appear memorable and worthy in themselves of being demonstrated. But there is another obstacle, viz.: that it is not easy to demonstrate all the axioms, and to reduce demonstrations wholly to intuitive knowledge. And if we had chosen to wait for that, perhaps we should not yet have the science of geometry." G. W. Leibnitz, *New Essay on Human Understanding*, Langley, bk. 4, chaps. 2, 8.

REGRESSION COEFFICIENTS AS MEANS OF CERTAIN RATIOS

By E. L. DODD, University of Texas

It has been found convenient to deal **[1, 2, 3]*** with means which may be either internal or external. Thus, the root-mean-square of -1 and -7 is usually computed as $\{[(-1)^2+(-7)^2]/2\}^{1/2}=5$. Upon noticing that 5 is not intermediate between -1 and -7 this 5 may perhaps be changed to -5. But it would seem better to recognize at once that a root-mean-square is a two-valued function. With reference to data, one value may be internal and the other external. We may, as in this present paper, wish to exhibit certain functions as means without determining their internality or externality.

Chisini [1] discarded internality as an essential feature of means used for statistical purposes, and defined means by an implicit relation. This leads to the following definition:

Definition. A function F of n measurements, $x_1, x_2, \cdots, x_n$,

$$M = F(x_1, x_2, \cdots, x_n), \tag{1}$$

is said to be a mean of these n measurements if F is so constructed that for any value c which the measurements can take on,

$$F(c, c, \cdots, c) = c, \tag{2}$$

or at least one value of F is c.

Upon the function F we place here no further restriction. It may be infinitely multiple-valued, like means encountered in mean-value theorems.

A number of authors **[4, 5, 6, 7, 8, 9]** have used this requirement (2), or a special case of it, such as $F(1, 1, \cdots, 1)=1$, as one of a set of conditions to define axiomatically some sort of mean.

If two sets of variates, $x_1, x_2, \cdots, x_n$, and $y_1, y_2, \cdots, y_n$, are each measured from their arithmetic means as origins, then the Pearsonian regression equation of y on x is, as is well known,

$$\text{Estimated} \quad y = \frac{\sum x_i y_i}{\sum x_i^2}\, x, \tag{3}$$

with the summation taken from $i=1$ to $i=n$ in each case.

It is perhaps fairly common to regard the coefficient of x in this equation as some sort of mean; e.g., a mean of slopes. At all events, it is indeed a mean of the ratios y_i/x_i expressed in the form, x_iy_i/x_i^2, if we agree to consider such a fraction as zero if $x_i=0$, or if we simply postulate that $x_i\neq 0$, a condition usually satisfied even if in the original data some zero appears. Indeed, it is well known that if from fractions a/b, c/d, $\cdots$ with positive denominators, the fraction $(a+c+\cdots)/(b+d+\cdots)$ is formed, this new fraction will be intermediate in value between the given fractions. While intermediacy is not of

* Numbers in brackets refer to references at the end of the paper.

fundamental importance in this paper, it is to be noted that this mean-forming process of adding numerators for a new numerator, and adding denominators for a new denominator does lead to this most simple form of regression coefficient, $\sum x_i y_i / \sum x_i^2$.

Suppose now that there are s variables $w, x, y, \cdots, u$ each taking on n values, $w_1, w_2, \cdots, w_n$; $x_1, x_2, \cdots, x_n$; etc., measured now from any *arbitrary* origins. And let ω be an estimate of w from the other variables $x, y, \cdots, u$, in the form of a linear equation,

$$\omega = a + bx + \cdots + ku. \tag{4}$$

Then

$$\omega_i = a + bx_i + \cdots + ku_i, \qquad (i = 1, 2, \cdots, n). \tag{5}$$

Now, with

$$S = \sum (\omega_i - w_i)^2, \tag{6}$$

let the coefficients, $a, b, \cdots, k$, be obtained by the least square method, setting:

$$0 = \partial S/\partial a = \partial S/\partial b = \cdots = \partial S/\partial k. \tag{7}$$

The s equations thus obtained are:

$$\begin{aligned} na + \sum x_i b \;\; + \cdots + \sum u_i k \;\; &= \sum w_i, \\ \sum x_i a + \sum x_i^2 b \;\; + \cdots + \sum x_i u_i k &= \sum x_i w_i, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ \sum u_i a + \sum x_i u_i b + \cdots + \sum u_i^2 k \;\; &= \sum u_i w_i. \end{aligned} \tag{8}$$

In general, these equations give unique values to $a, b, \cdots, k$, by Cramer's Rule, as quotients of two determinants. If now we arbitrarily assign to each w_i the constant value c, it will be noticed that the right members of the equations (8) become equal respectively to c times the coefficients which appear in the first column. For that particular value of the w_i's, Cramer's Rule leads to $a = c$. From the definition that has been given, it follows that a is a mean of the given w_i's, whatever they are, since the function F delivering this mean is so constructed that $F(c, c, \cdots, c) = c$.

Next, suppose that we give arbitrarily the value c to each of the ratios w_i/x_i, making $w_i = cx_i$. The right members of the s equations (8) are now respectively c times the coefficients in the second column. And thus, by Cramer's Rule, $b = c$. Hence, b is a mean of the ratios w_i/x_i. Likewise, k is a mean of the ratios w_i/u_i.

It was stated that the constant term a, as just found by the least square method, is a mean of the given w_i's. This should *not* be understood as signifying the *arithmetic* mean of the w_i's, nor even necessarily some *internal* mean. Indeed, if we fit the straight line $\omega = a + bx$ to the three points (x_i, w_i) given by (1, 0), (2, 0), and (3, 1), the result is:

$$\omega = -2/3 + x/2, \tag{9}$$

with its constant term $a = -2/3$ which is *not internal* to the w_i values, 0, 0, and 1. Here a is an *external* mean of 0, 0, and 1. Likewise, although $b = 1/2$ is a mean of the w_i/x_i ratios, 0, 0, and 1/3, this b is *not internal* to them; it is an *external* mean.

It is beyond the scope of this paper to discuss the special cases that arise when Cramer's Rule becomes inapplicable because of a vanishing determinant.

The foregoing general discussion applies at once to the *linear* regression equation of a variable w upon $(s-1)$ other variables. But the $x, y, \cdots, u$, may be functions—algebraic, trigonometric, exponential or otherwise—of certain other variables. In general, the ω will not be a linear function of these new variables; but the foregoing results may nevertheless be applied. As an illustration, we may consider:

$$\omega = \alpha + \beta x + \gamma y + \delta x^2 + \epsilon xy + \zeta y^2. \tag{10}$$

Then δ is a mean of the values w_i/x_i^2, and ϵ is a mean of the values w_i/x_iy_i, if these parameters, δ and ϵ, have been found by the least square method.

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"When the greatest of American logicians, speaking of the powers that constitute the born geometrician, had names Conception, Imagination, and Generalization, he paused. Thereupon from one of the audience there came the challenge, "What of reason?" The instant response, not less just than brilliant, was: "Ratiocination—that is but the smooth pavement on which the chariot rolls." C. J. Keyser, *Lectures on Science, Philosophy and Art*, New York, 1908, p. 31.

THE RESULTANT MATRIX OF TWO POLYNOMIALS*

By M. M. FLOOD, Princeton University

1. *Introduction.* Frobenius† has shown that if A is a matrix whose characteristic function is $A[x]$ and if $B[x]$ is a second polynomial then their resultant is the determinant of the matrix $B[A]$. In particular, if A is non-derogatory,‡ the present author§ has shown that the degree of the highest common factor of $A[x]$ and $B[x]$ is the same as the nullity‖ of $B[A]$.

In this paper the matrix A is taken to be the companion matrix¶ of $A[x]$, which is non-derogatory, and it is shown how the highest common factor of $A[x]$ and $B[x]$ can be found from their "resultant matrix" $B[A]$. A numerical example is then given to illustrate this result and an extension of it which is not proved in this paper.

2. *Preliminary remarks.* Suppose $A[x]=x^a-\sum_{j=0}^{a-1}A_jx^j=\prod_{k=1}^{a}[x-a_k]$ and let $[jk]$ denote the sum of the products of the zeros $a_{j+1}, a_{j+2}, \cdots, a_a$ taken k at a time. Define the matrices A, A^*, and E as follows††:

$$A=\left\|\begin{matrix} A_{a-1} & A_{a-2} & \cdots & A_2 & A_1 & A_0 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{matrix}\right\|, \quad A^*=\left\|\begin{matrix} a_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & a_2 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1 & a_a \end{matrix}\right\|,$$

$$E=\left\|\begin{matrix} 1 & -[11] & [12] & -[13] & \cdots & (-1)^{a-2}[1\ a-2] & (-1)^{a-1}[1\ a-1] \\ & 1 & -[21] & [22] & \cdots & (-1)^{a-3}[1\ a-3] & (-1)^{a-2}[1\ a-2] \\ & & 1 & -[31] & \cdots & (-1)^{a-4}[1\ a-4] & (-1)^{a-3}[1\ a-3] \\ & & & \cdots & \cdots & \cdots & \cdots \\ & & & & & 1 & -[a-1\ 1] \\ & & & & & & 1 \end{matrix}\right\|.$$

It follows that $A=E^{-1}A^*E$ and so $B[A]=E^{-1}B[A^*]E$. As a matter of fact, A and A^* are simply rational and irrational canonical forms, respectively, for all non-derogatory matrices with the characteristic function $A[x]$.

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* The present paper grew out of a consideration of a paper by W. V. Parker, The degree of the highest common factor of two polynomials, this MONTHLY, vol. 42, 1935, pp. 164–166. An abstract of it appeared in the Bulletin of the American Mathematical Society, vol. 42, 1936, p. 179.

† Frobenius, Journal für Mathematik, vol. 84, 1878, p. 11.

‡ Sylvester has called a matrix "non-derogatory" when its characteristic function and minimum function are the same.

§ M. M. Flood, this MONTHLY, vol. 43, 1936, p. 562.

‖ The "nullity" of a matrix is the difference between its order and rank.

¶ The "companion matrix" of $A[x]$ is the matrix A defined in §2.

†† This method was suggested by the paper by W. V. Parker.

Now if W is any square matrix of order w, $M_{jk}[W]$ will denote the minor of order j made up from its last j rows, first $j-1$ columns and $[w-k]$th column. Let p_j denote the degree of the polynomial $P_j[W]=\sum_{k=0}^{w-j} M_{jk}[W]x^k$ and set $R_j[W]=P_j[W]/M_{jp_j}[W]$. Because of the triangular form of E it follows immediately that $R_j[EW]=R_j[W]$.

It will be convenient to fix the notation further so that the highest common factor of $A[x]$ and $B[x]$ is $D[x]=\prod_{k=1}^{d}[x-a_k]$ where $B[x]=D[x]\prod_{k=1}^{b-d}[x-b_k]$. The principal result of this paper can now be easily stated.

3. *Theorem.* $D[x]=R_{a-d}[B[A]]$.

The theorem is clearly true if $d=0$. Suppose the theorem true for all cases in which the degree of the highest common factor involved is less than some positive integer d.

Define polynomials $\bar{A}[x]$, $\bar{B}[x]$, and $\bar{D}[x]$ by the relations:

$$A[x] = \bar{A}[x][x-a_1], \quad B[x] = \bar{B}[x][x-a_1], \quad D[x] = \bar{D}[x][x-a_1].$$

Then $\bar{D}[x]$ is the highest common factor of $\bar{A}[x]$ and $\bar{B}[x]$ and since this degree is less than d it follows by the hypothesis of induction that $\bar{D}[x]=R_{a-d}[\bar{B}[\bar{A}]]$, where $\bar{A}$ denotes the companion matrix of $\bar{A}[x]$. The matrices E and A^* may be partitioned so that

$$E = \begin{vmatrix} 1 & s \\ 0 & \bar{E} \end{vmatrix} \qquad \text{and} \qquad A^* = \begin{vmatrix} a_1 & 0 \\ u & \bar{A}^* \end{vmatrix},$$

where $\bar{A}^*$ denotes the irrational canonical form of $\bar{A}$, and $\bar{E}$ is the triangular matrix which transforms this irrational form into the rational one. Finally, let J and $\bar{J}$ be matrices of orders a and $a-1$, respectively, of the form

$$\begin{Vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{Vmatrix}.$$

It is easily verified that

$$\begin{aligned} B[A^*]E &= \begin{Vmatrix} 0 & 0 \\ \bar{B}[\bar{A}^*]u & \bar{B}[\bar{A}^*][\bar{A}^* - a_1] \end{Vmatrix} \begin{Vmatrix} 1 & s \\ 0 & \bar{E} \end{Vmatrix} \\ &= \begin{Vmatrix} 0 & 0 \\ 0 & \bar{B}[\bar{A}^*] \end{Vmatrix} \begin{Vmatrix} 0 & 0 \\ u & us + [\bar{A}^* - a_1]\bar{E} \end{Vmatrix}. \end{aligned}$$

Now since $\bar{A}\bar{E}^*=\bar{E}\bar{A}$, $\bar{A}=\bar{J}-us$, and $u=\bar{E}u$, it follows that

$$us+[\bar{A}^*-a_1]\bar{E}=\bar{J}-\bar{A}+\bar{E}\bar{A}-a_1\bar{E}=\bar{J}+[\bar{E}-1]\bar{J}-a_1\bar{E}=\bar{E}[\bar{J}-a_1]$$

and so

$$B[A^*]E = \left\| \begin{matrix} 0 & 0 \\ 0 & \overline{B}[A^*] \end{matrix} \right\| \left\| \begin{matrix} 0 & 0 \\ \overline{E}u & \overline{E}[\overline{J} - a_1] \end{matrix} \right\| = \left\| \begin{matrix} 1 & 0 \\ 0 & \overline{E} \end{matrix} \right\| \left\| \begin{matrix} 0 & 0 \\ 0 & \overline{B}[\overline{A}] \end{matrix} \right\| [J - a_1].$$

However,

$$R_{a-d}[B[A]] = R_{a-d}[B[A^*]E] = R_{a-d}\left[\left\| \begin{matrix} 0 & 0 \\ 0 & \overline{B}[\overline{A}] \end{matrix} \right\| [J - a_1] \right].$$

A close inspection of this last polynomial shows that it is simply $[x-a_1]R_{a-d}[\overline{B}[\overline{A}]]$, so necessarily $R_{a-d}[B[A]] = [x-a_1]\overline{D}[x] = D[x]$ and the induction is completed.

COROLLARY. *If $M_{j\ a-j}$ is the first non-zero member of the sequence M_{a0}, $M_{a-1\ 1}$, $M_{a-2\ 2}, \cdots$, then $j=d$.*

4. *A numerical example.* Set $A[x]=x[x^2-2]^2$; $B[x]=[x^2-2][x-1]^2[x-2]$. Then $D[x]=x^2-2$, and

$$A = \left\| \begin{matrix} 0 & 4 & 0 & -4 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{matrix} \right\|, \qquad B[A] = \left\| \begin{matrix} -8 & 16 & 16 & -32 & 0 \\ 8 & -8 & -16 & 16 & 0 \\ -4 & 8 & 8 & -16 & 0 \\ 4 & -4 & -8 & 8 & 0 \\ -1 & 4 & 0 & -8 & 4 \end{matrix} \right\|.$$

Now

$$\begin{aligned} P_1[B[A]] &= -x^4 + 4x^3 - 8x + 4, \\ P_2[B[A]] &= 4[3x^3 - 2x^2 - 6x + 4], \\ P_3[B[A]] &= 32x^2 - 64, \\ P_4[B[A]] &= P_5[B[A]] = 0, \text{ and so} \\ R_3[B[A]] &= x^2 - 2 \text{ which is } D[x]. \end{aligned}$$

The ordinary euclidean algorithm for $B[x]$ and $A[x]$ may be displayed as:

$B[x] = A[x] + S_1[x] = A[x] + [-x^4 + 4x^3 - 8x + 4]$,

$A[x] = Q[x]S_1[x] + S_2[x] = [-x-4]S_1[x] + [12x^4 - 8x^2 - 24x + 16]$,

$S_1[x] = Q_1[x]S_2[x] + S_3[x] = [-x/12 + 5/18]S_2[x] + [2x^2/9 - 4/9]$.

It follows that $P_1[B[A]]=S_1[x]$, $P_2[B[A]]=S_2[x]$, $P_3[B[A]]=144S_3[x]$.

This latter example illustrates the general situation. If $S_k[x]$ are the remainders obtained in the euclidean algorithm and $T_k[x]$ are the distinct polynomials in the sequence $P_k[B[A]]$ then it is true that, except for a constant factor, $S_k[x]=T_k[x]$. To complete a proof of this it is necessary only to consider the case in which $d=0$ since the induction given in the proof of Theorem 3 can be immediately extended to hold for this more general statement.

More precisely, it can be shown that $S_{k+1}[x]=c_{k+1}P_{a+1-t_k}[B[A]]$ for $k=0, 1, 2, \cdots$, where $t_0=a$, $t_{k+1}=\text{degree}\{P_{a+1-t_k}[B[A]]\}$, and the constants c_{k+1} are different from zero. Furthermore, if u_{k+2} denotes the sign of the leading

coefficient of $P_{a+1-t_k}[B[A]]$, if b_{k+2} denotes the sign of c_{k+1}, and if $r_{k+2}=t_k-t_{k+1}$; then $b_2=1$, $b_3=[-u_2]^{r_2-1}$, and the remaining b_k are determined by the relations:

$$b_{k+1} = [-1]^{m_k}b_k \quad \text{if } r_k \equiv 1 \pmod 2, \quad (k = 3, 4, \cdots),$$
$$b_{k+1} = -b_{k-1}u_ku_{k-1} \text{ if } r_k \equiv 0 \pmod 2, \quad (k = 3, 4, \cdots),$$

where m_k denotes the number of even integers in the sequence $r_2, r_3, \cdots, r_{k-1}$. It follows that the polynomials $b_{k+2}P_{a+1-t_k}[B[A]]$, for $k=0, 1, 2, \cdots$, are the remainders in the euclidean algorithm for $B[x]$ and $A[x]$ except for possible *positive* factors of proportionality. It can also be shown that the only other non-zero polynomials in the sequence $P_k[B[A]]$ are the polynomials $P_{a-t_k}[B[A]]$ for $k=0, 1, \cdots, t$, where $t_t=d$.

All of these results can be proved by elementary and completely rational methods. However, the proof given in §3 of this paper, although irrational, is much shorter than the one which seems to be necessary for the more general case and has been given because it extends the result which W. V. Parker obtained and uses methods of proof similar to those which he used.

ON SOME SERIES ARISING FROM A DEFINITION OF THE EXPONENTIAL FUNCTION

By J. K. L. MacDONALD and F. R. SHARPE, Cornell University

The exponential function e^x may be defined variously; for example, as an infinite series $e^x=x^0/0!+x^1/1!+x^2/2!+\cdots$; as the inverse function for $\log_e x=\int_1^x dt/t$; as the solution of a boundary problem such as $(d/dx)y(x)=y(x)$ subject to $y(0)=1$; and as

$$(1) \qquad e^x = \lim_{N\to\infty}\left(1+\frac{x}{N}\right)^N.$$

A. Here we shall examine the last mentioned form and we shall establish the following main results:

$$\left(1+\frac{x}{N}\right)^N = S_N + \left(\frac{-1}{N}\right)\left(\tfrac{1}{2}x^2S_{N-2}\right) + \left(\frac{-1}{N}\right)^2\left(\tfrac{1}{3}x^3S_{N-3}+\tfrac{1}{4\cdot 2}x^4S_{N-4}\right)$$
$$+\left(\frac{-1}{N}\right)^3\left(\tfrac{1}{4}x^4S_{N-4}+\tfrac{1}{5}\left(\tfrac{1}{2}+\tfrac{1}{3}\right)x^5S_{N-5}+\tfrac{1}{6\cdot 4\cdot 2}x^6S_{N-6}\right)$$
$$+\cdots,$$

where

$$S_m = \frac{x^0}{0!}+\frac{x^1}{1!}+\cdots+\frac{x^m}{m!}, \qquad m \geqq 0,$$
$$S_m = 0, \qquad m < 0.$$

More explicitly

$$(2)\qquad \left(1+\frac{x}{N}\right)^N = S_N + \sum_{p=1}^{N-1}\left(\frac{-1}{N}\right)^p \sum_{r=1}^{p} c_{p,r}x^{p+r}S_{N-p-r},$$

where the c's are constant coefficients which will be considered below. (The r-sum in (2) effectively stops at r equal to the smaller of $N-p$ and p, since $S_m=0$ for $m<0$.)

B. Let $P_{p,r}$ denote the sum of all distinct products formed by taking, p at a time, the integers $1, 2, \cdots, r$; that is,

$$(3)\qquad \begin{aligned} P_{p,r} &= (1\cdot 2\cdot \cdots \cdot p) + \cdots + ((r-p+1)\cdot(r-p+2)\cdot \cdots \cdot r) \\ &= \sum a_1 a_2 \cdots a_p, \end{aligned}$$

where the summation ranges over the integral values of the a's for which $r \geqq a_p > a_{p-1} > \cdots > a_1 \geqq 1$. Then

$$(4)\qquad P_{p,r} = \sum_{t=1}^{p} c_{p,t}\frac{(r+1)!}{(r-p-t+1)!},$$

where the c's are as in (2) and (5), and where $1/n!$ is interpreted to mean zero when* $n<0$ (that is, the t-sum effectively stops at t equal to the smaller of p and $r-p+1$).

C. The coefficients $c_{1,1}, c_{2,1}, c_{2,2}, \cdots$ in (2) and (4) can be determined as follows: $c_{p,q}$ is the element in the p-th row and q-th column in the following "number triangle:"

$$(5)\qquad \begin{array}{llll} \frac{1}{2} \\ \frac{1}{3}, & \frac{1}{4}\cdot\frac{1}{2} \\ \frac{1}{4}, & \frac{1}{5}\cdot(\frac{1}{2}+\frac{1}{3}), & \frac{1}{6}\cdot\frac{1}{4}\cdot\frac{1}{2} \\ \frac{1}{5}, & \frac{1}{6}\cdot(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}), & \frac{1}{7}[\frac{1}{4}\cdot\frac{1}{2}+\frac{1}{5}(\frac{1}{2}+\frac{1}{3})], & \frac{1}{8}\cdot\frac{1}{6}\cdot\frac{1}{4}\cdot\frac{1}{2} \\ \cdots & \cdots & \cdots & \cdots \end{array}$$

The first column is formed by taking successive reciprocals as shown (that is, $c_{p,1}=1/(p+1)$); the element $c_{p+1,q+1}$ in the $(p+1)$-th row and $(q+1)$-th column is $1/(p+1+q+1)$ times the sum of the elements in the preceding column (the p-th) summed down to the preceeding row (the q-th) inclusive. Explicitly,†

$$(5')\qquad \begin{aligned} c_{p,1} &= \frac{1}{p+1}; \\ c_{p+1,q+1} &= \frac{1}{p+q+2}(c_{q,q}+c_{q+1,q}+\cdots+c_{p,q}), \qquad 1 \leqq q \leqq p. \end{aligned}$$

* Compare $1/\Gamma(n+1)$ for $n<0$.

† More explicitly, but less usefully

$$c_{p+1,q+1} = \frac{1}{p+q+2}\sum_{2q}^{p+q}\frac{1}{n_1}\sum_{2(q-1)}^{n_1-2}\frac{1}{n_2}\cdots\sum_{2}^{n_{q-1}-2}\frac{1}{n_q}.$$

By induction (5′) yields the inequality*

$$0 < c_{p,q} \leqq \tfrac{1}{2}. \tag{6}$$

The above results are proved as follows. We first establish form (4), a form suggested by consideration of special cases for (3) (that is, cases $p=1, 2, r-1, r$). Our plan is to show that the P's defined by (3) can be generated by the same recursion formulas as the P's defined by (4) and (5′). The formulas concerned are

$$P_{1,r} = \frac{r(r+1)}{2}, \quad P_{r,r} = r!, \quad P_{p+1,r+1} - P_{p+1,r} = (r+1)P_{p,r}, \quad r \geqq p+1. \tag{7}$$

Equations (7) follow readily from definition (3), and it is easily seen that they suffice to determine $P_{p,r}$ for all relevant p, r (viz., $r \geqq p \geqq 1$). To show that equations (7) also follow from (4) and (5′) we note that (5′) can be replaced by

$$c_{1,1} = \tfrac{1}{2}; \quad (p+q+2)c_{p+1,q+1} - (p+q+1)c_{p,q+1} = c_{p,q}, \quad 0 \leqq q \leqq p, \tag{8}$$

where $c_{p,0} = 0 = c_{p,p+1}$. Forms (8) are immediately justified by substitution of (5′) in (8). Now using (4) and (8) and simple rearrangements, we have for $r \geqq p+1$,

$$\begin{aligned}
&\frac{1}{(r+1)!}\left(P_{p+1,r+1} - P_{p+1,r} - (r+1)P_{p,r}\right) \\
&= \sum_{t=1}^{p+1} \frac{(r+2)c_{p+1,t} - (r-p-t+1)c_{p+1,t} - (r+1)c_{p,t}}{(r-p-t+1)!} \\
&= \sum_{t=1}^{p+1} \frac{(p+1+t)c_{p+1,t} - (r+1)c_{p,t}}{(r-p-t+1)!} = \sum_{t=1}^{p+1} \frac{(p+t)c_{p,t} + c_{p,t-1} - (r+1)c_{p,t}}{(r-p-t+1)!} \\
&= -\sum_{t=0}^{p} \frac{(r-p-t+1)c_{p,t}}{(r-p-t+1)!} + \sum_{t=1}^{p+1} \frac{(r-p-t+2)c_{p,t-1}}{(r-p-t+2)!} = 0.
\end{aligned}$$

The last part of (7) is therefore proved. The first two parts follow directly from (4) and from the first formula of (5′).

Having thus established (4) we shall use the binomial theorem, definition (3), and equation (4) to yield (2). We have

$$\left(1 + \frac{x}{N}\right)^N = 1 + \frac{N}{1!}\left(\frac{x}{N}\right) + \frac{N(N-1)}{2!}\left(\frac{x}{N}\right)^2 + \cdots + \frac{N(N-1)\cdots 1}{N!}\left(\frac{x}{N}\right)^N$$

* More closely bounding inequalities for $c_{p,q}$ are easily found but are more cumbersome than (6). For example:

$$\log^q\left(\frac{p+q+2}{2q+1}\right) \leqq (p+q+2)\, q!\, c_{p+1,q+1} \leqq \log^q(p+q+2),$$

$$\left(1+\frac{x}{N}\right)^N = 1+\frac{x}{1!}+\left(1-\frac{1}{N}\right)\frac{x^2}{2!}+\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\frac{x^3}{3!}+\cdots$$
$$+\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\cdots\left(1-\frac{N-1}{N}\right)\frac{x^N}{N!}$$
$$= 1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^N}{N!}+\sum_{p=1}^{N-1}\left(\frac{-1}{N}\right)^p\sum_{r=p}^{N-1}\frac{x^{r+1}}{(r+1)!}P_{p,r}$$
$$= S_N+\sum_{p=1}^{N-1}\left(\frac{-1}{N}\right)^p\sum_{r=p}^{N-1}x^{r+1}\sum_{t=1}^{p}\frac{c_{p,t}}{(r-p-t+1)!}.$$

Now, by first summing over r and then over t, we have

$$\left(1+\frac{x}{N}\right)^N = S_N+\sum_{p=1}^{N-1}\left(\frac{-1}{N}\right)^p\sum_{t=1}^{p}c_{p,t}\sum_{r=p+t-1}^{N-1}x^{p+t}\frac{x^{r-p-t+1}}{(r-p-t+1)!},$$

where we have observed the usual convention $1/(r-p-t+1)!=0$ unless $N-1\geqq r\geqq p+t-1$. Equation (2) then follows directly.

In conclusion we shall obtain bounds for the "remainder"

$$R_N = \left(1+\frac{x}{N}\right)^N - S_N. \tag{9}$$

Using $|x|=X$ and the properties $|S_m|\leqq e^X$, (2), and (6), we have

$$|R_N| \leqq \sum_{p=1}^{N-1}\left(\frac{1}{N}\right)^p\sum_{r=1}^{p}\tfrac{1}{2}X^{p+r}e^X = \frac{e^X}{2}\sum_{r=1}^{N-1}X^r\sum_{p=r}^{N-1}\left(\frac{X}{N}\right)^p.$$

Now, with no important loss of generality, we assume $X^2<N$ and use

$$\sum_{p=r}^{N-1}\left(\frac{X}{N}\right)^p = \left(\frac{X}{N}\right)^r\sum_{t=0}^{N-1-r}\left(\frac{X}{N}\right)^t \leqq \left(\frac{X}{N}\right)^r\bigg/\left(1-\frac{X}{N}\right)$$

and

$$\sum_{r=1}^{N-1}X^r\left(\frac{X}{N}\right)^r = \frac{X^2}{N}\sum_{t=0}^{N-2}\left(\frac{X^2}{N}\right)^t \leqq \left(\frac{X^2}{N}\right)\bigg/\left(1-\frac{X^2}{N}\right)$$

to give finally the bounding inequality* for the remainder R_N of (9):

$$|R_N| \leqq \frac{X^2e^X}{2N}\left(1-\frac{X}{N}\right)^{-1}\left(1-\frac{X^2}{N}\right)^{-1} \quad \text{where} \quad |x| = X < \sqrt{N}. \tag{10}$$

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* R_N can be bounded much more closely by use of rather cumbersome forms using logarithms. Note, for example, for $x=1$ and $N\geqq 2$, $|R_N|<e/(N-1)$, by use of (10).

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions, and Notes in the Monthly is open to all forms of activity in collegiate mathematics including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

ON MATHEMATICAL NOMENCLATURE

By N. A. COURT, University of Oklahoma

In a recent issue of this MONTHLY (December, 1936, p. 630) D. L. Mackay, armed with chapter and verse, proves that the centers of similitude of two circles are also called their "homothetic centers." He thinks that the latter term alone will suffice and proposes to reserve the first term to designate any point on the circle of similitude of the two circles.

This is a reasonable suggestion and has much to recommend it. But that it will be adopted, and especially that it will be adhered to, is, to say the least, doubtful. Even in cases where the need is much more pressing, the desired specialization in terms is difficult of achievement. Witness, for instance, the indiscriminate use of the terms "imaginary number" and "complex number," a practice that necessitates the retention of the otherwise superfluous term "pure imaginary number." Moreover, the number of properties of the points of the circle of similitude is not sufficiently large to make the consistent use of a special term imperative.

It is nevertheless refreshing to see somebody attempting to improve our geometrical terminology. If fault is to be found with Mr. Mackay's suggestion, it is that it does not go far enough. Trying to distinguish between two different kinds of points, it proposes for both of them the same name "center," with different modifiers. It follows in this respect a tradition which is just as persistent as it is unfortunate. The linguistic imagination of makers of new geometrical terms hardly ever reaches beyond the word "center," when they wish to name a point. And so we have the center of a circle, the center of inversion, the center of a pencil of rays, the radical center, and so on. The result is that when misfortune leads to the consideration of several of these "centers" at the same time, only titanic efforts of skill and patience can keep ambiguity out of the corresponding sentences. The success is more often than not predicated on the sympathetic indulgence of the much overtaxed reader.

A great deal of this confusion could be avoided by a little daring in the way of language. Let two circles have their two time-honored centers, but instead of the two homothetic centers let them have two "hubs." More generally we could speak of the "hub" of any two homothetic figures, instead of their homothetic center. The new term would invoke a quite convenient image. Again it would be appropriate to refer to the "pivot" of a pencil of rays instead of the ever present center. Many physical analogies would be suggested by a ray turning about a fixed "pivot" and thus describing a pencil of rays. Barbaric

innovations these—granted. It may be objected, for instance, that hubs and pivots are much too bulky to represent geometrical points. Quite so. However, a line has a pole for a conic, or a great circle has a pole on a sphere, and we somehow manage not to attribute to them the size of a telegraph pole, not even of one of those short ones seen in some of the Western States. Nor have we set the parabola afire and aflame because we provided it with a focus.

Other words of the general vocabulary that could with profit be incorporated in our geometrical terminology will no doubt occur to any reader, upon reflection. But such words have little chance of adoption. Verbal habits are very tenacious. Nobody will either care or dare to tie the bell of linguistic change to the stiff neck of the cat of hoary tradition, even in matters of minor importance. Let us take this example. We speak of homological figures and refer to the relation between two such figures as being an "homology." On the other hand, we have homothetic figures, their homothetic center, but the noun corresponding to the adjective "homothetic" is not to be found in our books. The French writers use the term "homothétie" quite freely, but in English we have only circumlocutions for it, for no reason whatever. Who will dare to overcome the inertia and be the first to use the noun "homothecy"?

A Note on Conics Intersecting at Given Angles

By A. D. Campbell, Syracuse University

Let us consider the two conics

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0, \tag{1}$$

$$a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy = 0, \tag{2}$$

where the non-homogeneous coördinates $X = x/z$ and $Y = y/z$ are a rectangular system of coördinates in the plane.

The tangents to these conics at a point of intersection $P'(x', y', z')$ are respectively

$$(ax' + hy' + gz')x + (hx' + by' + fz')y + (gx' + fy' + cz')z = 0, \tag{3}$$

$$(a'x' + h'y' + g'z')x + (h'x' + b'y' + f'z')y + (g'x' + f'y' + c'z')z = 0. \tag{4}$$

These two tangents are mutually perpendicular if

$$\begin{aligned}(ax' + hy' + gz')&(a'x' + h'y' + g'z') + (hx' + by' + fz')(h'x' + b'y' + f'z')\\ &\equiv (aa' + hh')x'^2 + (bb' + hh')y'^2 + (gg' + ff')z'^2\\ &\quad + (ah' + a'h + bh' + b'h)x'y' + (ag' + a'g + hf' + h'f)x'z'\\ &\quad + (hg' + h'g + bf' + b'f)y'z' = 0.\end{aligned} \tag{5}$$

If the tangents at all of the points of intersection of (1) and (2) are to be mutually perpendicular, then equation (5), with x', y', z' replaced by x, y, z, must be an equation of a conic belonging to the pencil of conics determined by

(1) and (2); for (5) must then intersect (1) and (2) at all points of intersection of (1) with (2). Therefore, we have

$$
\begin{aligned}
&\lambda a + \mu a' = aa' + hh', && \lambda f + \mu f' = \tfrac{1}{2}(hg' + h'g + bf' + b'f),\\
(6)\qquad &\lambda b + \mu b' = bb' + hh', && \lambda g + \mu g' = \tfrac{1}{2}(ag' + a'g + hf' + h'f),\\
&\lambda c + \mu c' = gg' + ff', && \lambda h + \mu h' = \tfrac{1}{2}(ah' + a'h + bh' + b'h).
\end{aligned}
$$

To illustrate the use of (6) we shall find all the conics that intersect at right angles the circle

$$(7)\qquad x^2 + y^2 - r^2z^2 = 0.$$

Here $a = b = 1$, $c = -r^2$, $f = g = h = 0$. In this case (6) reduces to:

$$
\begin{aligned}
&\lambda + \mu a' = a', && \mu f' = \tfrac{1}{2}f',\\
(8)\qquad &\lambda + \mu b' = b', && \mu g' = \tfrac{1}{2}g',\\
&-r^2\lambda + \mu c' = 0, && \mu h' = h'.
\end{aligned}
$$

The value $\mu=1$ gives us $\lambda=0$, $f'=g'=c'=0$, but h', a', b' arbitrary. These conditions give us all the pairs of lines through the center of (7).

The value $\mu=1/2$ gives us $h'=0$, $a'=b'=2\lambda$, $c'=2r^2\lambda$, but f' and g' arbitrary. Dividing the resulting equation through by 2λ if $\lambda\neq 0$, we get

$$(9)\qquad x^2 + y^2 + r^2z^2 + 2\alpha yz + 2\beta zx = 0.$$

The value $\lambda=0$ gives the degenerate conics $2f'yz+2g'zx=0$, each consisting of a line through the center of (7) and the line at infinity. If $\mu\neq 1$ or $1/2$, we must have

$$f' = g' = h' = 0, \qquad a' = b' = \lambda/(1-\mu), \qquad c' = r^2\lambda/\mu,$$

giving us the circles $x^2+y^2+kz^2=0$, which intersect (7) only in the circular points at infinity. These circles have as common tangents the minimal lines through their common center, and these minimal lines are self-perpendicular. We see that we have now all the possible conics intersecting (7) at right angles at all their points of intersection.

In a similar way we could treat the hyperbola, ellipse, and parabola. The method is the main thing of interest here.

It is easy to see that in the case of orthogonal intersection this method generalizes at once to quadratic forms in n variables. Here we take $\cos\theta=0$, where θ is the angle between two tangent hyperplanes at a point of intersection P'. It is also evident how to treat the case in the plane where (1) and (2) are to intersect each other at a given angle $\theta\neq 90°$ at every common point P'. Here we replace (5) by the equation giving $\tan\theta=m$, where θ is the angle between (3) and (4). Unfortunately this last method does not generalize to n variables; also neither method generalizes to quantics of higher degree than the second. The complete discussion might be a good problem to propose to some advanced college student.

Examples in Finite Differences, Calculus and Probability. By H. Freeman. Cambridge, At the University Press; New York, The Macmillan Company, 1936. 86 pages. $2.50.

This supplement to the author's *Elementary Treatise on Actuarial Mathematics* contains 400 problems of which 131 are devoted to finite differences, 81 and 80 each to differential and integral calculus and 108 to probabilities and mean value. This publication will be of particular interest to those preparing for the examinations given by the various actuarial societies as they contain a variety of problems of more than average difficulty especially as regards finite differences and probabilities.

The notes and hints which follow the problems should enhance the value of the book considerably as they help to deepen the student's comprehension of the subject and teach him various methods of approach not contained in the textbook. The problems on finite differences contain quite a number of examples illustrating the application to life insurance and among the calculus problems are a number illustrating the relation between calculus and finite differences.

The answers found at the end of the book will serve as a check on the student's work. The reviewer did not have sufficient time to check the correctness of every answer but test checks made at random failed to reveal any errors.

This collection of examples is to be warmly recommended to all who wish to acquire facility in solving problems in the subjects of elementary finite differences, calculus and probabilities.

MARK KORMES

Pascal, The Life of Genius. By Morris Bishop. New York, Reynal & Hitchcock, 1936. xiv+398 pages. $3.50.

In reviewing a biographical work of this kind for a mathematical journal it is natural that special attention should be given to the chapter which treats of "the science venerable." On the other hand the reader may rightly expect that consideration should briefly be given to the framing of the picture—to the topics of interest which unite to make known the man himself. The activities of Pascal are so diverse, however, and the space for a review of a work like this of Mr. Bishop's is so limited, that little more can be done than to consider only a few of the incidents of Pascal's life as contemporary records reveal them.

Briefly stated, the author has here set forth the results of a genuine labor of love, revealing the life of one whom he has come to look upon as a marvel among men, or, to use his own characterization, as a genius. If, however, we may lay aside as useless the dictates of grammar, he pictures him as a genii—a combination of many geniuses. His thirteen chapters picture Pascal as a Prodigy, an Inventor, a Convert, a Physicist, a Mathematician, a Man of the World, a Lover, a Mystic, a Penitent, a Polemist, a Philosopher, a Saint, and a Man. Each of these chapters is in itself a biographical sketch of an individual of unusual attainments and a contributor to the knowledge of the divers affairs of life.

The book as a whole is a fair treatment of the subject. In it Pascal stands out as a combination of divers geniuses; as a mathematician who, had he not pursued any of his other lines of interest, might have ranked as a great creator of thought and a discoverer of the regions where later flourished the infinitesimal calculus and the modern geometry of our day.

Aside from a few cases in the typesetting of algebraic matter, from the lettering of a few geometric figures, and especially from the arrangement of the 349 notes, the mechanical features of the book are to be commended. Librarians will however, regret that there is no systematic list of the writings of Pascal and that the index is not more nearly complete. For example, they may wonder where readily to obtain information relating to such works as the *Traité de l'équilibre des liqueurs* and the *Traité de la pesanteur de la masse de l'air.*

DAVID EUGENE SMITH

Special Topics in Theoretical Arithmetic. By Joseph Bowden. Garden City, New York. 1936. vi+217 pages. $2.50.

A queer book. Planned partly as a sequel to the author's *Elements of the Theory of Integers*, Macmillan, 1903. The simplified spelling (ar for are, etc.) of the older book is maintained. The individual reader will have to decide whether he considers the introduction of a number of new symbols helpful or not ($\overset{.}{\underset{.}{>}}$ is divisible by; $\overset{.}{\underset{.}{<}}$ is a divisor of; $\perp\!\!\perp$ is prime to; $\otimes$, kor, the greatest common factor of; $\underline{\times}$, mul, the least common multiple of; $-=$, not equal; $\cdots$).

The main topics discussed are: (1) *Scales of Notation*, 64 pages. In this part many examples and tables for various bases are given. For the quaternary scale, some of the names suggested are: 10, four or wunfy; 11, wunfy wun; 101, wunsty wun, 110, wunsty wunfy; etc. (2) *Checks for division for any base r and for many special bases*, following the ordinary and natural method of attack by congruences, 88 pages. (3) *Some mathematical recreations*, 42 pages.

It is not easy to decide what class of readers will be attracted by this book. For the expert in number theory it is on too low a level, although he may find an occasional nugget. For the beginner the gain will probably not be quite worth the effort of following the terribly detailed, and, as it seems to me, frequently unnecessarily complicated, work.

It may offer suggestions to leaders of mathematics clubs.

A. J. KEMPNER

"I think it would be desirable that this form of word (mathematics) should be reserved for the applications of the science, and that we should use mathematic in the singular to denote the science itself, in the same way as we speak of logic, rhetoric, or (own sister to algebra) music." J. J. Sylvester, *Collected Mathematical Papers*, vol. 2, p. 659.

MATHEMATICS CLUBS

EDITED BY F. W. OWENS AND HELEN B. OWENS, State College, Pa.

All reports of club activities should be sent to F. W. Owens, 462 East Foster Ave., State College, Pennsylvania.

Club Reports

1935–1936

The Mathematics Club, Butler University

President, R. Fohl; Vice-President, Mary K. Mangus; Secretary, Helen Patrick; Treasurer, H. Carson. The club held a mathematical fair to which all seniors of Indianapolis high schools who were interested in mathematics were invited. Members of the club furnished a program including discussions of the abacus, polar planimeter, paper folding, the Rhind papyrus and conic sections. Demonstrations of cathode rays, the oscillograph and the telescope were also made. In addition to a Christmas party and picnic, nine meetings were held for the discussion of topics ranging from "The moral values of mathematics" to "Mathematical instruments" and "Mathematical short cuts."

Pi Mu Epsilon and The Mathematics Club, Kansas State College

Pi Mu Epsilon: Director, Professor W. T. Stratton; Vice-Director, Marjorie Lomas; Secretary, Alma Furman; Treasurer, T. C. Wherry; Librarian, J. York. The Mathematics Club: President, Marjorie Lomas; Vice-President, J. York; Secretary, S. Sjogren; Program Chairman, Betsy Sealer. Business meetings and a successful initiation banquet were restricted to chapter members but a large evening party and a lively picnic were enjoyed by the two organizations together. This cooperation of the two groups seems most inspiring to both. The meetings for papers are open to all those interested in the topics under discussion. These topics covered a wide field including such universal favorites as Trisection of an angle and Magic squares, and more unusual ones entitled: The ins and outs of a national mathematics meeting; Archimedes as an engineer; The origin of zero; Mathematics of bridge playing.

Undergraduate Mathematics Club, The State University of Iowa

Monthly meetings were held and the following topics discussed: Maps; Spherical trigonometry; The old age pension problem; The points of a circle; Calculus of variations; Some theorems in probability; Space fillers in the plane.

Mathematics Club, Connecticut College

President, Ruth W. Grodotzki, Secretary-Treasurer, Frances P. Wallis. As mentioned in an earlier issue the club successfully presented the play, "The evolution of numbers" written by Professor H. E. Slaught. A picnic in the spring was the real social event of the year. Program topics included: Paths on prisms; The making of maps; Digital reckoning; Nomograms; Mathematical computing machines; Triple ratio approximation.

Vanderbilt-Peabody Mathematics Club, Vanderbilt University and George Peabody College for Teachers

Officers were elected semi-annually as follows: First half year, President, Elizabeth Crane; Vice-President, Annette Beasley; Secretary-Treasurer, F. L. Moran; Social Chairmen, Agatha Scott, Sophronsia John. Second half year, President, M. Wicht; Vice-President, Buena Wilson; Secretary-Treasurer, Inez Roberson; Social Chairmen, Imogene Bratton, Julia McGee. Twelve meetings for papers and a wiener roast for fun were held. Topics of papers included: Trisection of an angle; Magic squares; Mathematical recreations; Development of the logarithmic series; Diophantine equations; Similitude; Divisions which occur in geometry. This interesting organization holds its meetings alternately on the two campuses.

Pi Mu Epsilon, University of Illinois

Director, R. M. Thrall; Vice-Director, R. E. Watson; Treasurer, L. Silverman; Recording Secretary, V. Westbrook; Corresponding Secretary, K. C. Schraut. The initiation banquet had Dr. H. J. Miles as toastmaster and Professor A. B. Coble as speaker. The papers at regular meetings were given by student members of the chapter. Topics included: Bayes's theorem of inverse probability; Vortex motion; An introduction to groups; Exterior ballistics; Finger mathematics and number form; A discussion of the gamma function; An introduction to the two body problem; Women mathematicians; Numerals; Mersenne numbers; Direction cosines in a plane; Vibrations; Finite differences.

Pi Mu Epsilon, Bucknell University

Director, Dr. W. T. MacCreadie; Vice-Director, C. Giles; Secretary, Amelia Ehlers; Treasurer, J. Logan. Six programs were held besides business meetings. The chapter wishes to acknowledge the cooperation of physics and engineering groups. Topics considered included: The nine point circle; Interpolation; The planimeter; Homogeneous coördinates; Trilinear coördinates.

Mathematics Club, Regis College

President, Doris Lebel; Vice-President, Ruth Heywood; Secretary, Doris Duchame; Treasurer, Dorathea Murray. This club, making its first report to the MONTHLY, includes in its programs discussions of: The Mathematics of photography; Mathematical prodigies; and Different number bases. Two social meetings with mathematical amusement furnished entertainment while the real work was furthered by a mathematics contest. Three hundred problems are published in six monthly installments of fifty each. Solutions may be submitted by any club member, a prize of ten dollars being awarded for the best set of solutions.

Mathematics Club, George Washington University

President, J. Goldman; Secretary-Treasurer, Mary R. Maciulla; Faculty Adviser, Professor J. H. Taylor. Meetings held twice each month gave opportunity for discussions of many topics which included: The mathematics of gambling; Two-dimensional flow problems; Infinite numbers; Applications of mathematics to economics; Logarithmic series; Mathematical theory of elasticity; Theory of statistical tests with small samples; Misuse of least squares; Sampling distribution of the coefficient of variation; The Apollonian circles of a triangle; Notion of a metric. At the annual banquet, Professor Tobias Dantzig of the University of Maryland, as guest of the club, spoke on "The history of zero." A Christmas party added fun to the year, and a pleasant afternoon at the summer home of Professor and Mrs. Taylor completed the club's activities.

Kappa Mu Epsilon, Athens College

President, Deweese Dunavant; Vice-President, Geraldine Whitt; Secretary, Mary Hursh; Corresponding Secretary, Dr. Kathryn Wyant; Treasurer, Florence Tilman. The program topics were taken from Ball's *Mathematical Recreations*, reports being made by the club members. A banquet held on Founders' Day was enlivened by the singing of "Rhythm in my X, Y, Z's," while all the talks were "geometrical." As general college activity the chapter assisted at the Athens College Stunt Night, presenting "The Classical Number Three," a shadow picture of "The Three Bears."

The Mathematics Club, The University of Alberta

Director, Dr. A. J. Cook. The Club held nine meetings and the annual midyear banquet at which Professor E. Sonet was guest speaker. Topics discussed included: Complex numbers; The meaning of probability; Properties of π; Geometry of crystals; Chinese mathematics; and A paradox of zero. A general discussion and criticism of the student papers was given by Professor E. S. Keeping. Each year Dr. A. J. Cook presents a prize for the best paper presented by a student.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 277. *Proposed by V. Thébault, Le Mans, France.*

A sphere (S) of constant size so moves as to always bear a fixed point P on its surface. If (C) is another sphere, fixed in size and position, and Q is the orthogonal projection of P on the radical plane of (S) and (C), prove that Q moves on the surface of a sphere.

E 278. *Proposed by Fred Discepoli, New York City.*

Prove the non-existence in the decimal system of a palindromic, four-digit square.

E 279. *Proposed by D. L. MacKay, Evander Childs High School, N. Y.*

Given two sides, construct a parallelogram whose angles equal the angles between its diagonals.

E 280. *Proposed by J. W. Cell, North Carolina State College.*

Prove that the determinant

$$\begin{vmatrix} a & a+d & a+2d \cdots & a+(n-1)d \\ a+d & a+2d & a+3d \cdots & a \\ a+2d & a+3d & a+4d \cdots & a+d \\ \cdots & \cdots & \cdots & \cdots \\ a+(n-1)d & a & a+d \cdots & a+(n-2)d \end{vmatrix}$$

has the value, $d^{n-1}n^{n-2}S_n(-1)^{n(n-1)/2}$, where S_n is the sum of the elements forming the arithmetic progression in the first column.

E 281. *Proposed by W. B. Clarke, San Jose, California.*

Let the incircle of triangle ABC touch sides a, b, and c at points D, E, and F respectively. Call the incenter, I. With A as center and AE as radius, swing an arc to cut DI produced, inside triangle ABC, at P. Similarly, let arcs centered at B and C cut EI and FI, inside the triangle, at Q and R. Let AP, BQ, and CR meet sides a, b, and c at points J, K, and L, respectively. Now prove or disprove the following: (1) AJ, BK, and CL are concurrent; (2) Triangles AJB and AJC, BKA and BKC, CLA and CLB have their incircles equal each to each in pairs.

SOLUTIONS

E 233 [1936, 494]. *Proposed by Virgil Claudian, Roumanian Mathematical Institute.*

If P is a point in the plane of the triangle ABC, and if circles are circumscribed about the triangles PAB, PBC, and PCA, prove that the tangents to these circles at P meet the respective sides AB, BC, and CA produced, in three collinear points whose common line is perpendicular to PO, where O is the center of the circle circumscribed about triangle ABC.

Solution by C. E. Springer, University of Oklahoma.

This problem can be extended to a linear space of n dimensions. Let

$$A_1(1, 0, 0, \cdots, 0), A_2(0, 1, 0, \cdots, 0), \cdots, A_{n+1}(0, 0, \cdots, 1)$$

be the $n+1$ vertices of the simplex of reference in S_n. The hypersphere through these $n+1$ points is represented by $a_{ij}x^ix^j=0$ $(i, j=1, 2, \cdots, n+1)$, $(a_{ii}=0)$. The equation $a_{ij}x^ix^j+b_k^hc_hx^hx^k=0$, where b_k^h is 1 or 0 according as h does or does not equal k, and where $h=1, 2, \cdots, n+1$; (h not summed) represents the family of hyperspheres through n vertices of the simplex omitting the vertex A_h.

The hypersphere S^h containing n of the vertices of the simplex of reference and the point $P(x_1^1, x_1^2, \cdots, x_1^{n+1})$, has the equation, $F^h \equiv b_k^h[x_1^hx_1^k a_{ij}x^ix^j - a_{ij}x_1^i x_1^j x^hx^k]=0$. The hypersphere S^h contains all the vertices of the simplex of reference except A_h, which is replaced by P. The tangent hyperplane T^h to the hypersphere S^h at the point P is given by $x_1^i(\partial F^h/\partial x^i)=0$; that is, by

$$(b_k^hx_1^hx_1^k)a_{ij}x^ix^j - (a_{rs}x_1^rx_1^s)b_i^hx_1^ix^h = 0, \quad (r, s = 1, 2, \cdots, n+1; h \text{ not summed}).$$

The intersections of these tangent hyperplanes T^h with the hyperplanes given by $x^h=0$ $(h=1, 2, \cdots, n+1)$ are contained in the hyperplane

$$(b_k^hx_1^hx_1^k)a_{ij}x_1^ix^j = 0, \qquad \text{or} \qquad a_{ij}x_1^ix^j = 0,$$

which is the polar hyperplane of the point P with respect to the hypersphere $a_{ij}x^ix^j=0$. Putting $n=2$ gives the solution for the problem proposed.

Also solved by W. B. Clarke, N. A. Court, D. K. Pease, V. Thébault, and Simon Vatriquant.

E 240 [1936, 575]. *Proposed by A. Gloden, Luxembourg.*

Find the largest perfect square of five digits, such that the sum of its digits is a cube.

Solution by C. W. Trigg, Cumnock College, Los Angeles.

If N^2 has five digits, their sum is less than 46, and since this sum is a cube, it is 8 or 27. No square number, $N^2 \equiv -1 \pmod 9$, so $N^2 \equiv 0 \pmod 9$ and $N \equiv 0 \pmod 3$. $N<317$, and the first number encountered below 317 whose digit sum is divisible by 3 happens to be the desired number, namely $(315)^2=99225$.

Also solved by W. E. Buker, Mary L. Constable, Fred Discepoli, Daniel Finkel, George Ligar, H. Nannei, R. F. Schnepp, E. P. Starke, W. R. Talbot, Simon Vatriquant, and Z. W. Wilchinsky.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

Editorial Note on 3817, 3818 [1937, 111]. The Newton line of a quadrilateral is the straight line through the mid-points of the three diagonals of a complete quadrilateral. The expression is seldom used. This information was furnished by N. A. Court and J. R. Musselman.

3821. [1937, 179] *Erratum.* Replace in l. 2 "inscribed" by "circumscribed."

3829. *Proposed by J. D. Hill, Michigan State College.*

Let C be a simple closed rectifiable plane curve and P an arbitrary point inside of C. (a) Show that there exist two points A and B on C such that P bisects the chord AB. (b) Does this property remain true if the curve is non-rectifiable?

3830. *Proposed by Otto Dunkel, Washington University.*

In n dimensional euclidean space, $n \geqq 2$, to the simplex S there corresponds a simplex S' such that the perpendiculars from each vertex A_i of S to the face of S' opposite A_i', $1 \leqq i \leqq n+1$, meet in a point P. Show that the perpendiculars from each vertex A_i' of S' to the face of S opposite A_i meet in a point Q; that is, S and S' are orthologic.

3831. *Proposed by V. Thébault, Le Mans, France.*

Given a triangle ABC and a variable point M on the circumcircle (O), show that (a) The parallels to MA, MB, MC drawn from the orthocenter H meet the corresponding sides in three points on the polar Δ of the point M', diametrically opposite to the point M on (O), with respect to the conjugate circle; and that this line passes through the mid-point of the straight line segment MH: (b) The line Δ envelops the conic with foci O and H inscribed in the triangle ABC: (c) The orthogonal projection of the point M on the line Δ is the orthopole of the reciprocal transversal of that straight line with respect to triangle ABC.

Editorial Note. A straight line d cuts the sides BC, CA, AB of a triangle in L, M, N. Let L' be the symmetric of L with respect to the mid-point of BC; M' and N' are defined in a similar manner. Then L', M', N' lie on a straight line d' which is called the reciprocal transversal of d with respect to triangle ABC. Let A', B', C' be the feet of the perpendiculars from the vertices of triangle ABC upon the straight line d in its plane. Then the perpendiculars from A' to BC, from B' to CA, from C' to AB meet in a point D which is called the orthopole of d with respect to ABC.

SOLUTIONS

3734 [1935, 256]. *Proposed by A. A. Bennett, Brown University.*

A car with $n(n>2)$ passengers of different speeds of mental reaction passes through a tunnel and each passenger acquires unconsciously a smudge of soot upon his forehead. Suppose that each passenger

(1) laughs and continues to laugh as soon as and only so long as he sees a smudge upon the forehead of a fellow passenger;

(2) can see the foreheads of all his fellows;

(3) reasons correctly;

(4) will clean his own forehead when and only when his reasoning forces him to conclude that he has a smudge;

(5) knows that (1), (2), (3), and (4) hold for each of his fellows.

Show that each passenger will eventually wipe his own forehead. (For the case of $n=3$, this has been proposed in conversation by Dr. Church of Princeton.)

I. *Solution by E. P. Starke, Rutgers University.*

Case 1. Consider the last man to clean his forehead. He realizes that every other passenger is laughing at someone with a smudge; but every other passenger has cleaned his forehead. Hence he himself must have the smudge, and he proceeds to clean his forehead.

Case 2. Prior to the above, there was a time when two had smudges. Each would think everybody was laughing at the other until he realized that if his own forehead were clean, the other would be reasoning and acting as in Case 1. As this is not the case, he himself has a smudge. The quicker of the two to reach this conclusion will clean his forehead.

Case 3. Sometime before that, there were three with smudges. Each one should realize that if his own forehead were clean, there would be only two smudges and within a reasonable time the action of Case 2 would result. As this does not occur, he, too, has a smudge.

Case k. Similarly, in general, if at a certain time there were k passengers with smudges, each of them should reason that, since he sees only $k-1$ smudges, if his own forehead were clean the case $k-1$ would hold. Since the other $k-1$ passengers are not acting accordingly, he himself has a smudge. When one has reached this conclusion, he will clean his forehead and the case $k-1$ then actually holds.

Thus the wiping of foreheads will proceed one by one as we go from case n to case $n-1$, to case $n-2$, etc., until (case 1) the last smudge is gone.

II. *Solution by G. M. Clemence, Bethesda, Md.*

Call the passengers A, B, $\cdots$, $N-1$, N; and assume that A is the quickest of perception. Then A will reason as follows: If I do not have a smudge, then B, whoever he may be, will reason as follows. If A and I are both without smudges, then C, whoever he may be, will reason as follows. If A, B, and I are all three without smudges, then $\cdots$. The chain continues to $N-1$, who reasons that he

must have a smudge since otherwise one member of the company would not be laughing. But the foregoing chain of reasoning is not carried through. This forces A to conclude that he himself has a smudge, and he wipes it off. Then B reasons as A expected he would, and wipes his own forehead. Next C follows, and the chain is carried through to N, who immediately perceives that the rest of the company is now laughing at him, and wipes his forehead.

It may be remarked that if N is large, some of the company will probably die laughing before they wipe their foreheads, thus defeating the desired end.

Solved also by W. E. Buker, J. H. Edmonston, L. M. Kelly, H. I. Lane, A. J. Lewis, H. A. Polachek, V. E. Pound, E. J. Purcell, and Leon Recht.

Editorial Note. Buker states in his solution that he proposed a similar problem for three persons in *School Science and Mathematics*; the solution appeared in that journal in vol. 35, 1935, p. 429.

Purcell numbers the passengers in order of decreasing rapidity of arriving at the conclusion that a smudge exists on the face of the reasoner. Then 1 arrives at the conclusion that he has a smudge by a reasoning which we have condensed in symbolic form as follows:

$$K_1K_2 \cdots K_{n-1}n \rightarrow (1, 2, \cdots, n-1),$$

which means Purcell's statement: 1 knows that 2 knows that $\cdots$ that $n-1$ knows that n does not think that he has a smudge, but sees a smudge on at least one of $1, 2, \cdots, n-1$. Since $n-1$ does not remove his smudge, we may write the second step of 1's argument

$$K_1K_2 \cdots K_{n-2}(n-1) \rightarrow (1, 2, \cdots, n-2).$$

A repetition of this argument gives in turn

$$K_1K_2 \cdots K_{n-3}(n-2) \rightarrow (1, 2, \cdots, n-3),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$K_12 \rightarrow (1).$$

After 1 has removed his smudge; 2 will remove his own, and so on.

In some of the solutions the time required for a passenger's reasoning and action is mentioned, for example, in phrases such as, after a reasonable time. Each knows that each reaches his final conclusion by correct reasoning from the existing data. If we assume that each knows that each can conclude his reasoning before the end of a time t, measured from the beginning of the episode, then all smudges will have been removed before the end of t. For, if there are still smudges on any faces at the time t, then the others will know that these have reasoned incorrectly. But this contradicts the assumption above; hence all smudges will have been removed before the end of t.

It will be observed that there are solutions in which a particular form of reasoning is assigned to the passengers. If they use some other argument, say some have arguments different from the others, each must reach the conclusion of the assigned particular form of argument since each reasons correctly.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items of interest to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

Dr. S. F. Barber, of the State University of Iowa, has been promoted to an assistant professorship.

Dr. Daniel Buchanan, dean of the Faculty of Arts and Sciences at the University of British Columbia, will conduct courses in astronomy during the summer session at the University of California at Los Angeles.

Professor Richard Courant has been appointed head of the department of mathematics in the Graduate School of New York University.

Professor D. R. Curtiss, of Northwestern University, has assumed the duties of managing editor of the *Bulletin of the American Mathematical Society*, succeeding Professor E. R. Hedrick whose promotion at the University of California (see page 274) necessitates his withdrawal from that position.

Associate Professor R. A. Hefner, of the Georgia School of Technology, has been promoted to a professorship.

Dr. Jack Levine, of North Carolina State College, has been promoted to an assistant professorship.

Professor E. H. McAlister, of the Oregon State College, has retired with the title of professor emeritus of mathematics. He has been connected with the Oregon state system of higher education for forty-six years.

Associate Professor N. H. McCoy, of Smith College, has been granted leave of absence for the first semester of the coming scholastic year.

Associate Professor Susan M. Rambo, of Smith College, has been promoted to a professorship. She has been granted leave of absence for the second semester of the scholastic year 1937–38.

Dr. Nathan Schwid, formerly of the University of Wisconsin, has been appointed adjunct professor of mathematics at the Texas College of Mines and Metallurgy.

Assistant Professor J. H. Simester, of the University of Louisville, has been promoted to an associate professorship.

Dr. F. C. Smith has been promoted to an assistant professorship in mathematics at the College of St. Francis.

Professor M.H. Stone of Harvard University is spending the second semester in Mexico and Central America.

The following appointments to instructorships in mathematics are announced:

Brooklyn College: Mr. Thomas Nicholson
Brown University: Dr. M. L. Kales
University of Illinois: Dr. W. S. Turpin
Massachusetts Institute of Technology: Dr. Norman Levinson
University of North Carolina: Dr. Nathan Jacobson
Princeton University: Mr. R. H. Fox, Dr. N. E. Steenrod
University of Wisconsin: Dr. C. B. Allendoerfer

Dr. C. W. Crockett, professor emeritus of mathematics and astronomy at Rensselaer Polytechnic Institute, died on December 30, 1936. At the time of his retirement in 1934 he had completed fifty years of service on the faculty.

Professor R. R. Hitchcock, of the University of North Dakota, died on March 11, 1937, at the age of fifty-six. He went to the University of North Dakota in 1910, and had been head of the department since 1915. He was a charter member of the Mathematical Association of America.

The following courses in mathematics are announced for the summer of 1937 (see also pp. 270–274):

University of California at Los Angeles. By Professor G. E. F. Sherwood: Third course in calculus. By Professor Glenn James: Foundations of arithmetic, Fundamental infinite processes. By Professor R. D. James: Selected topics in the theory of numbers.

University of Nebraska. The following courses will be offered during the summer session of 1937: Algebra, Trigonometry, Analytic geometry, Calculus, Mathematics of finance, Advanced euclidean geometry, Advanced analytic geometry, Differential equations, and Mathematical analysis.

University of Pennsylvania. In addition to the usual elementary courses the following advanced courses are offered: By Professor Caris: Modern analytical geometry. By Professor Babb: Differential equations. By Professor Hallett: Advanced calculus. By Professor Hallett: Theory of groups of finite order. By Dr. Clarkson: Abstract spaces.

University of Virginia. The following graduate courses will be offered: First term, June 21 to July 31: By Professor Whyburn: Foundations of geometry; Higher algebra. Second term: August 2 to September 4: By Professor Whyburn: Functions of real variables; Higher algebra.

Washington University. The following advanced courses will be offered during the summer quarter, 1937: Analysis; Projective geometry; Advanced descriptive geometry; Advanced differential equations.

A member or subscriber wishing to have his copy of the June-July MONTHLY forwarded when it appears should leave four cents postage with the local post office or carrier with instructions for remailing.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, E. J. MOULTON, Northwestern University, Evanston, Ill.

BOOKS FOR REVIEW should be addressed to REVIEW EDITOR, American Mathematical Monthly, 531 West 116th Street, New York, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of addresss to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-first Summer Meeting, Pennsylvania State College, Sept. 6–7, 1937.
Twenty-second Annual Meeting, Indianapolis, Ind., December 30–31, 1937.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1937 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Waynesburg, Pa., May 1.
ILLINOIS, DeKalb, May 14–15.
INDIANA, Greencastle, April 30–May 1.
IOWA, Dubuque, April 16–17.
KANSAS, Wichita, April 3.
KENTUCKY, Louisville, May 1.
LOUISIANA-MISSISSIPPI, Hammond, La., March 5–6.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Lynchburg, Va., May 8.
MICHIGAN, Ann Arbor, March 20.
MINNESOTA, St. Paul, May 15.
MISSOURI.
NEBRASKA, Lincoln, May 7.
OHIO, Columbus, April 1.
OKLAHOMA, Tulsa, February 5.
PHILADELPHIA, Haverford, Nov. 27.
ROCKY MOUNTAIN, Greeley, Colo., April 16–17.
SOUTHEASTERN, Nashville, Tenn., April 16–17.
SOUTHERN CALIFORNIA, Los Angeles, March 6.
SOUTHWESTERN, State College, N.M., April 2–3.
TEXAS, Houston, April 23–24.
WISCONSIN, Milwaukee, May 8.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

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The editors announce with deep regret
the death of
Herbert Ellsworth Slaught
on May 21, 1937.
At the time of his death
Professor Slaught was Honorary President
of the Mathematical Association
of America
and had served continuously
as an editor of this Monthly
for thirty years.

NEW MEMBERS OF THE MATHEMATICAL ASSOCIATION OF AMERICA

The following forty-six persons and one institution have been elected to membership in the Association on applications duly certified:

To Individual Membership

Sister M. ANTONINA, A.B.(St. John's Univ.) Teacher, Queen of All Saints High School, Brooklyn, N. Y.

JOSÉ BARRAL-SOUTO, Sc.D. Prof., Statistics, Fac. of Econ. Sciences, Univ. of Buenos Aires, Buenos Aires, Rep. Argentina.

Brother BERNARD ALFRED, A.M.(Catholic Univ.) Instr., Manhattan Coll., New York, N. Y.

H. D. BOUTELLE, B.S., Ch.E.(Worcester Poly. Inst.) Asst. Prof., Massachusetts State Coll., Amherst, Mass.

C. B. BOYER, A.M.(Columbia) Instr., Brooklyn Coll., Brooklyn, N. Y.

JOHN BREILAND, M.S.(Iowa) Instr., Luther Coll., Decorah, Iowa.

H. W. CHARLESWORTH, A.M.(Colorado State Coll. of Educ.) Chm. of Dept., East High School, Denver, Colo.

W. B. COLEMAN, A.M.(Lehigh; Harvard) Instr., Georgia School of Tech., Atlanta, Ga.

E. G. H. COMFORT, Ph.D.(Brown) Instr., Tulane Univ., New Orleans, La.

W. A. CORDREY, A.M.(Peabody) Grad. student, George Peabody Coll., Nashville, Tenn.

W. E. COX, M.S.(Tulane) Instr., Mississippi State Coll., State College, Miss.

R. C. DRAGOO, A.M.(Oklahoma) Prof., Southeastern State Teachers Coll., Durant, Okla.

E. D. EAVES, A.M.(Oklahoma) Instr., Univ. of Tennessee, Knoxville, Tenn.

W. H. FAGERSTROM, Ph.D.(Columbia) Instr., Coll. of the City of New York, New York, N. Y.

RUBY S. FLANNERY, A.B.(Colorado) Teacher, East High School, Denver, Colo.

B. H. GERE, A.M.(Syracuse) Instr., Massaachusetts Inst. of Tech., Cambridge, Mass.

M. E. GILLIS, A.M.(Chicago) Instr., Univ. of Tennessee, Knoxville, Tenn.

G. A. HEDLUND, Ph.D.(Harvard) Asso. Prof., Bryn Mawr Coll., Bryn Mawr, Pa.

CHARLES HOPKINS, Ph.D.(Illinois) Instr., Tulane Univ., New Orleans, La.

P. M. HUMMEL, Ph.D.(Ohio State Univ.) Instr., Univ. of Alabama, University, Ala.

L. P. HUTCHISON, Ph.D.(Kentucky) Instr., Indiana Tech. Coll., Fort Wayne, Ind.

E. D. JENKINS, Ph.D.(Ohio State Univ.) Asst. Prof., E. Kentucky State Teachers Coll., Richmond, Ky.

V. V. JOHNSTON, B.S.(Geneva Coll.) Rate and Cost Investigator, National Tube Co., Ellwood City, Pa.

P. W. KETCHUM, Ph.D.(Illinois) Member, Inst. for Advanced Study, Princeton, N. J.

M. L. MANNING, M.S.(Pittsburgh) Research Engineer, Westinghouse Elec. & Mfg. Co., Sharon, Pa.

C. F. MYERS, JR., B.S.(Citadel) Asso. Prof., The Citadel, Charleston, S. C.

E. B. OGDEN, Ph.D.(Boston Univ.) Prof., Head of Dept. Math. and Physics, Atlantic Union Coll., South Lancaster, Mass.

W. A. PATTERSON, Ph.D.(Ohio State Univ.) Instr., Fenn Coll., Cleveland, Ohio

VERTIE D. PRINCE, A.M.(Georgia) Prof., Flora Macdonald Coll., Red Springs, N. C.

D. G. RABINOW, M.S.(Mass. Inst. of Tech.) Grad. student, Harvard Univ., Cambridge, Mass.

W. C. RANDELS, Ph.D.(Brown) Instr., Northwestern Univ., Evanston, Ill.

C. G. SCHILLING, Ph.D.(Illinois) Instr., Purdue Univ., LaFayette, Ind.

E J SCOTT, A.M.(Vanderbilt) Teaching fellow, Vanderbilt Univ., Nashville, Tenn.

W. E SEWELL, Ph.D.(Harvard) Instr., Georgia School of Tech., Atlanta, Ga.

JULIA SIMPSON, A.M.(Columbia) Chm. of Dept., Grover Cleveland High School, Ridgewood, N. Y.

L. W. SIMS, M.S.(Northwestern) Teacher, Community High School, St. Charles, Ill.

A. R. Sloan, A.M.(Vanderbilt) Asso. Prof., Carson-Newman Coll., Jefferson City, Tenn.

S. Roscoe Smith, Ph.D.(Indiana) Instr., Carnegie Inst. of Tech., Pittsburgh, Pa.

H. E. Spencer, Ph.D.(Cornell) Prof., Presbyterian Coll., Clinton, S. C.

E. M. Starr, A.B.(Indiana) Asso. Prof., Carnegie Inst. of Tech., Pittsburgh, Pa.

Elizabeth C. Strayhorn, A.M.(Peabody) Asst. Prof., W. Kentucky State Teachers Coll., Bowling Green, Ky.

S. T. Vadhana, B.S. in E.E., Ch.E.(Tri-State Coll.) Grad. student, Univ. of Michigan, Ann Arbor, Mich.

Edgar Wax, A.M.(Oklahoma) Prof., Head of Dept., Central State Teachers Coll., Edmond, Okla.

D. L. Webb, Ph.D.(Calif. Inst. of Tech.) Instr., Georgia School of Tech., Atlanta, Ga.

J. T. C. Wright, A.M.(Peabody) Head of Dept., Appalachian State Teachers Coll., Boone, N. C.

J. E. Yenni, A.M.(St. Louis Univ.) Instr., Spring Hill Coll., Spring Hill, Ala.

To Institutional Membership

Mount St. Joseph College, Chestnut Hill, Philadelphia, Pa.

W. D. Cairns, *Secretary-Treasurer*

THE APRIL MEETING OF THE OHIO SECTION

The twenty-second annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, April 1, 1937, with an afternoon session, a dinner, and an evening session. Professor J. H. Weaver, chairman of the Section, presided at these sessions.

Seventy-four persons registered attendance, fifty-four of whom were members of the Association, namely: R. B. Allen, W. E. Anderson, F. R. Bamforth, Grace M. Bareis, I. A. Barnett, P. E. Baur, H. M. Beatty, H. A. Bender, Henry Blumberg, M. G. Boyce, J. B. Brandeberry, Foster Brooks, O. E. Brown, R. S. Burington, F. E. Carr, Rufus Crane, Wayne Dancer, O. L. Dustheimer, T. M. Focke, N. A. Gilbert, B. C. Glover, R. C. Hildner, E. J. Hirschler, F. C. Jonah, E. M. Justin, L. C. Knight, H. W. Kuhn, A. C. Ladner, Lincoln LaPaz, R. H. MacCullough, R. H. Marquis, Florentina Mathias, Sister Mercedes, Max Morris, J. R. Musselman, J. R. Overman, Jesse Pierce, Tibor Radó, S. E. Rasor, F. W. Reed, Hortense Rickard, R. F. Rinehart, S. A. Rowland, George Sauté, W. G. Simon, Mary E. Sinclair, H. E. Stelson, C. F. Thomas, C. C. Torrance, J. H. Weaver, Fern Welker, F. B. Wiley, C. O. Williamson, C. H. Yeaton.

The following officers were elected for the coming year: Chairman, O. E. Brown, Case School of Applied Science; Secretary-Treasurer, Rufus Crane, Ohio Wesleyan University; Member of Executive Committee, S. E. Rasor, Ohio State University; Members of Program Committee, F. B. Wiley, Denison University, and Lincoln LaPaz, Ohio State University.

It is expected that the next meeting will be held at the Ohio State University either March 31 or April 7, 1938.

The following papers were read:

1. "A generalization of the circles of Apollonius and some resulting properties" by the chairman of the Section, Professor J. H. Weaver, Ohio State University.

2. "Miscellanea on Diophantine equations" by Professor I. A. Barnett and Dr. C. W. Mendel, University of Cincinnati.

3. "Geometric proofs of multiple angle formulas" by Professor Wayne Dancer, University of Toledo.

4. "A note on tensor analysis" by Professor F. C. Jonah, Western Reserve University.

5. "Pythagorean numbers" by Professor O. E. Brown, Case School of Applied Science.

6. "Solution of a differential equation of the first order and first degree in terms of infinite series of solutions of linear differential equations" by Professor Jesse Pierce, Heidelberg College.

7. "Inconsistent uses of functional notation" by Dr. C. C. Torrance, Case School of Applied Science.

8. "A generalization of Newton's identities" by H. Reingold, University of Cincinnati, introduced by Professor Barnett.

9. "Recent developments in the conjugate endpoint problem in the calculus of variations" by Professor George Sauté, Western Reserve University.

10. "The technique of generalization" by Professor Henry Blumberg, Ohio State University.

Abstracts of some of the papers follow, the numbers corresponding to those in the list of titles:

1. If the sides A_iA_j of a triangle are divided internally and externally into segments whose lengths are proportional to the sides adjacent to the segments, two points A_k' and A_k'' are determined. If then the segments $A_k'A_k''$ are used as diameters, the circles thus determined are coaxal and are known as the circles of Apollonius. Professor Weaver has generalized the problem by dividing the sides A_iA_j in the ratios $m_i:m_j$ or $m_j:m_i$, where the product of the three ratios is unity. Under these conditions the circles determined on $A_k'A_k''$ are also coaxal and the radical axis always passes through the circumcenter of the triangle. Certain special cases yield a number of well known points and lines associated with the triangle.

2. Let a proper primitive Pythagorean set be a triple of positive integers having no common factor and satisfying the equation $x^2+y^2=z^2$. The first problem discussed by Professor Barnett is the number of such sets lying in a given plane $Ax+By+Cz+D=0$, where A, B, C, D are integers. It is found that the solution of this problem depends upon the nature of $\Delta\equiv A^2+B^2-C^2$. The second problem is the determination of all the rational solutions of the equation $r^2+s^2+t^2+2rst=1$, which arises from an addition formula in trigonometry. The final problem is the Diophantine equation $x^4+y^4+64=z^4$. Certain interesting facts were pointed out concerning the origin of the problem and methods of attacking it. No method of solving the equation completely has yet been found.

3. This paper appears in this issue of the MONTHLY.

4. In this paper, Professor Jonah developed a method for obtaining the derivatives of the general unitary-vectors in terms of the Riemann-Christoffel

3-index symbols without the necessity of using results from differential geometry.

5. Professor Brown showed that every solution in integers of the equation

$$x_1^2 + \cdots + x_s^2 = y_1^2 + \cdots + y_t^2$$

may be obtained from a trivial solution $a_1^2 + \cdots + a_n^2 = a_1^2 + \cdots + a_n^2$ by repeated application of the transformations T_h: $X_1 = s - x_1$, $Y_1 = s - y_1$; $X_2 = s - x_2$, $Y_2 = s - y_2$; $\cdots$ $X_{h+2} = s - x_{h+2}$, $Y_h = s - y_h$; $X_{h+3} = x_{h+3}$, $Y_{h+1} = y_{h+1}$; $\cdots$ $X_n = x_n$, $Y_n = y_n$; $s = x_1 + x_2 + \cdots + x_{h+2} - y_1 - y_2 - \cdots - y_h$, and trivial transformations.

6. The right-hand member of the differential equation under consideration by Professor Pierce is a power series in the dependent variable x, and the coefficient functions $f_n(t)$ satisfy certain integrability conditions but are not necessarily analytic or continuous or bounded on the path of integration. The solution function contains an arbitrary parameter c_1 and hence is the general solution. This parameter may or may not be the initial value of x. In one case the initial value of x is zero for all values of c_1.

7. Dr. Torrance discussed briefly the pedagogic difficulties involved in regarding the symbol $f(x)$ as denoting both a function of x as such and an arbitrary value of that function. He suggested as a possible method of avoiding these difficulties, that, in elementary mathematics, we use the single letter f to denote a function, and that we reserve the symbol $f(x)$ exclusively to denote some value of that function.

8. Mr. Reingold discussed an interesting generalization of the well known Newton identities. Let $\|c_{ij}\|$, $(i, j = 1, 2, \cdots, n)$, be a square matrix. The identities obtained in this paper are

$$s_k + a_1 s_{k-1} + a_2 s_{k-2} + \cdots + a_{k-1} s_1 + k a_k = 0, \qquad (k = 1, 2, \cdots, n-1),$$

$$s_k + a_1 s_{k-1} + a_2 s_{k-2} + \cdots + a_n s_{k-n} = 0, \qquad (k = n, n+1, \cdots),$$

where s_m is the trace (sum of the diagonal elements) of the m-th power of the matrix c_{ij} and where $a_1, a_2, \cdots, a_n$ are the coefficients of the characteristic equation of this matrix, i.e.,

$$| c_{ij} - \lambda I | = (-1)^n (\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots + a_{n-1} \lambda + a_n).$$

For the particular square matrix $\|c_{ij}\| \equiv x_i \delta_{ij}$, where $\delta_{ij} = 1$ when $i = j$, and $\delta_{ij} = 0$ when $i \neq j$, the above relations reduce to the ordinary Newton identities. Mr. Reingold then discussed some applications of the above identities.

9. Professor Sauté reported that necessary and sufficient conditions for the fixed end point problem in m-space have been formulated. The necessary condition is essentially a generalization of the well known condition in the plane that no part of the envelope of the family of extremals neighboring g may extend from the second endpoint B toward the initial endpoint A if g is to furnish a minimum for the integral; it is given in analytic form. On the other hand, the formulation of the sufficient conditions is geometric. Marston Morse has given

it with details of proof in the *Bulletin of the American Mathematical Society*, vol. 42, February, 1936.

10. A certain part of the "creative" process, widely so regarded, of securing generalizations is amenable to analysis, and something of a useful technique for deriving generalizations is susceptible of formulation. In matters of technique the mere knowledge of a principle is of slight value; what counts is how operative the principle becomes in our consciousness. The paper by Professor Blumberg treats of a variety of principles potentially of use in the art of generalization. Following are some of the ideas discussed: determining the essence of a concept; manifoldness and singleness of property; relation of rigorous formulation to intuitive essence; the importance of the processes of decomposition, change of form, approximation, and of the principle of indetermination in the art of generalization; the passage from the absolute to the relative; the process of extension of class; postulate systems in relation to the technique of generalization; pseudo-generalizations; dangers of abstraction.

RUFUS CRANE, *Secretary*

THE FOURTH ANNUAL MEETING OF THE OKLAHOMA SECTION

The fourth regular meeting of the Oklahoma Section of the Mathematical Association of America was held at the Central High School in Tulsa, Oklahoma, on Friday morning, February 5, 1937, Professor L. W. Johnson presiding.

The number in attendance was sixty-five, including the following sixteen members of the Association: J. C. Brixey, J. H. Butchart, N. A. Court, Mildred Dolezal, R. C. Dragoo, L. A. Dwight, H. L. Hall, J. O. Hassler, E. E. Heimann, L. W. Johnson, Clarence McCormick, Dora McFarland, W. T. Short, C. E. Springer, R. W. Veatch, B. S. Whitney.

The following officers were elected for the next year: Chairman, H. L. Hall, Northwestern State Teachers College: Vice-Chairman, R. C. Dragoo, Southeastern State Teachers College; Secretary, C. E. Springer, University of Oklahoma. The next meeting will be held at Oklahoma City in February 1938 in connection with the Oklahoma Education Association.

The following papers were presented:

1. "Logarithmic solution of cubic and quartic equations" by Professor W. T. Short, Oklahoma Baptist University.

2. "New methods for computing meteor heights" by B. S. Whitney, Capitol Hill Junior College, Oklahoma City.

3. "Associativity conditions for division algebras defined by non-Abelian groups of three generators" by Professor Dora McFarland, University of Oklahoma.

4. "The triangles determined on three lines by parallel planes" by Professor J. H. Butchart, Phillips University.

5. "The null forms $Ax^2+By^2+Cz^2+Du^2$ which represent all integers" by Professor J. C. Brixey, University of Oklahoma.

6. "Should colleges offer elementary plane geometry for credit?" by Professor R. C. Dragoo, Southeastern State Teachers College.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Professor Short solved the reduced cubic $x^3+px+q=0$ by the substitutions $r^2=-4p/3$ and $\cos 3\theta=|4q/r^3|$ for the case in which p is negative and $|4q/r^3|\leqq 1$. The roots x_1, x_2, x_3 are given by $x_1=r\cos\theta$, $x_2=r\sin(\theta+30°)$, and $x_3=r\cos(\theta+60°)$. In case p is negative and $|4q/r^3|>1$, $r^2=-4p/3$ and $\cosh 3\theta=|4q/r^3|$, in which case the roots are given by $x_1=\cosh\theta$, $2x_2=-y_1+iy\sqrt{3}$, $2x_3=-x_1-iy\sqrt{3}$, where $y=r\sinh\theta$. If p is positive, $r^2=4p/3$ and $\sinh 3\theta=|4q/r^3|$. He extended this method to the solution of the reduced quartic and exhibited several examples.

2. The methods described by Mr. Whitney were devised to make allowance for inevitable errors in visual observations of meteors. In the first method, the coordinates of a point on the common perpendicular to the lines of observation (determined by the coordinates of the two observing stations and the directions of corresponding points on the observed paths of the meteor) are so determined that the directions of observation are adjusted through equal angles. The second method is a least-squares solution for the case of observations from n stations, which may be used for computing heights of multiple-observed meteors. The complete derivations have been published in *Monthly Notices of the Royal Astronomical Society*, vol. 96, no. 5, p. 544.

3. In her paper, Professor McFarland traced the development of knowledge of division algebras since 1905, and followed this by a discussion of the general requirements for an algebra to be associative and the methods used in working out these conditions in detail for the three generator problem.

4. Professor Butchart showed that the locus of the centroid of a triangle determined by the traces of three skew lines on a plane moving parallel to itself is a fourth line and that the locus of the circumcenter, ordinarily a space quartic, may become a hyperbola or even a line. Some applications in the plane were considered.

5. Professor Brixey gave a review of his University of Chicago dissertation, "*The null forms $Ax^2+By^2+Cz^2+Du^2$ which represents all integers.*" It was shown, beginning with any known solution x, y, z, u, not all zero, of the null form $F\equiv Ax^2+By^2+Cz^2+Dy^2=0$, F satisfying certain conditions, that from this solution it is possible (1) to construct solutions of $F=g$, g arbitrary, or (2) to construct solutions of $F=0$ such that from these solutions we can obtain solutions of $F=g$, g arbitrary. Hence, every null form F having certain properties is universal.

6. Professor Dragoo concluded colleges should offer elementary plane geometry for credit since (a) plane geometry is not required for graduation from Oklahoma high schools and in some of them it is not offered, (b) the courses in composite mathematics, as taught in the high schools, do not afford the opportunity to learn the nature of a proof in plane geometry, (c) at least one-fourth

of the liberal arts colleges and more than one-half of the teachers colleges of the United States do not require credit in plane geometry for entrance, and (d) a survey of eleven of the colleges of Oklahoma reveals that nearly ten per cent of freshmen in these colleges have not received credit in plane geometry. Professor Dragoo argued that colleges exist for the benefit of the student, that his conclusion is supported by abundant precedent in the case of other high school studies, and that plane geometry requires a type of thinking which is of college grade.

C. E. SPRINGER, *Secretary*

THE FOURTEENTH ANNUAL MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The fourteenth annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at Southeastern Louisiana College and Hammond High School, Hammond, Louisiana, March 5–6, 1937. The chairman, Professor D. S. Dearman of State Teachers College, Hattiesburg, Mississippi, presided and introduced the guest speaker, Professor W. L. Miser of Vanderbilt University. Professor Miser's subject at the banquet was "Applications, the better half of mathematics."

The attendance was approximately sixty, including the following nineteen members of the Association: Nola L. Anderson, W. E. Cox, Mrs. A. P. Daspit, D. S. Dearman, Virginia I. Felder, Elizabeth Freas, Deborah M. Hickey, Dorothy McCoy, A. C. Maddox, W. L. Miser, I. C. Nichols, Arthur Ollivier, W. V. Parker, S. T. Sanders, C. D. Smith, H. L. Smith, P. K. Smith, V. B. Temple, J. F. Thomson.

The following officers were elected for the year 1937–38: Chairman, Dorothy McCoy, Belhaven College; Vice-Chairman for Louisiana, Nola L. Anderson, Sophie Newcomb College; Vice-Chairman for Mississippi, C. D. Smith, Mississippi State College; Secretary, W. V. Parker, Louisiana State University.

The following six papers were read:

1. "Remarks on the transcendence of e and π" by G. F. Cramer, Tulane University, introduced by the Secretary.
2. "A problem in the computation of state and federal taxes" by Professor J. F. Thomson, Tulane University.
3. "Harmonic section of the complex line" by Professor B. E. Mitchell, Millsaps College.
4. "On certain types of correspondence in euclidean geometry" by Professor C. D. Smith, Mississippi State College.
5. "History of mathematics as an aid in teaching" by Professor W. L. Miser, Vanderbilt University.
6. "Development of certain frequency curves in Charlier's type A series" by Professor Arthur Ollivier, Mississippi State College.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles.

1. Mr. Cramer presented some of the history of transcendence proofs and an outline of a proof essentially the same as that of Gordan. There was also a short discussion of the relation between these proofs and the ancient problem of squaring the circle.

2. In Louisiana the Federal tax is deductible from the net income before computing the state tax, and likewise the state tax is deductible from the net income before computing the Federal tax. Professor Thomson raised the question as to the net income taxable under Federal law after deducting the state tax, also as to the net income subject to state tax after the Federal tax has been deducted. Formulas were derived for these two quantities in terms of the original net incomes taxable and the rates of taxation.

3. The problem, given three real points on a (real) line to find the harmonic conjugate of one of the points with respect to the other two, is a familiar construction. Professor Mitchell carried this problem into the complex domain. All imaginary points were given real representation by the Laguerre method. By a series of nested theorems the final, inclusive theorem was reached: *The necessary and sufficient condition that four complex points shall constitute a harmonic range on a complex line is* (1) *that the quadrilaterals determined by the initial points of the real representation and by the terminal points of the same shall be both inversely similar and concyclic and* (2) *that the vertices of each quadrilateral shall constitute a harmonic set on its circumscribing circle.*

4. In this paper, Professor Smith began with a theorem of Euclid which states that three circles which intersect two by two form a system which has a radical center. A theorem for quadrilaterals followed which led to a type of continuous correspondence analogous to harmonics. The theory serves to connect certain classic theorems which to date have been treated from different points of view without apparent connections. Additional relations also appear between points interior and points exterior to the triangle.

5. From the doctrine that the education of the child must accord both in mode and arrangement with the education of mankind as considered historically, it is argued that the history of the origin and development of mathematics can be made an effective aid in teaching. Professor Miser considered some points of mathematics which are better taught when they are looked on historically and concluded that, when free use is made of readily available mathematical history in teaching mathematics by the most interesting and effective methods, then mathematics will appeal to students as a rich heritage and become a practical and basic part of their knowledge.

6. The distributions considered by Professor Ollivier consist of certain sums and products of ordinary binomials and may be designated as generalized Poisson or Lexis distributions. The characteristic function associated with each distribution function is set forth and the semi-invariants and moments found by

differentiation. This method may be used for any function for which the characteristic function can be found. With the semi-invariants known, the distribution function may be approximately represented by a Charlier Type A series.

DOROTHY MCCOY, *Secretary*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The seventeenth regular meeting of the Southern California Section of the Mathematical Association of America was held at the Los Angeles Junior College, Los Angeles, California, Saturday, March 6, 1937. Professor S. E. Urner presided.

The attendance was sixty-two, including the following thirty members of the Association: L. J. Adams, O. W. Albert, L. D. Ames, Clifford Bell, E. T. Bell, L. T. Black, Elizabeth J. Breuer, Jessie R. Campbell, Myrtie Collier, P. H. Daus, D. C. Duncan, Iva B. Ernsberger, H. H. Gaver, Harriet E. Glazier, E. R. Hedrick, M. R. Hestenes, G. H. Hunt, C. G. Jaeger, Glenn James, Ada A. McClellan, A. D. Michal, W. B. Orange, Lena E. Reynolds, J. M. Robb, G. E. F. Sherwood, D. V. Steed, C. W. Trigg, S. E. Urner, W. M. Whyburn, Euphemia R. Worthington.

The following officers were elected for the next year: Chairman, A. D. Michal, California Institute of Technology; Vice-Chairman, W. M. Whyburn, University of California at Los Angeles; Program Committee: L. J. Adams, Santa Monica Junior College, and C. G. Jaeger, Pomona College. The next meeting was tentatively scheduled for March 5, 1938, at Pomona College.

The following six papers were read:

1. "Spherical harmonics with spin" by Professor L. J. Adams, Santa Monica Junior College.

2. "Descartes and three centuries of analytic geometry" by Professor A. D. Michal, California Institute of Technology.

3. "Strong summability of sequences" by Dr. Hugh Hamilton, Pomona College, introduced by Professor Jaeger.

4. "The extension of Horner's method to the solution of n equations in n unknowns" by Professor L. E. Gurney, University of Southern California, introduced by Professor Ames.

5. "Early contributors to the calculus of variations" by Dr. M. R. Hestenes, University of California at Los Angeles.

6. "Some autopolar configurations" by Dr. D. C. Duncan, Compton Junior College.

Abstracts of the papers follow, the numbers corresponding to those in the list of titles.

1. Spherical harmonics with spin are defined in Hermann Weyl's book, *Theory of Groups and Quantum Mechanics*. Professor Adams gave explicit formu-

FOURTEENTH ANNUAL MEETING OF THE INDIANA SECTION

The fourteenth meeting of the Indiana Section of the Mathematical Association of America was held Friday and Saturday, April 30 and May 1, 1937, at DePauw University, Greencastle, Indiana.

Eighty persons registered at the different sessions including the following twenty-eight members of the Association: W. C. Arnold, L. G. Black, G. E. Carscallen, J. E. Dotterer, Olive M. Draper, W. E. Edington, P. D. Edwards, G. H. Graves, H. E. H. Greenleaf, S. G. Hacker, Lawrence Hadley, C. T. Hazard, F. H. Hodge, L. P. Hutchison, M. W. Keller, Mayme I. Logsdon, Florence Long, Juna M. Lutz, T. E. Mason, H. A. Meyer, D. H. Porter, H. R. Pyle, C. K. Robbins, C. G. Schilling, L. S. Shively, W. O. Shriner, Anna K. Suter, K. P. Williams.

At the business session on Saturday the following officers were elected for next year: Chairman, W. O. Shriner, Indiana State Teachers College; Vice-Chairman, Florence Long, Earlham College; Secretary, P. D. Edwards, Ball State Teachers College. The fifteenth annual meeting will be held at Indiana State Teachers College, Terre Haute, in May 1938.

Professor P. D. Edwards made the report for the committee appointed to encourage and recognize superior preparation for the teaching of secondary mathematics. On the basis of examinations conducted April 17 and April 24, 1937, a Certificate of Merit was awarded by the Indiana Section to Eugene Grenling of Butler University and Eugene Brazier of Earlham College.

Following the annual dinner on Friday evening the members of the Association met with the members of the Indiana Philosophical Association where the following joint program was presented:

1. "The logical structure of a four-dimensional space" by Professor Mayme I. Logsdon, University of Chicago.

2. "Mathematics and empirical science" by Professor Rudolf Carnap, University of Chicago.

Abstracts of these papers follow:

1. Preliminary to the discussion of the logical structure of a four-dimensional space Professor Logsdon compared the logical structure of Euclid's axiomatic geometry of three-space with the logical structure of an analytical geometry of three-space. The latter is a purely arithmetical theory and is logically equivalent to the former if suitable definitions of the distance function and of the sub-spaces are agreed upon. A space is curved if the angle sum of a triangle whose sides are geodesics is not 180°. A definition was given of a general analytical space, and it was observed that phenomena of our world of experience can be explained by more than one theory, and that it is unwise to accept the predictions of any mathematical theory with regard to situations which cannot be checked by observation.

2. Professor Carnap presented the view that theorems of mathematics sometimes have the same grammatical form as sentences with factual content

(*e.g.*, "7 is a prime number," like "Chicago is a large city"). However, their truth—like that of the theorems of logic—does not depend upon the existence of any facts but upon our conventions as to the structure of language. Within the whole of scientific language, mathematical symbols (*e.g.*, '1', '2', '+', '=', etc.) play the same role as logical symbols (*e.g.*, 'or', 'and', 'not', 'every', etc.); they do not refer to anything but serve to connect other symbols. The chief part of the scientific language consists of the synthetic, factual, sentences of empirical sciences. The theorems of mathematics and logic are instruments for facilitating the operations with such factual sentences.

At the sessions on Saturday the following program was presented:

1. "Vitalizing mathematics" by Professor W. E. Edington, DePauw University, retiring chairman of the Indiana Section.
2. "A class in Fluxions" by Professor H. E. H. Greenleaf and members of the Napierian Club, DePauw University.
3. "A new polynomial approximation for the Gamma Function" by Professor Cornelius Lanczos, Indiana University, introduced by Professor K. P. Williams.
4. "A problem in phase rule" by Zenon Szatrowski, Indiana University, introduced by Professor K. P. Williams.
5. "Homogeneous functionals and Euler's theorem" by Dr. Richard Duffin, Purdue University, introduced by Professor T. E. Mason.
6. "On the derivatives of polynomials" by Dr. A. C. Schaeffer, Purdue University, introduced by Professor T. E. Mason.
7. "What do students think of mathematics?" by Professor G. H. Graves, Purdue University.
8. "Mathematical prefaces and advertisements" by Professor T. E. Mason, Purdue University.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles.

1. Professor Edington presented the following five suggestions as possible means for improving the teaching of mathematics and counteracting the trend away from mathematics: (1) Conscious, continual and consistent reference to the analogies between mathematical processes and the ordinary processes met with in the work-a-day world; (2) Dissemination of information on the wide application of mathematics in the various fields of human endeavor, including not only the fields of physical science, engineering and business, but also such fields as agriculture, physiology, medicine and psychology; (3) Dissemination of knowledge of the parallel cultural development of mathematics with art, literature, and science; (4) Teaching of certain fundamental concepts of mathematics earlier in the student's training; (5) The use of favorable propaganda based on facts and on the opinions of authorities and others who have recognized the values of mathematics as a cultural as well as a utilitarian subject.

2. Professor Greenleaf and four members of the Napierian Club of DePauw University demonstrated the notation and language of Newton by conducting a class recitation using as a text "Principles of Fluxions" by Rev. S. Vince. This text was printed in America in 1812. Most of the demonstrations were in the form of Euclidean algorithms. This form of demonstration appears much more strange today than the difference in notation and raises the interesting question as to whether we profit by the retention of this form in the study of elementary geometry in the United States long after we have abandoned it in all other branches of mathematics and after its abandonment in the study of elementary geometry in the continental countries of Europe.

3. Professor Lanczos discussed a method by which the ordinary asymptotic series of Stirling in the expansion of the Gamma function may be replaced by a strictly convergent series representing the function with even accuracy in a large, finite, a priori given range. The expansion is arranged in Tchebychef's polynomials and yields for the logarithm of $\Gamma(x)$ in the range $1 \leqq x \leqq \infty$ the successive accuracies 10^{-4}, 10^{-5}, $4\ 10^{-7}$, 10^{-10} by using two, three, four and eight terms of the series. The coefficients of the approximation have been determined by the method of trigonometric interpolation, making use of the tabulated values of $\log \Gamma(x)$ at 12 selected points of the interval. The same series remains an effective approximation in large portions of the complex domain.

4. The problem discussed by Mr. Szatrowski is to find the composition of the vapor given off by a perfectly miscible binary liquid mixture. It was shown that the composition of vapor in equilibrium with a liquid mixture can be expressed as a function of (1) the composition of the liquid, (2) the composition of an amount of vapor given off during a definite change in the boiling temperatures, and (3) the rates of change of these compositions with the temperature. The composition of the vapor in equilibrium with the liquid mixture is determined by means of this relationship, where the composition of the liquid mixture and the composition of an amount of vapor formed during a definite change in the boiling temperatures have been determined experimentally.

5. Dr. Duffin spoke on the extension of Euler's Theorem to "homogeneous functionals." Suppose the product of two functions is integrated over a fixed region. If one of the functions is homogeneous, it may be replaced by Euler's well known expressions, but this replacement might be valid in some cases even if the function were not homogeneous. A modified definition of homogeneous was given which is a sufficient condition for this replacement. An application was given to potential theory.

6. Dr. Schaeffer acknowledged credit to Dr. Duffin for assistance in the preparation of this paper. Let $P(z)$ be a polynomial of degree n with real coefficients and not greater than 1 in absolute magnitude in the interval $(-1, 1)$ of the real axis. The nth Tchebychef Polynomial, $T_n(z) \cdot \equiv \cdot \cos(n \cos^{-1} z)$, satisfies these conditions. It is shown that at certain points inside the unit circle of the z plane and at all points outside the unit circle $|P^k(z)| \leqq |T_n^k(z)|$, superscripts denoting differentiation.

7. During three years twenty-one classes of students were asked, "What, in your opinion, are the values in studying mathematics?" The students were predominantly freshman and sophomore engineers, but there were two classes of science freshmen and three classes included juniors and seniors expecting to teach mathematics. Professor Graves stated there was much evidence that students were seeking to relate their study of mathematics to their total experience. Besides the predominant reply that mathematics is needed in engineering and science many emphasized that the procedure of solving a problem was of value in a wide range of situations. Most students seem to assume the transfer of training but some appreciate that the problem of transfer demands special attention.

8. Professor Mason discussed the changes that took place in the purposes of authors in writing mathematical books as illustrated by the prefaces in forty books distributed over the period from 1585 to 1859. Extracts from prefaces were quoted as illustrative of authors' purposes. He illustrated also the nature of advertisements printed in mathematical books during this period.

P. D. Edwards, *Secretary*

THE MAY MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The eighth regular meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at Waynesburg College, Waynesburg, Pennsylvania, on Saturday, May 1, 1937. Professor L. L. Dines, chairman of the Section, presided at both the morning and afternoon sessions. Following the afternoon session a very enjoyable social hour was spent at a tea arranged through the courtesy of the Waynesburg College department of mathematics. The date of the fall meeting was set for October 23, 1937, at the University of Pittsburgh.

The number of those in attendance was fifty-one, including the following twenty-seven members of the Association: C. S. Atchison, O. F. H. Bert, H. L. Black, A. M. Bryson, Helen Calkins, L. L. Dines, H. L. Dorwart, F. A. Foraker, C. H. Graves, R. P. Johnson, V. V. Johnston, A. V. Karpov, W. A. Klein, M. L. Manning, David Moskovitz, L. T. Moston, E. G. Olds, F. W. Owens, Helen B. Owens, J. B. Rosenbach, E. A. Saibel, S. R. Smith, E. M. Starr, R. G. Sturm, J. S. Taylor, Bird M. Turner, E. A. Whitman.

Following a welcoming address by President P. R. Stewart of Waynesburg College, the following five papers were read:

1. "Purposive selection" by W. A. Klein, Carnegie Institute of Technology.
2. "A theorem concerning certain surface paths on the rectangular parallelopiped" by V. V. Johnston, National Tube Company.
3. "Elementary electrodynamics of the cathode ray oscillograph" by E. R. Whitehead, Duquesne Light Company, introduced by the Secretary.

4. "Concerning the osculants of the rational plane quartic with compound singularities" by Gerald Kraus, Carnegie Institute of Technology, introduced by Professor Neelley.

5. "A mathematical theory of depreciation" by Professor J. S. Taylor, University of Pittsburgh.

Abstracts of these papers follow, the numbers corresponding to the numbers in the list of titles:

1. In this paper Mr. Klein dealt with the problem of estimating the mean by dealing with a sample that is taken from the universe of data by the method of Purposive Selection. This method, which is advocated by Bowley, Neyman, and others, consists of guiding the selection of a sample by means of some "control" which is a characteristic of the data that is correlated with the characteristic being investigated. The new method presented consists of estimating the mean for each group from the regression equation and using the weighted average for all such groups as the estimate of the mean for the total. A concrete application of this method was presented.

2. Mr. Johnston compared the distances from one end to the other of the space diagonal of a rectangular parallelopiped by various paths measured on the surface. In particular, it was proved that if the three dimensions of the parallelopiped are expressible as products ab, ac, and bc, where a, b, and c are the two legs and the hypothenuse of a right triangle, then the least distance via the diagonal of one face plus the edge common to the two other faces equals the greatest distance via the diagonal of two adjacent faces considered as one rectangle.

3. Mr. Whitehead described briefly the origin and present form of the cathode ray oscillograph and discussed in detail the motion of the electron in the magnetic and electric fields in this instrument. Particular stress was laid on the development of the relativity expressions for the kinetic energy and the longitudinal and transverse masses of an electron travelling at high speed; for this purpose familiar laboratory equipment provides the electrical concepts involved and these are treated within the scope of elementary mathematics.

4. In this paper Mr. Kraus examined the osculants at the singular points of the quartic with a tacnode, the curve with a ramphoid cusp, the oscnodal quartic, the curve with an oscnodal cusp, and the knot. The osculants are classified as proper or degenerate. Due to the use of parametric representation, the curves which become degenerate are observed in both ternary point and line forms to determine their types of degeneracy.

5. In this expository paper Professor Taylor presented a method of treating depreciation based upon the following: "A statistical theory of depreciation" by J. S. Taylor, Quarterly Publication of the American Statistical Association, December, 1923; "A general mathematical theory of depreciation" by Harold Hotelling, Journal of the American Statistical Association, September, 1925.

J. S. TAYLOR, *Secretary*

4. Mr. Griffin presented new formulas which he has derived for simplified computation of partial and multiple regression coefficients. He asserted that his new formulas are easier for students to set up and follow without error than are the customary formulas, and that, in a problem involving fifteen variates, an accurate answer will be obtained in from ten to twenty-five per cent of the time required by determinants or Yule's formulas.

5. Two fundamental formulas involving incomplete numerical functions, discovered by Uspensky (Bulletin of the American Mathematical Society, 1930), have been shown by the author (W. A. Dwyer, American Journal of Mathematics, April, 1927) to result from the paraphrase of a certain theta function identity. Related theta-identities give rise to other formulas of the type of Uspensky's, and Mr. Dwyer here presented some arithmetical results of one of these formulas.

T. A. PIERCE, *Secretary*

THE SPRING MEETING OF THE MICHIGAN SECTION

The fourteenth spring meeting of the Michigan Section of the Mathematical Association of America was held at the University of Michigan on Saturday, March 20, 1937. The chairman of the Section, Professor C. C. Richtmeyer presided.

The attendance was about eighty, including the following forty-two members of the Association: W. L. Ayres, W. D. Baten, F. A. Beeler, J. W. Bradshaw, J. B. Brandeberry, R. V. Churchill, C. J. Coe, A. H. Copeland, C. C. Craig, S. E. Crowe, P. S. Dwyer, J. P. Everett, Peter Field, C. H. Fischer, K. W. Folley, R. E. Gaskell, V. G. Grove, T. H. Hildebrandt, J. D. Hill, L. A. Hopkins, Ralph Hull, E. E. Ingalls, L. S. Johnston, H. S. Kaltenborn, Theodore Lindquist, Roy MacKay, E. D. McCarthy, A. L. Nelson, L. F. Ollmann, L. C. Plant, J. E. Powell, G. Y. Rainich, C. C. Richtmeyer, L. J. Rouse, E. R. Sleight, D. E. South, G. G. Speeker, T. O. Walton, J. V. Wehausen, Fern Welker, E. T. Welmers, R. L. Wilder.

The annual business meeting was held at the luncheon at the Michigan Union. The following officers were elected for the year 1937–1938: Chairman, Professor V. G. Grove, Michigan State College; Secretary-Treasurer, Professor C. C. Craig, University of Michigan. The third member of the Executive Committee will be the retiring chairman, Professor C. C. Richtmeyer, Central State Teachers College. It was announced that the Executive Commitee had accepted the invitation of Wayne University to hold the fall meeting there in 1937. The invited address was given by Professor Tibor Radó of Ohio State University.

The following papers were presented at the morning and afternoon sessions:

1. "New light on the first mathematician of the New World, and on the first English mathematician of the New World" by Professor L. C. Karpinski, University of Michigan.

2. "On the introduction of real numbers" by G. B. Dantzig, University of Michigan, introduced by Professor G. Y. Rainich.

3. "Conversion of factors in engineering" by Professor F. J. Linsenmeyer, Department of Mechanical Engineering, University of Detroit, introduced by Professor L. S. Johnston.

4. "A problem in the elastic solid theory" by Professor V. C. Poor, University of Michigan, introduced by the Secretary.

5. "Early American arithmetics" by Professor E. R. Sleight, Albion College.

6. "Strong summability of sequences" by Dr. H. J. Hamilton and Dr. J. D. Hill, Michigan State College; Dr. Hamilton introduced by Dr. Hill.

7. "An alternative to the coefficient of correlation" by Professor A. H. Copeland, University of Michigan.

8. "An actuarial study of the teachers retirement fund of Detroit" by Dr. C. H. Fischer, Wayne University.

9. "Simple canonical forms of the equations of quartic curves of genus 0, 1 or 2" by Professor V. G. Grove, Michigan State College.

10. "Rigor in college mathematics" by Professor Tibor Radó, Ohio State University.

Abstracts of these papers follow, the numbers corresponding to the numbers in the list of titles:

1. Professor Karpinski reported on early study of higher mathematics in America. The first Hollis Professor of Mathematics at Harvard University was Isaac Greenwood, a native-born graduate of the class in 1722. It was Greenwood's influence that inspired Hollis to create a chair of mathematics at Harvard. Greenwood published far more extensively on mathematics than any of his successors in the Hollis chair of mathematics up to 1840. In the outlines of Greenwood's courses of lectures on physics, on the nature of the universe and on the orrery, Greenwood shows an appreciation and understanding of the part of Newton in the development of modern mathematics.

2. Mr. Dantzig defined convergent sequences in terms of equivalence of sequences. Two sequences are equivalent if the absolute value of their corresponding terms except for a finite number are less than any preassigned positive number. A convergent sequence then is a sequence which is equivalent to every one of its subsequences. By a function of two sequences is meant a sequence in which each element is equal to the function of the corresponding elements of the two sequences. It can be shown that continuous functions transform convergent sequences into convergent sequences; and since the fundamental operations are considered as functions whose continuity can be demonstrated, convergent sequences are closed under the operations of addition, multiplication, subtraction, and division. If now convergent sequences of rational numbers are identified with real numbers, operations upon real numbers may be reduced to operations on convergent sequences.

3. In this paper Professor Linsenmeyer pointed out the need for teachers to give more attention to the units of engineering factors. A simple method was offered for conversion of units from one energy form to another and from English to metric units. The method, in addition, shows whether units are consistent, homogeneous or inconsistent. Some units are spoken of inaccurately and give rise to incorrect solutions; viz., B.t.u. per sq. ft. per hour per degree per inch in thickness; the incorrect part being per inch in thickness. Where a factor is given in unit variation, one may multiply or divide and by cancelling units arrive at the total quantity of the desired result.

4. Professor Poor discussed the problem of the existence of a solution of the equations of equilibrium in the elastic solid theory and the attitude of the mathematician and the applied elastician towards this problem.

5. In this paper Professor E. R. Sleight reviewed three of the outstanding early American arithmetics—namely, texts written by Nicholas Pike, Daniel Adams, and Warren Colburn. Pike's was the first written by an American to receive any great recognition. In his text Adams attempted to get away from the mechanical process of stating a rule and then solving problems accordingly, and introduced some plans "to make the scholar think." Colburn's text was based on Pestalozzi's "analytic plan of education," and represented a complete change in American arithmetics.

6. Being given in the complex domain a matrix (a_{mk}), a sequence $\{s_k\}$, and a positive real number p, it was said (following Szász, Brown University Lectures, 1934–35) that $\{s_k\}$ *is strongly summable of order p to s* if there exists a number s such that each of the series $S_m=\sum_k a_{mk}|s-s_k|^p$ is convergent, and if $\lim_m S_m=0$. If this holds for each convergent $\{s_k\}$ with $s=\lim s_k$, the method of summability was called *regular*. Necessary and sufficient conditions for regularity were shown to be (1) $\lim_m a_{mk}=0$ for each k, and (2) $\sum_k|a_{mk}|<M$ for all m. Any regular method having the property that the summability of any sequence to both s and t implies $s=t$, was said to be *unique*. For uniqueness it is necessary that (3) $\lim_m \sum_m a_{mk}=0$ be not true; and if $a_{mk}\geqq 0$ for all m, k (i.e., if the matrix is *positive*) this condition is also sufficient. Furthermore, (3) and the following condition (4) are sufficient in the general case: (4) There exist constants m_0, k_0, and right angled sectors R_m in the complex plane such that for each $m>m_0$, arg a_{mk} lies in R_m for all $k>k_0$. The necessity of (4) remains undecided. Finally it was shown that if $\{s_k\}$ is summable to s by a unique positive regular matrix, then s must be a limit point of $\{s_k\}$.

7. Professor Copeland was concerned with a measure of the functional dependence of a numerical variable upon a variable whose values are elements of an abstract space. The dependence of one variable upon n others was then immediately obtained by the introduction of a product space. The functional dependence is invariant under an arbitrary transformation of the independent variable provided the transformation has a unique inverse. The measure is closely related to the correlation ratio.

8. The results were given of a survey in which the probable future age dis-

tributions of employees for each of the next twenty years and the anticipated receipts and disbursements for this period were obtained. This type of report was suggested by Dr. Fischer as a useful supplement to the conventional type of actuarial survey.

9. A method of writing equations of quartic curves of genus 0, 1 or 2 was discussed by Professor Grove. The method was explained in some detail for quartic curves with a cusp. The triangle of reference was chosen so as to reduce the equation of a certain cusped cubic covariantly related to the quartic to a simple form. Restriction of the unit point reduces all of the coefficients of the quartic to absolute invariants. The method was outlined for quartic curves with a node.

10. The presentation in the elements of Euclid has always been considered as a model of mathematical rigor. It is obvious that this type of rigor cannot be attained in a college course in calculus. In particular it is impossible, and very likely undesirable, to give complete proofs for so-called pure existence theorems. As a rule such thorems are, and should be, taken for granted. The purpose of Professor Radó was to show that theorems of this type can be used to considerable advantage in presenting calculus on the college level. In the way of illustration, the theory of some of the elementary functions was discussed in detail.

C. C. CRAIG, *Secretary*

THE TWENTY-THIRD ANNUAL MEETING OF THE KANSAS SECTION

The twenty-third annual meeting of the Kansas Section of the Mathematical Association of America was held in conjunction with the annual meeting of the Kansas Association of Mathematics Teachers at the Allis Hotel, Wichita, Saturday, April 3, 1937. The two groups met together for the morning session, and for the first hour of the meeting in the afternoon. Mr. William Betz, Rochester, N. Y., was the featured speaker, and Professor R. G. Smith, Kansas State Teachers College, Pittsburg, chairman of the Section, presided at the joint sessions.

The attendance was one hundred eighty-nine, including the following thirty-five members of the Association: R. W. Babcock, Wealthy Babcock, Lois E. Bell, Florence L. Black, E. E. Colyer, R. D. Daugherty, Lucy T. Dougherty, D. D. Driver, W. A. Harshbarger, A. J. Hoare, A. S. Householder, Emma Hyde, W. C. Janes, H. E. Jordan, C. F. Lewis, W. H. Lyons, Anna Marm, U. G. Mitchell, Thirza Mossman, O. J. Peterson, C. B. Read, C. A. Reagan, B. L. Remick, D. H. Richert, J. A. G. Shirk, G. W. Smith, R. G. Smith, W. T. Stratton, Sister M. Helen Sullivan, C. B. Tucker, Henry Van Engen, W. G. Warnock, J. J. Wheeler, A. E. White, Ferna E. Wrestler.

At the business meeting of the Section the following officers were elected for the coming year: Chairman, W. G. Warnock, Kansas State College, Fort Hays; Vice-Chairman, C. B. Tucker, Kansas State Teachers College, Emporia; Secretary, Lucy T. Dougherty, Junior College, Kansas City, Kansas.

The following three papers were read:

1. "Some identities due to Cayley and their vector equivalents" by Professor A. S. Householder, Washburn College.

2. "Concerning Gamma Function expansions" by Dr. Henry Van Engen, Kansas State College, Manhattan.

3. "An unknown property of conics" by Professor G. W. Smith, University of Kansas.

Abstracts of the papers follow, numbered in accordance with their numbers in the list of titles.

1. Professor Householder gave two proofs by vectors of each of the Cayley identities relating the mutual distances of four complanar points with those of four concyclic points. The second proof exhibited the connection of one identity with the formula for the volume of a tetrahedron, and the other identity with the power of a point with respect to the circle through three others.

2. In a recent book by W. B. Ford on Asymptotic Development of Functions Defined by Maclaurin's Series, a function of the type

$$\Omega(z) = \frac{\Gamma(z+a_1) \quad \Gamma(z+a_2)}{\Gamma(z+b_1) \quad \Gamma(z+b_2)}$$

was developed asymptotically in the form

$$\text{(A)} \qquad \Omega(z) \sim 1 + \frac{c_1}{z+1} + \frac{c_2}{(z+1)(z+2)} + \frac{c_3}{(z+1)(z+2)(z+3)} \cdots \quad |\arg z| < \pi$$

under the restrictions that $a_1+a_2=b_1+b_2$. Dr. Van Engen's results consisted of: (1) Removal of the restriction $a_1+a_2=b_1+b_2$; (2) The determination of the c_n in (A) in terms of a_1, a_2, b_1, and b_2. (3) A proof that many well-known series expansions of quotients of Gamma Functions found in analysis are valid asymptotically throughout the sector of the complex plane for which $|\arg z| < \pi$. Previous results had indicated only convergence in half-planes or in some cases convergence for real values of the variable only.

3. Professor Smith gave the following property of conics and asked if it appears in the literature. Given a conic with its transverse axis parallel to the x-axis, let a line with slope m through any point A meet the conic in the points (real or imaginary) D and E and the line with slope $-m$ in the points D' and E' then $AD \cdot AE = AD' \cdot AE'$. Also DE' intersects $D'E$ in B such that $BD \cdot BE' = BD' \cdot BE$ and the slope of DE' is the negative of the slope of $D'E$. Likewise EE' intersects DD' in C such that $CD \cdot CD' = CE \cdot CE'$ and the slope of DD' is the negative of the slope of EE'. The author also discussed some relations of the loci of the points A, B, and C and pointed out some properties of conics easily obtained from the above property.

LUCY T. DOUGHERTY, *Secretary*

THINKING VERSUS MANIPULATION*

By W. B. CARVER, Cornell University

We would probably all agree that mathematics consists, not in the correct manipulation of symbols, but in clear and logical thinking about certain concepts and relationships which the mathematical symbols represent. It is characteristic of all symbolism that there is a constant tendency for the symbol to replace the thing symbolized; and it is very easy to allow mathematical symbols to obscure mathematical ideas. If one should memorize and reproduce correctly the sounds which constitute a sentence in the Russian language without knowing anything of their significance, he would not be speaking Russian—he would merely be making Russian noises. And if a student in the classroom goes through a correct piece of manipulation of mathematical forms without understanding the meaning and significance of the processes, his work is *not mathematics at all*—he is merely making mathematical marks. The vast short-hand symbolism we have invented to record and communicate mathematical ideas is only the *language* of mathematics, and language is a very doubtful asset to any one who has no ideas to communicate.

If this concept of mathematics is correct at all, it follows at once that as teachers of this subject our primary aim should be to help the student to a clear understanding of mathematical ideas and relationships, and to encourage him to develop the ability to think for himself so that he may arrive under his own power at conclusions in which he will have confidence.

I take it that you would all agree with what I have said so far, and are wondering why one should take the time of this group to emphasize the obvious. I am presuming to do so only because this is one of those things so obvious that it is easily overlooked. One says, "Of course"—and then forgets. Manipulation, even correct manipulation, is comparatively easy to teach, but it is always difficult to get students to think. We do our teaching in a gravitational field with a constant pull downward toward the easy thing, and a constant effort necessary to keep our work up to a worth-while level. And the best teachers slip at times and become tired and discouraged. In speaking of this matter recently an excellent mathematician and teacher said to me, "But do you believe it is possible to get most of our students in mathematics to think for themselves?" and his question implied a defeatist attitude toward what I have called the primary aim of our teaching.

The obvious answer to my colleague's question is, of course, "More or less." Our American educational system is very democratic while the distribution of intelligence is not so democratic. We have, as a rough classification, three grades of students: a few at the top are alert and eager and with a very little encouragement and guidance will think their own way ahead with enthusiasm; a large middle group will expect you to tell them dogmatically what is correct or incor-

* An address given by invitation at the meeting of the Mathematical Association of America at Durham, N. C., January 1, 1937.

rect, but are capable of seeing many things clearly for themselves and reaching their own conclusions in many cases if they find that this is what is expected of them; and finally there is a group at the bottom who will only occasionally show a spark of intelligence and real understanding. In most cases all three of these groups will be represented in the same class. In our educational system as organized, the teacher has a distinct obligation to *all* students placed in his classes. This obligation can never be fulfilled by any effort to bring them all to one dead level of attainment. We have no right to give all our time and effort to the weaklings while we hold back our best students or allow them to loaf; nor have we a right to assume the high-hat attitude that we are only interested in real students and cannot waste our time on the incompetent. Every student must be encouraged to think as far as he is capable of thinking. Even with our best students mechanical manipulation may be greatly overemphasized, while with the poorest of them the teacher is always tempted to be satisfied with memorized processes even though he knows that the student has no understanding of what he is doing.

Examination systems, in spite of all efforts to the contrary, seem to influence our teaching in the direction of formalism rather than insight; because it is easy to test a student's manipulative skill and extremely difficult to test his ability to think. As soon as organizations such as the College Entrance Exmination Board or our New York Board of Regents set up the machinery for mathematical examinations, it is perfectly natural for the secondary school teacher to ask what ground will be covered by the examination in Mathematics A or in Solid Geometry; and the answer is a syllabus or some sort of detailed definition of requirements for each examination. It is again only natural that the teacher takes this syllabus, not so much as a suggestion for the desirable content of a course in the subject, but as a sort of guarantee that questions concerning ideas not specifically mentioned in the syllabus will not be asked; and this makes it almost impossible to prepare an examination paper which will test a student's ability to think.

Let me give an instance of the kind of thing I have in mind. Several years ago one member of the committee preparing the New York Regents' examination in intermediate algebra wanted to include a question asking for the roots of the equation

$$x^3 - 3x^2 - 4x + 12 = 0.$$

The syllabus mentioned specifically the factoring of such polynomials (in fact the examination for the preceding year had called for the factoring of just such a cubic polynomial) and the solution of *quadratic* equations by factoring. But, with the exception of the one member, the committee was unanimous in the opinion that the syllabus would not allow a question calling for the solution of a cubic equation and that such a question would be quite unfair to the pupils! Of course the question would have been unfair only on the assumption that the pupils were being taught purely formal processes and were not to be expected to take a single step of the simplest kind on their own initiative.

It is only fair to say that the College Entrance Examination Board is aware of this difficulty, and is proposing to meet it by abolishing their subject examinations in favor of the new mathematics attainment tests. It remains to be seen whether or not the new system will accomplish the purpose for which it was designed. May I ask a question which I am not able to answer (at the risk that the mere asking of the question may stultify me in the opinion of my friends in Education): may it not be possible that the only really important objective in our teaching of mathematics is something that we will never be able to measure satisfactorily by any kind of test or examination? Of this much, at least, I am convinced: that there is a wide difference between helping students to acquire a clear and useful understanding of mathematics and "preparing" them for any form of mathematical examination that has been so far devised.

But I have been talking so far in very general terms. Not many teachers will be willing to admit that they are teaching formal processes and not trying to help their students to a clear understanding of what it is all about. I want therefore to give a few simple examples from elementary mathematics to show how easy it is for students to overlook the real meaning of what they are doing, and how easy it is for teachers, even very good teachers, to accept formal processes without finding out whether or not they are understood.

You may find it interesting to try the following experiment with a class which seems to show this tendency to manipulate without thinking. Take any two numbers (members of the class may choose them), say 12 and 5; add them and subtract them obtaining 17 and 7; multiply this 17 and 7 obtaining the result 119. Now begin again and square the original numbers, getting 144 and 25, and when we subtract we again have the result 119! Students in my class thought this was quite peculiar, and proceeded to try it out on some other pairs of numbers. One boy asked whether it "would work" if one started with fractions. It was fully ten minutes before one of the best students in the class said (rather disgustedly), "Why of course!" This is a striking example because the very poorest student who has had a first course in algebra is familiar with the relation $(x+y)(x-y)=x^2-y^2$; at least he is familiar with the appearance of this arrangement of letters and symbols. But it is possible that many of your pupils fail to realize the significance of this very simple symbolism and will be surprised when their attention is called to the fact that numbers really behave in this way.

Students make many manipulative errors in algebra because they are not thinking of the letters as numbers. They "clear of fractions" the expression $x+3/x$ and tell you it is equal to x^2+3, but the same students would never tell you that $4\frac{1}{2}$ is equal to 9 or that a dollar and a quarter is the same as five dollars. Algebra (at least the algebra of the secondary school and freshmen year in college) has to do with *numbers* and relations between numbers, and a student who is learning to manipulate letters and symbols, however skilfully, without any understanding of the number relations represented is *not learning algebra at all.*

In the application of algebra to the solution of problems it would seem that the student would be forced to think things through for himself. But there is a strong temptation to devise schemes to save him from thinking—fool-proof schemes which will enable the weak student to get the right answer with a minimum of thought. Problems are classified into standard types—rate problems, work problems, clock problems—and stereotyped forms are often given to students for the handling of these types, thereby defeating the only possible useful purpose of such problems. A few years ago I listened to a high-school teacher struggling to elicit from a boy the answer to the question how long would it take a man to travel 120 miles at the rate of $x+2$ miles per hour. The boy was completely helpless until the teacher suggested "the time-rate-distance formula," and he then mumbled something about $DR=T$ or $R/T=D$. He finally made the guess $T=D/R$ and arrived at $120/(x+2)$ as the answer to his question. He would have been equally happy (or unhappy) with the formula $DR=T$ if it had been acceptable to the teacher, because he was thinking merely of three letters and not at all about actual motion at a constant speed. The boy was not really stupid. Out on the street, free from an artificial classroom atmosphere, if you had asked him how long it would take to drive 120 miles at 40 miles per hour he would *not by any possibility* have multiplied 120 by 40.

In our elementary geometry courses we like to believe that our pupils are not only becoming familiar with the facts of geometry but that they are also being trained to reason accurately and logically. But I doubt whether we have grounds for any great complacency with regard to what we are accomplishing in the latter direction. Too many pupils are merely memorizing book proofs, some of which we know to be logically unsound. The measure of our success in developing the power of our students in geometry to reason for themselves is the percentage of them who can do anything with a reasonably simple original proposition (and by "original" I mean, of course, a proposition which the student has not seen before).

If my examples so far have been chosen from the secondary school program. it is only because I believe that the development of the pupil's power to think for himself cannot begin too early. The six-year-old, adding his units column to 24, is quite capable of *understanding why* he should "put down 4 and carry 2" rather than put down 2 and carry 4. But the tendency to teach mechanical manipulation in mathematics rather than to encourage and insist upon thinking is not confined to the secondary schools. There is probably no mathematical subject in which this tendency is stronger than in the first course in calculus. What percentage of our students emerge from this course with a fair understanding of the meaning of a derivative or a definite integral? We start out, to be sure, with a definition of a derivative, and the student computes the derivative of a few simple functions by forming the fraction $\Delta f(x)/\Delta x$ and finding its limit; but how quickly he forgets all this as soon as he has memorized some formulas that enable him to write the derivative at once. They may remember that dy/dx gives the slope of the tangent to the curve $y=f(x)$, but could they

give even a fairly plausible explanation as to why this is true? To solve maximum and minimum problems they "put the derivative equal to zero"; but how many of them could explain *why* a zero value for dy/dx is related to the question of a maximum or minimum value for y, or why the same process seems to give a maximum value when one wants a maximum and a minimum value when that is wanted.

The greatest temptation to pure formalism arises when the formal process is very simple and the significance of the process rather deep-lying and difficult to understand. The outstanding example of this is the matter of the derivative of an implicit function. It is so easy:

$$4x^2 + 9y^2 = 36,$$

$$8x + 18y\frac{dy}{dx} = 0,$$

$$\frac{dy}{dx} = -\frac{4x}{9y};$$

but does it mean anything? In the earlier editions of a widely used textbook in calculus there was the problem to find dy/dx, given

$$\tan(x + y) + \tan(x - y) = 0.$$

This is almost as simple as the example given above; and following the same procedure one obtains the "answer" given in the text,

$$\frac{dy}{dx} = \frac{\sec^2(x - y) + \sec^2(x + y)}{\sec^2(x - y) - \sec^2(x + y)}.$$

The only trouble with all this is that *it means absolutely nothing at all.* The given equation does not define y as a function of x, the corresponding graph being a series of lines parallel to the y axis; hence dy/dx can have no meaning. That such a problem should get into a good textbook is something of a joke, not in itself very significant or important. But it is vitally important if it is true that we are teaching calculus in such a way that we could give our students dozens of such problems and they would be quite satisfied to go through meaningless manipulations on all of them and get meaningless results.

As teachers we can not allow ourselves to become discouraged because our best efforts to get our students to think meet with only partial success. We must continually remind ourselves that the only worth-while objective of our teaching is the encouragement and stimulation of each student to think for himself to the extent of his individual ability. If you teach a boy to carry through, however accurately and correctly, certain processes which he does not understand, what have you accomplished? Machines are being built to do that kind of arithmetic and calculus. But you have done a far greater thing when you have helped the most stupid student you ever had in your classroom to think just a little.

AN ELEMENTARY DEVICE IN DIOPHANTINE ANALYSIS

By E. T. BELL, California Institute of Technology

1. The following obvious remark may lead to results of some interest concerning the unsolvability of homogeneous diophantine equations of a special type.

Let K denote the field of rational numbers and let $\alpha, \beta, \cdots, \mu$ be algebraic numbers, linearly independent with respect to K, of the respective degrees $a, b, \cdots, m$, defined by the (irreducible) equations

$$\begin{aligned} \alpha^a + A_1\alpha^{a-1} + \cdots + A_a &= 0, \\ \beta^b + B_1\beta^{b-1} + \cdots + B_b &= 0, \\ \cdots\cdots\cdots\cdots\cdots & \\ \mu^m + M_1\mu^{m-1} + \cdots + M_m &= 0, \end{aligned}$$

with coefficients $A_1, \cdots, M_m$ in K. The result of adjoining α to K is denoted by $K(\alpha)$, and similarly for $K(\beta), \cdots, K(\mu)$. The general number in $K(\alpha)$ is of the form $x_0+x_1\alpha+ \cdots +x_{a-1}\alpha^{a-1}$, where $x_0, \cdots, x_{a-1}$ are independent variables in K. By the irreducibility of the defining equation, $1, \alpha, \cdots, \alpha^{a-1}$ are linearly independent with respect to K. Hence $k_0= \cdots =k_{a-1}=0$ is the only solution in K of

$$k_0 + k_1\alpha + \cdots + k_{a-1}\alpha^{a-1} = 0.$$

The a roots of the equation defining α(α included) are the conjugates of α. Similarly for $K(\beta), \cdots, K(\mu)$.

We write

$$\begin{aligned} x_0 + x_1\alpha + \cdots + x_{a-1}\alpha^{a-1} &\equiv X(\alpha), \\ y_1\beta + \cdots + y_{b-1}\beta^{b-1} &\equiv Y(\beta), \\ \cdots\cdots\cdots\cdots\cdots & \\ u_1\mu + \cdots + u_{m-1}\mu^{m-1} &\equiv U(\mu), \\ X(\alpha) + Y(\beta) + \cdots + U(\mu) &\equiv L(\alpha, \beta, \cdots, \mu), \qquad (ab\cdots m \equiv n), \end{aligned}$$

and form the product $\Pi L(\alpha, \beta, \cdots, \mu)$, extended to all the conjugates of $\alpha, \beta, \cdots, \mu$. The result (being a symmetric function of all the conjugates) is a homogeneous polynomial P of degree n in $x_0, \cdots, u_{m-1}$ with coefficients in K. From the linear independence of the powers of $\alpha, \beta, \cdots, \mu$ occurring in $L(\alpha, \beta, \cdots, \mu)$ it follows that $x_0= \cdots =u_{m-1}=0$ *is the only rational solution of* $P=0$.

In the equation $P=0$ we may take some of the variables to be identically zero, or we may express some as polynomials in others with rational coefficients. For example, all the variables in each of several sets may be taken equal, those in different sets being unequal. The modified equations will have as their only rational solutions those in which all variables are zero.

In forming the initial equation $P=0$ we may proceed by calculating the symmetric functions of the roots, or by eliminating $\alpha, \beta, \cdots, \mu$ from the defining equations in conjunction with

$$w - L(\alpha, \beta, \cdots, \mu) = 0;$$

the term independent of w in the eliminant is P. The second method is frequently used in forming circulants or norms of algebraic integers, and is due to Lagrange.*

2. Any equation $P=0$ of the kind described can be shown independently to have $0, \cdots, 0$ as its only rational solution by resolving P into its linear factors. However, without an indication of the origin of P, it may not be easy to find the factors. We give an example.

Let α be a root of the irreducible cubic

$$\theta^3 - a\theta^2 + b\theta - c = 0,$$

where a, b, c are rational numbers. Taking $L(\alpha) = x + y\alpha + z\alpha^2$, we find for $P=0$ the equation

$$\begin{aligned} x^3 + cy^3 + c^2z^3 + (ab - 3c)xyz + ax^2y + (a^2 - 2b)x^2z \\ + by^2x + acy^2z + (b^2 - 2ac)z^2x + bcz^2y = 0, \end{aligned} \tag{1}$$

of which the only rational solution is $(x, y, z) = (0, 0, 0)$.

To specialize the last, let a, b, c be integers, of which c is either 1 or a prime. The only possibilities for a rational root θ are then $\theta = \pm 1, \pm c$. Excluding these, we see that, *if c is either* 1 *or a prime, and if a, b are integers such that*

$$(c^2 - ac + b - 1)(c^2 + ac + b + 1)(a - b)(a + b + 2) \neq 0, \tag{2}$$

the only rational solution of the equation (1) *is* $(x, y, z) = (0, 0, 0)$.

From this we can construct an infinite chain of equations of the same form as (1), with different coefficients, whose only rational solution is $(x, y, z) = (0, 0, 0)$. First, taking $x=y$, $y=z$, $z=x$ in turn in (1), and making a suitable change of notation on the remaining two variables in each case, we find that *if a, b, c are as in* (2), $(x, y) = (0, 0)$ *is the only rational solution of each of the equations*

$$x^3 - a_1x^2y + b_1xy^2 - c_1y^3 = 0, \tag{3}$$

in which a, b, c have the values in any line of the table

a_1	b_1	c_1
$-(a^2+a-2b)$	$ab-2ac+b^2+b-3c$	$-c(a+b+c+1)$
$-(b^2+ac)/c$	$(ab+a+bc-3c)/c$	$-[(a-c)^2+(b-1)^2]/c$
$-(b^2-2ac+bc)/c^2$	$(ab+a^2+ac-2b-3c)/c^2$	$-(a+b+c+1)/c^2$.

* Oeuvres, vol. 7, p. 178; (Additions aux Éléments d'Algèbre d'Euler, 1798). See also R. D. Carmichael, Diophantine Analysis, 1915, Chap. 2.

Hence, if $\theta = x/y$, *each of the three equations*

$$\theta^3 - a_1\theta^2 + b_1\theta - c_1 = 0 \tag{4}$$

is irreducible, and we can proceed with any one of them as in obtaining (1). The result is (1) with (a, b, c) replaced by (a_1, b_1, c_1), and we can continue in the same way. Thus, attending only to the first line of the table and the corresponding equation (4), we write

$$
\begin{aligned}
&(a_0, b_0, c_0) \equiv (a, b, c), \\
a_s &\doteq - a_{s-1}^2 - a_{s-1} + 2b_{s-1}, \\
b_s &= a_{s-1}b_{s-1} - 2a_{s-1}c_{s-1} + b_{s-1}^2 + b_{s-1} - 3c_{s-1}, \\
c_s &= - c_{s-1}(a_{s-1} + b_{s-1} + c_{s-1} + 1),
\end{aligned}
$$

which determine a_s, b_s, c_s as polynomials in a, b, c on taking $s = 1, 2, \cdots, s$. In (1) we replace (a, b, c) by (a_s, b_s, c_s); the resulting equation has as its only rational solution $(x, y, z) = (0, 0, 0)$. This holds if a, b, c are as in (2), or are merely such that the initial equation used in getting (1) is irreducible.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions, and Notes in the Monthly is open to all forms of activity in collegiate mathematics including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

NOTE CONCERNING DEAN CARMICHAEL'S PAPER

Concerning his paper entitled, "On Numbers of the Form $a^2+\alpha b^2$," published in the February number of this MONTHLY, Dean Carmichael writes as follows:

"Through the kindness of Professor H. S. Vandiver my attention has been called to the fact that the two principal theorems in my paper in the current volume of this MONTHLY (pp. 81–86) have been treated by earlier authors, as follows: L. Aubry: Association francaise pour l'avancement des sciences, vol. 40, 1911, pp. 55–60; H. S. Vandiver, this MONTHLY, vol. 34, 1937, pp. 86–88."

Dean Carmichael's paper remains of value, however, as an expository presentation of results of marked interest in number theory.—E.J.M.

GEOMETRIC PROOFS OF MULTIPLE ANGLE FORMULAS

By WAYNE DANCER, University of Toledo

The following construction for the trisection of an angle, accomplished by means of a marked straight edge, has been well known since the time of Archimedes.* The purpose of this note is to show that this same figure may be

* Heinrich Dörrie, Triumph der Mathematik, Breslau, 1933, pp. 170–171.

utilized to establish and illustrate several common trigonometric identities.

Let the angle GCH, of magnitude $3x$, be placed at the center of a circle of radius unity. Mark on a straight edge a segment AE of unit length. Lay the straight edge on the figure to determine a line through G such that one end of the marked segment, E, will fall on the circumference, and the other end, A, will fall on the diameter extended. Draw the radius CE. Then $\angle EAC = \angle ACE$

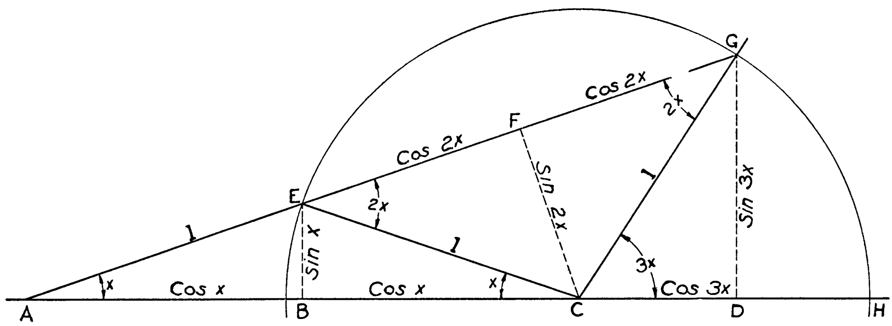

FIG. 1

$=x$, and $\angle GEC = \angle CGE = 2x$. From points E and G drop perpendiculars EB and GD respectively to the horizontal diameter, and from the center C drop a perpendicular CF to the line EG. It is observed that all segments in the figure are either of unit length, or equal in length to the sine or cosine of one of the angles x, $2x$, or $3x$, as shown. Application of the Theorem of Pythagoras to proper triangles in the figure illustrates the identity $\sin^2 \alpha + \cos^2 \alpha = 1$ for the three cases where $\alpha = x, 2x, 3x$.

In the triangle ACF, we have $AF = AC \cos x$, whence $1 + \cos 2x = 2 \cos^2 x$, or, transposing, $\cos 2x = 2\cos^2 x - 1$; and $CF = AC \sin x$, or $\sin 2x = 2 \sin x \cos x$.

Similarly, in the triangle ADG, we have $AD = AG \cos x$, that is,

$$2 \cos x + \cos 3x = (1 + 2 \cos 2x) \cos x,$$

or

$$\begin{aligned} \cos 3x &= 2 \cos x \cos 2x - \cos x \\ &= 2 \cos x(2 \cos^2 x - 1) - \cos x \\ &= 4 \cos^3 x - 3 \cos x. \end{aligned}$$

In this same triangle, $DG = AG \sin x$, or,

$$\begin{aligned} \sin 3x &= (1 + 2 \cos 2x) \sin x \\ &= \sin x + 2(1 - 2 \sin^2 x) \sin x \\ &= 3 \sin x - 4 \sin^3 x. \end{aligned}$$

The figure may be extended to include higher multiples of x, but the formulas developed are the fundamental ones.

from the Brocard vertices (foci) upon the sides of the original triangle. From familiar properties of the parabola it is easy to show that the circle through F with center at A is cut by the circle with center at B and radius BF, and by the circle with center at C and radius CF, in the points at which these circles are tangent to the directrix. Lines drawn through B and C, parallel to the median AX, pass through these points, as do lines through A parallel to the focal radii, BF and CF. The common chords of the circles are the perpendiculars dropped from F upon the sides of the original triangle. Through the feet of the perpendiculars upon the tangent sides, of course, passes the vertex tangent of the parabola, and further, perpendiculars at F to the focal radii intersect the tangent sides on the directrix. The perpendicular through F upon the third side is the line

$$4ahx + 4aby + b^2h + h^3 = 0,$$

which passes through the point $(-[b^2+h^2]/4a,\ 0)$. Thus,

(8) *A perpendicular through a vertex of Brocard's "second triangle," drawn to the corresponding third side of the original triangle, passes through the intersection of the nine-point circle and the corresponding median.*

Note on an Operational Formula

By C. A. Hutchinson, University of Colorado

In H. T. H. Piaggio's *Elementary Differential Equations* (Second Edition, 1928, page 44, example 26 (iii)), is given the operational formula

$$\text{(A)} \qquad \frac{1}{F(D)}[xV] = x\frac{1}{F(D)}V - \frac{F'(D)}{[F(D)]^2}V,$$

where V is a function of x, $D \equiv d/dx$, and $F(D)$ is a polynomial in D.

The object of this note is to call attention to the fact that formula (A), as it stands, may give incorrect results. A correct formula is stated in Murray's *Differential Equations* (1913, page 80):

$$\text{(B)} \qquad \frac{1}{F(D)}[xV] = x\frac{1}{F(D)}V - \frac{1}{F(D)}\cdot F'(D)\cdot\frac{1}{F(D)}V.$$

One example will be sufficient. Let us find a particular integral of the differential equation $(D^2+1)y = x\cos x$ by using the formula

$$y = \frac{1}{(D^2+1)}x\cos x.$$

Formula (B) gives us*

$$\begin{aligned}\frac{1}{D^2+1}\, x \cos x &= x\,\frac{1}{D^2+1}\cos x - \frac{1}{D^2+1}\cdot 2D\cdot\frac{1}{D^2+1}\cos x\\ &= \frac{x^2 \sin x}{2} - \frac{1}{D^2+1}(x\cos x + \sin x)\\ &= \frac{x^2\sin x}{4} + \frac{x \cos x}{4}.\end{aligned}$$

Direct substitution into the differential equation shows that the result is correct.

Two interpretations of formula (A) are possible, both leading to incorrect results:

$$(1)\qquad \begin{aligned}\frac{1}{D^2+1}\, x\cos x &= x\,\frac{1}{D^2+1}\cos x - \frac{1}{(D^2+1)^2}\,2D\cos x\\ &= \frac{x^2\sin x}{2} + \frac{2x^2}{2^2\cdot 2!}\sin(x-\pi) = \frac{x^2\sin x}{4},\end{aligned}$$

which does not solve the differential equation; and

$$(2)\qquad \begin{aligned}\frac{1}{D^2+1}\, x\cos x &= x\,\frac{1}{D^2+1}\cos x - 2D\cdot\frac{1}{(D^2+1)^2}\cdot\cos x\\ &= \frac{x^2\sin x}{2} - 2D\cdot\frac{x^2}{2^2\cdot 2!}\cos(x-\pi)\\ &= \frac{x^2\sin x}{4} + \frac{x\cos x}{2},\end{aligned}$$

another incorrect result.

The usual method of verifying (B) shows at once why (A) is not safe.

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* In this work I have used two formulas which I have not seen in print, namely,

$$\frac{1}{(D^2+a^2)^n}\sin ax = \frac{x^n}{(2a)^n\cdot n!}\sin(ax - n\pi/2),$$

$$\frac{1}{(D^2+a^2)^n}\cos ax = \frac{x^n}{(2a)^n\cdot n!}\cos(ax - n\pi/2).$$

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EDITED BY W. R. LONGLEY, Yale University

All books for review should be sent directly to the editor of this department, at the American Mathematical Monthly, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Leçons d'algèbre et de géométrie. by R. Garnier, Volume 2. Coniques et quadriques. Paris, Gauthier-Villars, 1936. 207 pages. 40 fr.

A School Algebra. By R. M. Carey. Parts 1 and 2 with answers. London, Longmans, Green, and Company, 1936. 8+288+33 pages. $1.50.

An Examination of Logical Positivism. By J. R. Weinberg. (International Library of Psychology, Philosophy, and Scientific Method.) New York, Harcourt, Brace, and Company, 1936. 7+311 pages. $3.75.

Analysis of Thin Rectangular Plates Supported on Opposite Edges. By D. L. Holl. (Bulletin 129, Iowa Engineering Experiment Station.) Ames, Iowa, Iowa State College, 1936. 100 pages.

Mechanics, Molecular Physics, Heat, and Sound. By R. A. Millikan, D. Roller, and E. C. Watson. Boston, Ginn and Company, 1937. 14+498 pages. $4.00.

Second Year Algebra. By H. B. Kingsbury and R. R. Wallace. Milwaukee, The Bruce Publishing Company, 1936. 10+428 pages. $1.40.

Numbers and Numerals. By D. E. Smith and J. Ginsburg. (Contributions of Mathematics to Civilization, edited by W. D. Reeve, No. 1.) New York, Bureau of Publications, Teachers College, Columbia University, 1937. 8+52 pages. 24 cents.

Lectures on College Algebra. By S. B. Dandekar. (A textbook for the use of intermediate students of Indian universities.) Indore, Vinayak, 1936. 12+402 pages.

A First Course in Statistical Method. By G. I. Gavett. Second edition. New York, McGraw-Hill Book Company, 1937. 9+400 pages. $3.50.

Analytic Geometry. By W. A. Wilson and J. I. Tracey. Alternate edition. With Answers. New York, D. C. Heath and Company, 1937. 17+296+18 pages. $2.12.

A First Course in the Differential and Integral Calculus. By W. B. Ford. Revised edition. New York, Henry Holt and Company, 1937. 7+369 pages. $3.00.

Actualités scientifiques et industrielles. Paris, Hermann et Cie, 1935–1936. The following additional titles have been received. See this MONTHLY, vol. 42, 1935, pp. 40–41, for an earlier list of titles in this series.

198. *Sur les groupes de transformations analytiques.* By H. Cartan. 53 pages. 15 fr.
206. *Arithmétique et géométrie sur les variétés algébriques.* By André Weil. 16 pages. 6 fr.

210. *Quelques propriétés des variétés algèbriques se rattachant aux théories de l'algèbre moderne.* By Paul Dubreil. 32 pages. 10 fr.
219. *Le problème de la dérivée oblique en théorie du potentiel.* By G. Bouligand, G. Giraud, and P. Delens. 80 pages. 18 fr.
229. *Charakterisierung des Spektrums eines Integraloperators.* By J. von Neumann. 20 pages. 7 fr.
252. *Integralgeometrie.* By W. Blaschke. 22 pages. 7 fr.
270. *Les involutions cycliques appartenant à une surface algèbrique.* By Lucien Godeaux. 48 pages. 12 fr.
274. *Les définitions modernes de la dimension.* By Georges Bouligand. 48 pages. 12 fr.
285. *Étude statistique de la fécondite matrimoniale.* By A. C. Mukherji. 80 pages. 16 fr.
302. *Les théorèmes de la moyenne pour les polynomes.* By J. Favard. 52 pages. 15 fr.
305. *Séries lacunaires.* By S. Mandelbrojt. 40 pages. 12 fr.
323. *Arithmétique dans les algèbres de matrices.* By Claude Chevalley. 36 pages. 10 fr.
325. *Cinématique du solide et théorie des vecteurs.* By Ch. Platrier. 56 pages. 12 fr.
326. *La masse en cinématique et théorie des tenseurs du second ordre.* By Ch. Platrier. 84 pages. 18 fr.
327. *Cinématique des milieux continus.* By Ch. Platrier. 36 pages. 8 fr.
329. *Les conditions de monogénéité.* By D. Menchoff. 56 pages. 15 fr.
331. *Les fonctions polyharmoniques.* By M. Nicolesco. 51 pages. 15 fr.
333. *Propriétés générales de l'équation d' Euler et de Gauss.* By Edouard Goursat. 96 pages. 20 fr.
356. *L'emploi des observations statistique: méthodes d'estimation.* By G. Darmois. 31 pages. 10 fr.
357. *Integralgeometrie 5: über das kinematische Mass im Raum.* By L. A. Santalo. 54 pages. 18 fr.
358. *La topologie des groupes de Lie.* By E. Cartan. 28 pages. 10 fr.
362. *L'algèbre abstraite.* By Oystein Ore. 55 pages. 15 fr.
427. *Les axiomes de la mécanique Newtonienne.* By Ch. Platrier. 59 pages. 14 fr.
436. *Sur la théorie mathématique des jeux de hasard et de refléxion.* By R. de Possel. 44 pages. 10 fr.

REVIEWS

Gewachsene Raumlehre. By B. Petermann and Karl Hagge. Herder and Co., Freiburg, 1935. 8+164 pages +plates. 4.40 marks.

This stimulating book, conceived by the psychologist Petermann (University of Hamburg) and the teacher of mathematics Hagge (Kiel) and prepared for publication by Petermann after Hagge's death in 1932, presents a novel and highly original program of instruction in geometry for the "Volksschule." It

is written, not for the pupil, but for the teacher, and consists of six lectures, of which the last four cover respectively the four years of instruction in geometry in the "Volksschule," while the second is devoted to the preparatory work in the lower school, and the first presents the fundamental pedagogical ideas at the basis of the program.

The program aims at an extension of intuitive methods in the teaching of geometry to what Petermann describes as "pragmatic-empirical" methods. Geometric facts and formulas are derived empirically, from models of solids made by paper folding, from constructions by ruler and compasses, and from figures cut from paper. The work proceeds from the simple to the composite and a certain degree of continuity is attained by choice of material which lends itself to immediate development and exploitation.

A few examples suffice to illustrate the methods. The instruction begins with the problem of how to make a model of a cube by paper folding. This leads to the discussion of other regular polyhedra, particularly those with triangular faces, and the problem of making models of them in the simplest ways. The network of regular hexagons thus presents itself, and with it systems of circles, which in turn bring in, among other things, perpendicular and parallel lines.

Angles are compared as to size by measurement of circular arcs intercepted by them. From the measure of an angle with vertex at the center of a circle in terms of the intercepted arc, the usual measurements of angles with vertices arbitrarily situated with respect to the circle are readily obtained. Hence, it is easily shown, for example, that the sum of the angles of a triangle is two right angles, that an angle inscribed in a semicircle is a right angle, and that the angles opposite the equal sides of an isosceles triangle are equal.

The unit for measuring area is not the unit square, but a generalization of it, the α-unit rhombus, that is, the rhombus with unit sides and angles equal to α and $180°-\alpha$. The relationship between this generalized unit, for a given α, and the square unit may be readily obtained approximately by graphical methods, and the obvious advantages of it are too numerous to mention. One advantage which is not obvious is worthy of special reference. It leads in a rather simple and elegant way to the Pythagorean Theorem.

W. C. GRAUSTEIN

Mathematics of Finance. By T. M. Simpson, Z. M. Pirenian, and B. H. Crenshaw. Second Edition. New York, Prentice-Hall, Inc., 1936. 13+330 pages +tables 126 pages. $3.75.

The first edition of this text was reviewed in the November, 1932, issue of this MONTHLY. The general plan of the previous edition has not been altered, but several additions and a number of minor changes have been made.

A short chapter on statistics has been added to the first part, and in Part II the chapter on life annuities and life insurance has been somewhat amplified. Three new tables, which are slight modifications of those ordinarily used, have

been prepared for the purpose of facilitating computations involving general annuities, and an attempt has been made to simplify the method of treating such annuities. The problems have been revised, and are reasonably numerous.

H. E. ARNOLD

Report on the Mathematics Attainment Test of June 1936. By J. M. Stalnaker. College Entrance Examination Board. Research Bulletin No. 7. New York, 1936. 62 pages. 20 cents.

The Mathematics Attainment Test will soon supplant all the other mathematical examinations given by the College Board, and it is therefore incumbent on teachers of mathematics in both secondary and collegiate fields to acquaint themselves with this new type of test, first given last June. A careful perusal of Professor Stalnaker's excellent report cannot fail to give a clear idea of the aims and procedures of this test.

The report first considers the principles underlying the examination, the methods of assembling and testing the items, the organization and form of the examination, and the distribution of subject matter. The impression which this part of the report leaves is admirable. It is a calm impartial account of an intelligent and honest attempt to achieve certain well defined objectives. Of great interest is the statement that the readers agreed that the June examination had been so constructed that thorough knowledge of fundamentals was more valuable to a candidate than a superficial knowledge on a wide range of topics. This statement should go far to allay the apprehensions which many teachers felt about the inclusion of analytic geometry and the calculus.

The report goes on to consider the scoring of the examination, the conversion procedures, general statistics, and interpretation of scores. The conversion procedures may seem at first to be somewhat complicated and arbitrary, but if we grant the principles underlying the examination, these procedures are essentially prescribed.

The report includes finally one-third of the short objective items used in Part I of the examination, and all the essay-type questions of Part II, with statistics as to the difficulty and validity of the objective items. Owing to the great expense in gathering and pre-testing objective items, the Board felt it was unable to publish all the objective items of Part I. The reviewer feels that this decision is unfortunate, and is likely to create considerable prejudice against what seems to him a forward step in examination procedures, a prejudice which might have been largely obviated if the whole examination were allowed to speak for itself. Despite the omission of two-thirds of the items, the reviewer is impressed with the nature of the objective questions in Part I. On the other hand, Part II, especially the Gamma Part II, seems definitely inferior to Part I. This is hardly the place for a detailed analysis of individual questions. It would appear, however, that in the future it is this part of the examination which will require further study and modification. The attempt to change the candidates'

presentations of geometric proofs as outlined at the beginning of Part II appears to have been ill-timed.

Although the report states that it desires to present a full description of each step in the development of the examination, only brief mention is made of the procedure of reading the questions of Part II solely in terms of categories. This is unquestionably an important new development, and the report states that the readers liked it and recommended its continued use. It is to be hoped that studies will be made of this procedure, for it seems to have great promise; but since relatively little is as yet known about this method, Professor Stalnaker has wisely not stressed it, nor has he made any extravagant claims with regard to it.

The entire report is a model of clear exposition, with no slightest hint of special pleading.

B. H. Brown

Analytic Geometry. By P. H. Graham, F. W. John, and H. R. Cooley. New York. Prentice-Hall Inc., 1936, xx+294 pages. $2.35.

The preface says that this book does not attempt to offer a novel development of the subject, but to present the explanations and arrange the material in a manner that will be easily grasped by the average student and that will ensure a highly flexible text in the hands of the instructor. In this the authors have succeeded, as have so many writers of quite similar textbooks. It is difficult to see why this book is any better than other books of the kind written in the last forty years, but it is probably just as good. It gives the main facts about rectangular and polar coordinates, the straight line, circle, parabola, ellipse, and hyperbola, and the relation of the conics to the equation of the second degree and to the cone, a few of the most important transcendental curves, enough solid analytics to introduce planes, straight lines, and the simpler quadrics. It also has a chapter on curve tracing (under the title "General Locus Problems") and a chapter on empirical equations.

The book is attractively arranged, with the usual topics where it is most convenient to look for them. The problems are of the kind the average student best likes to do, and seem to avoid all the troublesome exceptional cases on which the reviewer comments below. The typography is excellent, and the diagrams very good. The references are given by the numbers of articles, which are hard to find, as they are in the same type as the numbers to the problems, and are not indicated at the tops of the pages. The historical notes given at the ends of some of the chapters are brief and interesting. It is a pity that they are ignored in the table of contents, and that the index refers to them only under some of the proper names that appear in them.

The two-foci definitions are given when the ellipse and hyperbola are dealt with separately, and the focus-directrix definition of the curves is reserved until they are shown (with numerous admirable diagrams) to be sections of a cone.

The tangent to the circle is found from the property that it is perpendicular to its radius; in the cases of parabola and ellipse, the slope of the tangent is found by what used to be called the "secant method," that is by using h and k for increments and making no mention of the differential calculus.

Determinants of the second, third, and fourth orders appear at appropriate places, and the student is referred to an algebra (by the same publisher) for their use and meaning.

The chapter on empirical equations gives little about types, only a footnote about least squares, and a surprisingly inadequate definition of "best fitting curve." According to this chapter, the lines $x-y=0$, $2x+y=1$, and $x+2y=1$ are all best fitting straight lines for the set of points (0, 0), (1, 0), and (0, 1). Double and single logarithmic transformations are used and the corresponding ruled papers are mentioned but not exhibited.

There is a four page index, but it does not include some of the technical terms used in the text, such as: best fit, branch, conjugate, curve, determinant, direction numbers (given under N only), discussion, identity, infinity, invariant, normal, quadric, revolution, symmetric, vertex. The cycloid is the only curve indexed other than the conics.

The reviewer wonders if the average student can be expected to have learned enough about identities, imaginaries, infinity, zero-divisors, and the distinction between signless and sign-bearing numbers to warrant passing over these topics in a text on analytic geometry where the ideas behind these words have a more vivid significance than in previous subjects.

The authors surely have in mind an average student who does not wish to have any notice taken of the exceptional or troublesome cases. They nowhere point out that the definition of slope (p. 8), the two-point equation (p. 22), the intercept form (p. 25), the slope-intercept form (p. 26), fail in the case of vertical lines: that the formula for ϕ (p. 10) fails when $1+m_1m_2=0$, and that the proof that $A_1A_2+B_1B_2=0$ for perpendicular lines (p. 31) fails when $m_2=0$. On p. 169, under $y=\sec x$, they say "For a value of x at the asymptote, we may say that the function is equal to $\pm\infty$," and on p. 167 we find "tan $x=y$, $x=\pi/2$, $y=\pm\infty$." But on p. 196, "$\rho\theta^n=1$. We cannot let $\theta=0$." But such things do not bother the average student.

W. R. Ransom

Five Place Tables. By P. Wijdenes. Logarithms of integers, logarithms and natural values of trigonometric functions in the decimal system for each grade from 0 to 100 grades with interpolation tables. Groningen (Holland), P. Noordhoff, 1937, 167 pages. 2.50 fl.

This book is an English version of school tables, published by Mr. P. Wijdenes, well known author of mathematical books for school use in the Netherlands. Instead of the division of the right angle into 90 degrees, and each degree into 60 minutes and 3600 seconds, the division adopted by these tables is that

in which the right angle has 100 grades (gr.), which are subdivided in decigrades (dgr.), centigrades (cgr.), milligrades (mgr.) and decimilligrades (dmgr.). The author, in these tables, has advanced by milligrades up to 1, 20 gr., beyond that point by centrigrades. Proportional interpolation tables appear on every page.

The book has a handy size, which makes it fit for school use, and costs 2.50 Dutch florins, a little more than one dollar at the present rate.

A French version, which also contains tables for the old sexagesimal division, has recently appeared under the title: *Tables de logarithmes et de valeur naturelles des rapports trigonomêtriques*. It is by H. Commissaire and is published by Masson et Cie, Paris.

D. J. STRUIK

Elementary Analytical Conics. By J. H. S. Bailey, D.D., London, Oxford University Press, 1936. 378 pages. $2.75.

The older generation of mathematicians in this country who were brought up on Salmon and C. Smith and Loney, will pick up this book with interest to see what changes the passing years have brought. They will lay it down assured of the fundamental stability of the British Empire. The thing we remember about the texts of our youth is that they adapted methods to the situation, and did everything in the neatest possible way. This is true of Bailey; analytical conics is still a fine art.

Is it necessary or desirable in this day to devote so much time and effort to this subject? American universities and colleges have almost universally said no. Our large and restless clientele must be on to the calculus, though possessed of the barest minimum of algebra, trigonometry, and analytic geometry. If we may judge from this and from other texts, this is not so in Great Britain. With fewer, and more keenly interested students, the old ways are still maintained. And they are the old ways! "An asymptote of a curve is a straight line which meets it in two points at infinity, but which is not altogether at infinity. . . . We know that the equation $0x^2+0x+c=0$ has two roots equal and infinite. This may be proved by putting $x=1/y$ in the equation $cx^2+0x+0=0$ which has two equal roots of zero. For the equation $0y^2+0y+c=0$ thereby obtained must have as its equal roots the reciprocals of the equation whose two equal roots were zero, i.e. they are both infinite."

Many of us were exposed to this years ago. We liked it then, revolted from it later, and brought up our students in truth and purity. This may sound horrifying to our students, but to us there is a nostalgic flavor in consecutive points and infinite roots. Surely this is not as black as it has been painted. Fundamentally it is an unsophisticated adaptation of both algebra and geometry to fit each other better. Historically, mathematics was developed through just such methods; it is not obvious that these methods have today lost their pedagogic value.

B. H. BROWN

Leçons d'Algèbre et de Géométrie. By R. Garnier. vol. 2; coniques et quadriques. Paris, Gauthier-Villars, 1936. 207 pages. 40 fr.

The second volume of Garnier's course treats of poles and polars, anharmonic ratio, and pencils, with consideration first of conics and then of quadric surfaces. The treatment is largely analytic. A striking feature is the inclusion of very general theories from which specializations are made. Thus we find first a treatment of contact-transformations, both in the plane and in space, illustrated by polar reciprocation, and the process of determining pedal curves and surfaces. The order of difficulty is therefore very variable. A general theory, presented in an abstract, concise manner, is followed by a series of elementary theorems and simple constructions. The reaction of readers trained in schools other than the French, is that any student capable of understanding such general theory must certainly have encountered at an earlier date these elementary developments. The reviewer finds difficulty in seeing how such a course as this could fit into the program of an American university.

B. H. BROWN

MATHEMATICS CLUBS

EDITED BY F. W. OWENS AND HELEN B. OWENS, State College, Pa.

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to F. W. Owens, 462 East Foster Ave., State College, Pa.

THE MATH MIRROR

Brooklyn College *Math Mirror* is the name of the Annual Bulletin of the mathematics clubs of Brooklyn College which has appeared each February since 1933. From a set of mimeographed sheets it has developed into a dignified bound magazine with over six hundred subscribers among the students. The current issue includes five mathematical papers, discussion of elective courses in mathematics, news of the clubs and Pi Mu Epsilon and an interesting problem corner. The management and contributions are all the work of students. The five numbers show a steady and noteworthy increase in quality and interest of material.

LOAN LIBRARY

We wish to acknowledge the following recent additions to this collection:

Grabitall, a play in two acts, by Professor A. Marie Whelan, Hunter College. It includes the mathematics of the Townsend Plan.

Sad Ballad of the Jealous Cones. A song, tune Lorelei, by K. Lasswitz, translated by Professor A. E. Meder, New Jersey College for Women.

Cross word puzzles, real mathematical thought teasers; The Math Club, Brooklyn College.

CLUB REPORTS

1935–36

Informal Seminar, University of Florida

With frequent meetings at irregular intervals this group is informal in organization only. Topics discussed by students and faculty members included: Geometric probability; Trigonometric approximations; Relation of mathematics and economics. The guest speaker of the year was R. Walker of the Aetna Life Insurance Company who discussed "Mathematics and the actuary."

Kappa Mu Epsilon, Illinois State Normal University

President, Bernice Ramsey; Vice-President, H. McClintock; Secretary, Miriam Brown; Treasurer, Marjorie Burrow; Historian, Mary Bryant; Corresponding Secretary, Edith Atkins; Faculty Sponsor, Professor C. N. Mills. The chapter holds monthly meetings. One open meeting was devoted to "Numbers" with talks on Number sense, Number lore, Irrational numbers, and Playing with numbers. Professor N. E. Rutt, of Northwestern University, guest speaker at the Founders' Day banquet, spoke on "Mathematics as a tree of knowledge." Other interesting topics were: Bending moments by graphic methods; Decibels; The shoemaker's knife; The tractrix in designing loud speakers; The circle contact problem. In May the chapter assisted in greeting the Illinois Section of the Mathematical Association of America.

Women's College Mathematics Club, University of Delaware

President, Marianne Baldt; Secretary-Treasurer, Kathleen Spencer; Faculty Adviser, Edith McDougle. The general topic for the year's program was "Applications of mathematics and the use of instruments." Hence theory and practice were combined so that the level, the transit, the slide rule, and the telescope all added their bits of interest to the club's work. Applications of elliptic functions proved a profitable subject of discussion. Each year the club adds several volumes to the mathematical library of the College.

Mathematics Club, Los Angeles Junior College

Director, Dr. S. E. Urner. First Semester: President, E. Lofgren; Vice-President, M. Moran; Secretary, A. Bidek; Treasurer, J. Barber. Second Semester: President, G. Trapp; Vice-President, M. Huber; Secretary, J. Barber; Treasurer, F. Avila. The semi-monthly programs included: Determination of π with matches; Hydrographic surveying; Biological applications of mathematics; and Cosmic rays. An afternoon social with mathematical games and puzzles and a well conducted problem solution contest added diversity. Instructive expeditions to Griffith Park Observatory and Planetarium and to Mount Wilson Observatory, and a gay beach party rounded out a full year.

Pi Mu Epsilon, University of California

Director, F. G. Fisher; Vice-Director, E. Lingafelter; Secretary, Roberta Kneedler; Treasurer, R. Wakerling; Librarian, Dr. C. B. Morrey. Eight regular meetings, two initiation banquets at which, according to traditional custom, new members provided entertainment, and a spring picnic filled a successful year's program. Topics discussed included: Fowler's method in statistical mechanics; Some features of the three dimensional and Lorentz transformation groups; Mathematical theory of the planimeter; Application of projective geometry to the determination of the positions of the heavenly bodies; Group problems in general perturbations; The integrals of the differential equations of motion from the point of view of the theory of groups of contact transformations.

Euclid Club, College of William and Mary

President, R. Prince; Vice-President, E. Talley; Secretary, Elizabeth Tate; Treasurer, A. Sinclair; Program Chairman, Mildred Graves. The club has assumed a more formal organization with impressive initiation and banquet for new members. The group has gained added interest, and well attended monthly meetings were held. Topics of papers included: Magic squares; Monism in arithmetic; Photo-electric number cell; Early history of astronomy; Theory of areas and least squares; Fourth dimension.

Mathematics Society of Northeastern University

President, H. E. Rogers; Secretary, L. G. Mason; Treasurer, J. A. Tierney; Faculty Adviser, E. E. Haskins. Topics discussed include: The geometry of space; Modern theories of matter; Numerical and graphical methods of finding derivatives; The quantum theory; Mathematics in nature; Why is 9 a magic number? C. F. Taylor won the contest for the best paper by a student with a paper entitled "Perfect numbers."

The Case Mathematical Club, Case School of Applied Science

President, F. R. Kraft; Vice-President, L. D. Kovach; Secretary, C. H. Tindal; Treasurer, S. Foldes; Faculty Adviser, Professor M. Morris. This group, which awards prizes for the best papers presented by a student, held monthly meetings with topics including: The triangle; Proofs of the tangent law; Conformal and equiareal mapping; Mathematics and music. Twice the members were guests of Dean T. M. Foche, once when Dr. C. C. Torrance discussed "Distance" and again when Professor D. Rinehart of Ashland College spoke on "Generalized vectors and complex numbers."

Phi Chi Mu, Washington and Jefferson College

President, J. R. Bukey; Secretary-Treasurer, E. P. Albright; Faculty Adviser, Dr. C. S. Atchison. This active science club of honor students includes mathematics, physics, and chemistry as subjects for programs. Among its topics more nearly related to mathematics were: The relation of philosophy to science; Interesting numbers; Polar planimeter; Concepts of the universe. At the initiation banquet, Dr. R. C. Hutchinson, President of Washington and Jefferson College, discussed "The importance of science in education" and D. W. Ross of Findlay Clay Products spoke on "Distribution."

Pi Mu Epsilon, Marquette University

Topics discussed included: The relation of philosophy to science; Stokes' law; Elementary concepts of mathematics; Mathematics involved in music; Charting of distribution curves; Are Newton's laws still valid? The Frumweller contest was again successfully conducted. The lively mimeographed Bulletin appeared each month with brief discussions of the meetings, the plans, and mathematical problems. The chapter takes active part in the Intercollegiate Mathematical Association of Milwaukee. The annual banquet closed the year's activities.

The Junior Mathematical Club, The University of Chicago

Autumn-Winter: President, D. M. Dribin; Treasurer, C. H. Denbow; Program Chairman, H. H. Goldstine; Social Committee, S. Kathryn Cardwell, M. F. Smiley. Spring: President, H. H. Goldstine; Treasurer, C. H. Denbow; Program Chairman, M. F. Smiley; Social Committee, S. Kathryn Cardwell, F. A. Valentine. During the year the club sponsored one bridge dance, held a tea to welcome new students in the autumn and a social tea and bridge in the spring. Guest speakers were Dr. F. C. McLean, Department of Physiology, whose subject was "The law of mass action in biology" and Professor N. Rashevsky, Department of Psychology, with the topic "Mathematical biophysics." Topics at regular meetings included: The plane quadratic Cremona transformation; Dialectics and mathematics; Integer vectors; Some postulational bases for topology; Systems of quadrics at a point of a surface; The transfinite ordinal numbers; Recent developments in Waring's theorem; Boolean algebras; Relations between integral equations and differential systems; Some harmonic function problems in bio-physics. At one meeting Professor G. A. Bliss gave interesting reminiscences of Professor Oskar Bolza.

Mathematics Club, Milwaukee-Downer College

Secretary, Arlyne Lawrence. Topics discussed included: Magic squares and how to make them; The fourth dimension; Giant numbers; Regular solids; Magic polygons; Curve tracing; Construction of regular polygons with special reference to the 17-gon; Mathematical tombstones; Cryptograms; Mathematics in nature. The club held a tea for new students, an annual banquet, and it entertained the Intercollegiate Mathematical Association of Milwaukee.

Newtonian Society, Lehigh University

Director, Professor J. H. Ogburn. Topics discussed included: Newton's part in astronomical research; Aircraft design; Squaring the circle; Magic squares; Infinity; Mathematical oddities; Life and achievements of Euler; History of differential and integral calculus; Life of Newton; Mathematical paradoxes; Determination of π; Duodecimal system; A paradox in probability; Fourth dimension; Kepler's astronomy; Prime numbers.

The Evening Session Math Club, Brooklyn College

President, B. Miller; Vice-President, H. Rich; Secretary-Treasurer, P. Handelman. This youngest member of Brooklyn College's triumvirate of undergraduate mathematical organizations has included in its programs talks by faculty and student members on many topics including: Scales of notation; Regular and semi-regular solids; Diophantine equations; Differential geometry. Such favorites as "Magic squares" and "Trisection of angles" were not neglected.

Math Club, Brooklyn College

President, H. Wolf; Vice-President, A. Goodman; Secretary, Esther Glicker; Publicity Director, M. Reade; Social Director, D. Cohen. The weekly programs, covering a wide range of topics, included: Nomography; Line geometry; The fifteen puzzle; Heaviside operational calculus; A posteriori probability; Recurring series. The annual mathematics party was successful in "promoting sociability" among students and faculty. The Integration Contest is held each term. Each class in integral calculus enters one or more teams of five members. Scoring is done by individuals and by teams. The winner of the highest individual score receives a prize.

Mathematics Club of Oberlin College

President, D. Ransom; Vice-President, Frances Sherman; Secretary-Treasurer, Eleanor E. Gish; Chairman of Social Committee, Ruth Binning; Chairman of Program Committee, J. Friedman. Topics discussed included: The Institute for Advanced Study; Constructions with straight edge and compasses; Constructions with compasses alone; Different bases for number systems and the game of Nim; The algebra of Khowarizmi; The set of rational numbers is countable; The set of algebraic numbers is countable; The story of the algebraic equation; Polar reciprocation; Postulates of geometry; The harmonograph; Statistical correlation; Telescopes; Kepler's laws; Abstract groups; Problems in maxima and minima treated algebraically; Mathematics in social sciences; Three point problem in plane table surveying; Complex numbers. The social activities of the club included a tea, a Christmas party, a spring party, and a banquet at which Professor Tibor Radó of Ohio State University spoke on "The story of a research problem in mathematics."

Kappa Mu Epsilon, Kansas State Teachers College, Pittsburg, Kansas

President, H. Menne; Vice-President, Ruby Fulton; Secretary, Marie Monk; Treasurer, C. C. Foust; Corresponding Secretary, Professor W. H. Hill; Faculty Sponsor, Professor J. A. G. Shirk. The chapter held monthly meetings with papers on many favorite topics. Some of the more unusual ones were: Numbers represented by sequences; Evolution of a planetary system; Progression of measurement; Numerology; and Duo-decimal number systems. The program meetings are open to all students and are conducted under the name, The Math Club. Meetings are held at the homes of the faculty members of the chapter.

The Mathematics Club, Ball State Teachers College

President, W. L. Cook; Secretary, Alberta Harrell. This club combines its social events with its program meetings. Many clever stunts are devised for getting acquainted and arousing interest. A number of the stunts have been filed with the Loan Library of Stunts. Among the programs two of especial interest were an open forum on "Student teaching in mathematics" and a discussion of "History of mathematics in America before 1900."

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 282. *Proposed by A. Gloden, l'Athénée de Luxembourg.*

Find a seven-digit cube of the form, $1{,}000{,}000x+1{,}000y+z$, with $y=x+z+1$. Is the solution unique?

E 283. *Proposed by W. B. Clarke, San Jose, California.*

Find the smallest pair of differently shaped right triangles with integer sides and the same area.

E 284. *Proposed by V. Thébault, Le Mans, France.*

Find the set of numbers, each of which equals thrice the square of the sum of its digits.

E 285. *Proposed by D. L. MacKay, Evander Childs High School, New York.*

If in triangle ABC, $\sin^2 A+\sin^2 B+\sin^2 C=1$, prove that the circumcircle cuts the nine point circle orthogonally.

E 286. *Proposed by Mannis Charosh, New Utrecht High School, Brooklyn, New York.*

Prove that

$$\sum_{r=0}^{n-1} 2^r \tan (2^r x) = \cot x - 2^n \cot (2^n x).$$

E 287. *Proposed by W. B. Campbell, Ithaca, New York.*

Find the maximum and minimum values of the function Z, defined as follows:

when $0\leqq x\leqq \frac{1}{2}$ and $0\leqq y\leqq x$, or when $\frac{1}{2}\leqq x\leqq 1$ and $0\leqq y\leqq 1-x$,

$$Z = y - y^2/3x;$$

when $\frac{1}{2}\leqq x\leqq 1$ and $1-x\leqq y\leqq x$,

$$Z = y - y^2/3x - (x+y-1)^3/(6xy).$$

Also solved by Norma Ackerman, E. F. Allen, Harry Anisgard, K. W. Crain, Rosemary Culligan, Ruth Efrein, Ruth Ginsburg, Herman Greenberg, Evelyn Hoffman, Irene Karmin, Ruth Karp, Molly Kuris, Milton Leifer, Benjamin Liebowitz, Emanuel Mehr, Louis Shapiro, Moe Smolinsky, Milton Sobelman, C. E. Springer, Simon Vatriquant, Sylvia Wiener, Leon Weiss, G. A. Williams, and Harcourt Williams.

E 239 [1936, 575]. *Proposed by D. L. MacKay, Evander Childs High School, New York.*

Construct the triangle ABC, given c, angle C, and the ratio of b^2+a^2 to the area of the triangle.

Solution by the Proposer.

Let the area of triangle ABC be K, and set $(a^2+b^2)/K=m/n$. Since $K=\frac{1}{2}ab\sin C$ and $c^2=a^2+b^2-2ab\cos C$, then $(c^2+2ab\cos C)/\frac{1}{2}ab\sin C=m/n$, and $ab=2nc^2/(m\sin C-4n\cos C)$. Now the altitude from C equals $2K/c$, $=(ab/c)\sin C=2nc\sin C/(m\sin C-4n\cos C)=nc/(\frac{1}{2}m-2n\cot C)$, which is readily constructed. Having the base, altitude, and vertical angle, the construction of triangle ABC is reduced to a standard procedure.

Also solved by W. B. Clarke, Wm. Douglas, L. M. Kelly, E. P. Starke, Simon Vatriquant, and Z. W. Wilchinsky.

E 241 [1936, 575]. *Proposed by J. Rosenbaum, Hartford Federal College.*

On the sides A_0B_0, B_0C_0, and C_0A_0 of a triangle $A_0B_0C_0$, the points A_1, B_1, and C_1 are taken respectively, such that $A_0A_1/A_0B_0=B_0B_1/B_0C_0=C_0C_1/C_0A_0=1/n$, where n is a positive integer. A triangle $A_2B_2C_2$ is now obtained from triangle $A_1B_1C_1$ in the same manner that $A_1B_1C_1$ was obtained from $A_0B_0C_0$. We now consider the sequence of triangles, $T_0, T_1, T_2, \cdots, T_n$, formed in this manner. Let R_n be the ratio of the area of T_n to T_0, and determine the limit of R_n as n increases through integer values without limit.

Solution by C. E. Springer, University of Oklahoma.

If $A_0(1, 0, 0)$, $B_0(0, 1, 0)$, and $C_0(0, 0, 1)$ be taken as the vertices of the triangle of reference for a system of areal coordinates, then the vertices of $A_1B_1C_1$ are given by $A_1(1-1/n, 1/n, 0)$, $B_1(0, 1-1/n, 1/n)$, and $C_1(1/n, 0, 1-1/n)$. Similarly, the vertices of $A_2B_2C_2$ are given by $A_2[(1-1/n)^2, (2/n)(1-1/n), (1/n)^2]$, $B_2[(1/n)^2, (1-1/n)^2, (2/n)(1-1/n)]$, and $C_2[(2/n)(1-1/n), (1/n)^2, (1-1/n)^2]$. This process may be readily continued.

The area of triangle $A_1B_1C_1$ is given by the determinant

$$\begin{vmatrix} 1-1/n & 1/n & 0 \\ 0 & 1-1/n & 1/n \\ 1/n & 0 & 1-1/n \end{vmatrix} \cdot K = \frac{(n-1)^3+1}{n^3} \cdot K = Q\cdot K,$$

where K is the area of triangle $A_0B_0C_0$. Similarly, the area of triangle $A_2B_2C_2$

is given by $Q^2 \cdot K$, and the area of $A_nB_nC_n$ is given by $Q^n \cdot K$, where as before, $Q=[(n-1)^3+1]/n^3$. Since for $n=1$, triangle $A_1B_1C_1$ coincides with triangle $A_0B_0C_0$, we must have $R_n=Q^{n-1}$. Then, as n becomes infinite, R_n approaches the limit $1/e^3$.

Also solved by W. B. Clarke, K. W. Crain, D. L. MacKay, Simon Vatriquant, and the proposer.

E 242 [1936, 576]. *Proposed by V. Thébault, Le Mans, France.*

Find a palindromic number whose digits are all even, and whose square contains the ten digit-symbols once each. Show that the solution is unique.

Solution by Simon Vatriquant, Brussels.

If $N^2=(abcba)^2$ contains the ten digits, then the root $abcba$ must be $\equiv 0 \pmod 3$, and $3<a$. Thus $2a+2b+c\equiv 0 \pmod 3$, and since a, b, and c are even, we have $a+b\equiv c \pmod 6$. For each of the five possible values of c, namely 0, 2, 4, 6, and 8, there are five potential pairs of values for a and b complying with the conditions derived above. Of these twenty-five potential values of N we may discard seventeen by using a table of the squares of numbers under a thousand, examining the first three and the last three digits of the squares of our subjects for investigation and ruling out any which show a repeated digit in its square. Among the remaining eight numbers, only one satisfies all requirements. It is

$$84648^2 = 7165283904.$$

Also solved by W. E. Buker, Mary L. Constable, C. T. Oergel, Walter Penney, E. P. Starke, W. R. Talbot, C. W. Trigg, Z. W. Wilchinsky, and the proposer.

E 243 [1936, 576]. *Proposed by J. E. Trevor, Cornell University.*

The diameter of the tread of a driving wheel of a locomotive is five feet, and the diameter of its flange is five and a half feet. When the locomotive moves from west to east, a point on the rim of the flange moves in a curved path which at times runs from east to west. Disregarding the up and down motion on this retrograde path, the point is displaced each time a certain distance towards the west. When the locomotive is going sixty miles an hour, what is the sum of these westward displacements in one minute? (Give the answer to the nearest foot.)

Solution by W. B. Clarke, San Jose, California.

For convenience, let t represent the radius of the tread ($=30$ inches) and f represent the radius of the flange ($=33$ inches). Then, taking the origin at the point of contact of wheel and rail when the chosen point on the rim of the flange is at its lowest position, and letting θ be the angular measure of rotation of the wheel, the equations of the curve traced by this point P are

$$x = t\theta - f\sin\theta, \qquad y = t - f\cos\theta.$$

This curve is the trochoid, and the width of the loop which is symmetrical to the axis of y represents the retrograde displacement for one revolution of the wheel. To get the width of the loop, find the value of x when $y=0$, and double, since differentiation shows that the x-axis crosses the loop at its widest part, with vertical tangents there.

When $y=0$, $\cos\theta=t/f=.90909\cdots$, $\theta=24°37'=.4296$ radians, $\sin\theta=.4165$, and $x=t\,\theta-f\sin\theta=-.8565$ inches. (The minus reminds us that as the wheel rolls forwards, the bottom of the flange moves backwards.) The width of one loop is then 1.713 inches, and in going one mile (which takes one minute at sixty miles an hour), the wheel makes $5280\cdot 12/60\pi=336.1$ revolutions, which results in a total retrograde motion of forty-eight feet, with an error of less than a quarter of an inch.

Also solved by E. S. Smith and the proposer.

E 244 [1936, 576]. *Proposed by Virgil Claudian, Roumanian Mathematical Institute.*

Find by elementary methods, as n increases without limit, the limit of

$$[1^3a^{1/n}+2^3a^{2/n}+\cdots(n-1)^3a^{(n-1)/n}+n^3a]/n^4.$$

Solution by J. Rosenbaum, Hartford Federal College.

The expression in question can be written as

$$(1)\qquad (1/n)(1/n)^3a^{1/n}+(1/n)(2/n)^3a^{2/n}+\cdots+(1/n)(n/n)^3a^{n/n}.$$

If now the interval (0, 1) be divided into n equal parts, then denoting each part by Δx, (1) becomes

$$(2)\qquad \Delta x(\Delta x)^3a^{\Delta x}+\Delta x(2\Delta x)^3a^{2\Delta x}+\cdots+\Delta x(n\Delta x)^3a^{n\Delta x}.$$

From (2) it is seen that the limit in question equals the area under the curve, $y=x^3a^x$, from $x=0$ to $x=1$. Hence the required limit is

$$(3)\qquad L=\int_0^1 x^3a^xdx$$

which is found to be

$$(4)\qquad L=a/\log a-3a/(\log a)^2+6a/(\log a)^3-6a/(\log a)^4+6/(\log a)^4.$$

While this expression for L is not defined when $a=1$, we may replace a by 1 in (1), (2), and (3), and in that case (3) reduces to $L=1/4$. This, incidentally, is also the limit, as a approaches 1, of L as given by (4).

Also solved by E. P. Starke and Simon Vatriquant.

E 245 [1936, 638]. *Proposed by J. Rosenbaum, Hartford Federal College.*

Prove that the sum of the squares of the edges of an isosceles tetrahedron is equal to four times the square of the diameter of the circumscribed sphere. (A tetrahedron is isosceles when its faces are congruent.)

Solution by Simon Vatriquant, l'Athénée Royale d'Ixelles, Brussels, Belgium.

The parallelepiped circumscribed about this tetrahedron is rectangular. (N. A. Court, *Modern Pure Solid Geometry*, p. 94.) Its face diagonals are the edges of the tetrahedron, and its solid diagonal is the diameter of the circumsphere. Thus, if we denote the edges of the parallelepiped by x, y, and z, we will have $4R^2=x^2+y^2+z^2$, with $a^2=y^2+z^2$, $b^2=z^2+x^2$, and $c^2=x^2+y^2$. Consequently, $16R^2=4D^2=2a^2+2b^2+2c^2$.

N. A. Court remarks, as a bibliographical note, that this formula was obtained by J. Neuberg in the *Nouvelle Correspondence Mathématique*, vol. 2, 1876, p. 44, art. 2, and is ascribed by him to G. Dostor. It constitutes a part of exercise 18, p. 108 in N. A. Court's *Modern Pure Solid Geometry*.

Also solved by W. E. Buker, William Douglas, L. M. Kelly, W. J. Kirkham, C. W. Trigg, and the proposer.

E 246 [1936, 638]. *Proposed by V. Thébault, Le Mans, France.*

In what system of enumeration can a four-place number of the form *abab* be the square of a two-place number of the form *ba*? Ascertain if the solution is unique.

Partial Solution by E. P. Starke, Rutgers University.

Let r be the base of the system. Then by hypothesis we have $ar^3+br^2+ar+b=(br+a)^2$, or $(ar+b)(r^2+1)=(br+a)^2$. We seek solutions of this Diophantine equation in positive integers a, b, and r, for which $a<b<r$.

Since r^2+1 is the greater of two factors of a square, it must itself possess a square factor greater than unity. Put $r^2+1=kx^2$. Since -1 is a quadratic residue to an odd prime modulus only of the form $4n+1$, kx^2 must be a product of such factors, or twice such a product.

There is an infinite number of solutions of $r^2+1=kx^2$ for each value of k of the form h^2+1 (this MONTHLY, 1935, p. 573). Further, given any solution (r, x), we have other solutions $(mx^2\pm r, x)$ for all sufficiently large integer values of m. There are still other solutions, such as (70, 13). The following relations are valuable: (p_i represents a prime.)

(A) Let $r^2+1\equiv 0 \bmod p_1^2$; then $br+a\equiv 0 \bmod p_1$.

(B) Let $r^2+1\equiv 0 \bmod p_2$ but $r^2+1\not\equiv 0 \bmod p_2^2$, and let $2<p_2$. Then it is easy to show that $a\equiv b\equiv 0 \bmod p_2$.

(C) Let $r-1\equiv 0 \bmod p_3$; then $a+b\equiv 0$ or $2 \bmod p_3$.

(D) Let $r+1\equiv 0 \bmod p_4$; then $b-a\equiv 0$ or $2 \bmod p_4$.

Thus we find possible values for r to be 7, 18, 32 38, 41, 43. 57, 68, 70, 82, 93, 99, etc. For $r=7$, $k=2$ and $x=5$. Take $p_1=5$, $p_3=2$ or 3, $p_4=2$, and obtain from the above congruences $a=2$, $b=4$, a solution. The congruences show easily that each of the other stated values for r yields no solutions for a and b. This method does not determine whether or not another solution exists.

Also partially solved (base seven determined but uniqueness not shown) by W. E. Buker, Daniel Finkel, W. J. Kirkham, C. W. Trigg, and the proposer.

E 247 [1936, 638]. *Proposed by N. A. Court, University of Oklahoma.*

Show that the six planes perpendicular to the six edges of a tetrahedron, and passing through the mid-points of the projections of the respective edges upon a given plane, have a point in common.

Solution by C. E. Springer, University of Oklahoma.

Let the vertices of the tetrahedron be $O(0, 0, 0)$, $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$, and $C(c_1, c_2, c_3)$, and let the given plane have the equation $z=0$. Then the planes perpendicular to OA, OB, and OC, and passing through the points $(a_1/2, a_2/2, 0)$, $(b_1/2, b_2/2, 0)$, and $(c_1/2, c_2/2, 0)$ are given by

$$a_1x + a_2y + a_3z = \tfrac{1}{2}(a_1^2 + a_2^2),$$
$$b_1x + b_2y + b_3z = \tfrac{1}{2}(b_1^2 + b_2^2),$$
$$c_1x + c_2y + c_3z = \tfrac{1}{2}(c_1^2 + c_2^2).$$

These planes intersect in a point since OA, OB, and OC are not coplanar. The planes perpendicular to the edges BC, CA, and AB, and passing through the mid-points of the projections of these segments on the plane $z=0$, are given by

$$(b_1 - c_1)x + (b_2 - c_2)y + (b_3 - c_3)z = \tfrac{1}{2}[(b_1^2 + b_2^2) - (c_1^2 + c_2^2)]$$

and two other equations obtained by permuting the letters a, b, and c cyclically. Since these equations are linear combinations of the equations of the first set, these planes contain the point common to the first three planes.

Solution by the Proposer.

Let A', B', C', and D' be the projections of the vertices A, B, C, and D of the given tetrahedron $ABCD$ upon the given plane. The powers MA'^2, MB'^2 of the mid-point M of $A'B'$ with respect to the spheres (A), (B) having A, B for centers and AA', BB' for radii are equal, hence the plane passing through M and perpendicular to the edge AB is the radical plane of the two spheres (A) and (B).

Similarly for the other planes. Thus the six planes considered are the radical planes of the spheres (A), (B), and their analogues (C), (D), taken two-by-two. Hence the proposition. (N. A. Court, *Modern Pure Solid Geometry*, p. 201, art. 626.)

Also solved by W. T. Short and Simon Vatriquant.

E 248 [1936, 638]. *Proposed by Virgil Claudian, Roumanian Mathematical Institute.*

Show that for any plane triangle,

$$\sum \frac{a^2}{(r_a - r)r_b r_c} = \frac{2}{r},$$

where r is the radius of the inscribed circle, r_a the radius of the escribed circle tangent to the side a between B and C, etc.

Solution by Herbert Tate, McGill University.

It is well known that $r=K/S$ and $r_a=K/(S-a)$. (N. A. Court, *College Geometry*, Arts. 107, 108; and R. A. Johnson, *Modern Geometry*, Art. 298 a.) Here K is the area of the triangle ABC, and $2S=a+b+c$. Hence

$$\sum \frac{a^2}{(r_a - r)r_b r_c} = \sum \frac{a^2}{\left(\frac{K}{S-a} - \frac{K}{S}\right)\frac{K}{S-b}\,\frac{K}{S-c}}$$

$$= \sum \frac{a}{K} = \frac{a+b+c}{K} = \frac{2S}{K} = \frac{2}{r}.$$

Also solved by M. W. Aylor, W. E. Buker, W. B. Clarke, K. W. Crain, Hansraj Gupta, R. A. Johnson, L. M. Kelly, E. A. Nordhaus, Augustus Sisk, C. E. Springer, E. P. Starke, W. R. Talbot, and C. W. Trigg.

E 249. [1936, 638]. *Proposed by E. R. Ott, University of Buffalo.*

A circular track has a circumference of sixty miles. Three travelers, A, B, and C, start in the same direction from the same point, and travel around the track continuously at the rates of 26, 10, and 2 miles per day, respectively. When are they next all three abreast?

Solution by Althéod Tremblay, Laval University, Quebec.

Let the rates of A, B, and C be u, v, and w respectively, and let the length of the track be t. Then A and B will meet at the end of $rt/(u-v)$ days, where r is any positive integer. Also A and C will meet at the end of $qt/(u-w)$ days, with q any positive integer, and B and C will meet at the end of $pt/(v-w)$ days, with p any positive integer. Then any integer values of p, q, and r which satisfy the equations, $p/(v-w)=q/(u-w)=r/(u-v)$, will give a solution.

In the particular case given we have $p/8=q/24=r/16$, or $6p=2q=3r$. The smallest integer values of p, q, and r which satisfy these equations are $p=1$, $q=3$, and $r=2$. The required number of days is therefore $60p/8=60q/24=60r/16=7\frac{1}{2}$ days.

Also solved by J. A. Benner, W. E. Brooke, W. E. Buker, W. B. Clarke, Fred Discepoli, William Douglas, Daniel Finkel, G. E. Forsythe, K. W. Miller, C. W. Munshower, E. A. Nordhaus, Walter Penney, C. C. Richtmeyer, C. E. Springer, E. P. Starke, W. R. Talbot, C. W. Trigg, B. C. Zimmerman, and the proposer.

E 250 [1936, 638]. *Proposed by A. Gloden, l'Athênée de Luxembourg.*

Find a seven-place number whose exact cube root equals the millions digit plus the number formed by the last three digits, minus the number formed by the remaining three digits.

Solution by C. W. Trigg, Cumnock College, Los Angeles.

It is well known that if, beginning at the right, an integer is divided into triads, then the number is congruent to the sum of the odd triads less the sum of the even triads, modulo $7\cdot 11\cdot 13$. So in the present case $N^3\equiv N \pmod{7\cdot 11\cdot 13}$, or, $(N-1)N(N+1)\equiv 0 \pmod{7\cdot 11\cdot 13}$. Now N^3 has seven digits, so $100<N<216$. We now seek three consecutive integers within this range such that multiples of 7, 11, and 13 all appear in the set. This is most conveniently done by listing the nine multiples of 13 in this range and seeking multiples of 7 and 11 among the pentads centered on these numbers. In this way we find just two values of N, namely 155 and 209, to which correspond the cubes, 3723875 and 9129329.

Also solved by W. E. Buker, K. W. Miller, E. P. Starke, Simon Vatriquant, B. C. Zimmerman, and the proposer.

E 251 [1936, 639]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

After an exact long division had been made (in the decimal system), some of the digits, 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9, were replaced by the letter s wherever they occurred in the work, while all the other digits in the work were replaced by the letter t. This was the result:

$$\begin{array}{rcccccccl}
s\,t\,s\,) & t & t & t & s & t & t & (& s\,t\,s\,t \\
 & \underline{t} & \underline{t} & \underline{t} & & & & & \\
 & s & s & t & s & & & & \\
 & \underline{t} & \underline{s} & \underline{t} & \underline{t} & & & & \\
 & & t & s & t & t & & & \\
 & & & \underline{t} & \underline{t} & \underline{s} & & & \\
 & & & t & t & s & t & & \\
 & & & \underline{t} & \underline{t} & \underline{s} & \underline{t} & &
\end{array}$$

Reconstruct the original long division and show that the solution is unique.

Solution by B. C. Zimmerman, Corozal, British Honduras.

Call the divisor D, the minuends and partial products M_i and P_i, and the digits of the quotient in order, a, b, c, and d.

Since D has two different three-digit multiples distinct from itself, it is less than 333. Since $9D<3000$, each $P_i<3000$.

Since $M_3=1stt$, 1 is among the digits represented by t. If 2 is also in the t-set, $300<D$, $3400<$ the quotient Q (since $a<b$), and $10^6<D\cdot Q$. Since there are but six digits in the dividend, 2 is in the s-set. Consequently each $P_i<2000$, while $2000<M_2$, from the second subtraction. Furthermore, $M_2-P_2<200<M_1-P_1<D$. Since $M_3-P_3<200$, $M_3<1200$, and since $M_3=1stt$, $1010<M_3<1099$. Hence $M_2-P_2<110$, $M_2<2200$, and since $M_2=ssts$, $2000<M_2<2099$.

Since $D=sts$, $D\geqq 210$, and $M_1<1000-D<790$, and since $200<M_1-P_1$, we have $a\cdot D<790-200$, whence $D<295$.

Since $M_2-P_2<110$, and $P_2+110=b\cdot D+110>2010$, therefore $b\geqq 7$, and $a=2$.

Now if $c=4$, since $c\cdot D<1000$, $D<250$, $a\cdot D=2D=4tt$, and 4 would be both an s and a t, which is impossible. Therefore $c=3$.

Since $(10c+d)D=tsttt\geqq 10111$, $D>10111/39>259$. Since $b\cdot D<2000$, $b<8$. Therefore $b=7$. Furthermore, $(10c+d)>10111/295>35$. Since $d<b$, $10c+d=36$ and $d=6$; [$d<b$ because $b\cdot D>1900$, and $d\cdot D<2000$.]

Now $36\cdot D=tsttt=10ttt$, so that $280<D$. On the other hand, $7\cdot D=tstt\leqq 1988$, so that $D<284$. Since $d\cdot D=6D=ttst$, $D\neq 282$. Since $D=sts$, $D\neq 281$. Hence by elimination, $D=283$. We have already found $Q=2736$, so the reconstructed problem is

$$\begin{array}{r}
283\,)\,774288\,(\,2736\\
\underline{566}\\
2082\\
\underline{1981}\\
1018\\
\underline{849}\\
1698\\
\underline{1698}
\end{array}$$

The uniqueness of this solution is apparent from the method of finding it.

Also solved by Mary L. Constable, William Douglas, Robert Seamons, E. P. Starke, C. W. Trigg, and the proposer.

Advanced Problems

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3832. *Proposed by Don Wallace, Charlottesville, Va.*

Given the circle whose center is the circumcenter and which passes through the incenter: prove that (a) it cuts the altitudes at a distance from the vertices equal to the circumradius, and (b) it cuts the perpendicular bisectors of the sides at a distance twice the inradius from the reflections of the circumcenter in the sides.

3833. *Proposed by Emma Lehmer, Bethlehem, Pa.*

If U_n, $n=0, 1, 2, \cdots$, is a kth order recurring series defined by

$$U_{n+k} = a_1 U_{n+k-1} + a_2 U_{n+k-2} + \cdots + a_k U_n,$$
$$U_0 = U_1 = \cdots = U_{k-2} = 0, \qquad U_{k-1} = 1,$$

prove that for $n>0$

$$U_{n+k-1} = \sum a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k} \frac{(\alpha_1 + \alpha_2 + \cdots + \alpha_k)!}{\alpha_1!\alpha_2! \cdots \alpha_k!},$$

where the summation extends over all integral solutions of

$$\alpha_1 + 2\alpha_2 + \cdots + k\alpha_k = n.$$

3834. *Proposed by Paul Erdös, Budapest, Hungary.*

Let $a_1 < a_2 < \cdots < a_n \leqq 2n$ be a sequence of positive integers. Then

$$\min [a_i, a_j] < 6([n/2] + 1),$$

where $[a_i, a_j]$ denotes the least common multiple of a_i and a_j. This is the best possible estimate.

3835. *Proposed by Paul Erdös, Budapest, Hungary.*

Let $a_1 < a_2 < \cdots < a_n < 2n$ be a sequence of positive integers. Then

$$\max (a_i, a_j) > \frac{38n}{147} - c,$$

where c is independent of n, and (a_i, a_j) denotes the greatest common divisor of a_i and a_j. This is the best possible estimate.

3836. *Proposed by H. P. Thielman, College of St. Thomas, St. Paul, Minn*

Given Kelvin's function

$$\operatorname{bei} x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x/2)^{4n-2}}{[(2n-1)!]^2},$$

evaluate

$$\int_0^\infty \frac{\operatorname{bei} x}{x} dx.$$

3837. *Proposed by H. P. Thielman, College of St. Thomas, St. Paul, Minn.*

Evaluate

$$\int_0^\infty \int_0^x x^{-1} y \operatorname{bei} y \sin [(x^2 - y^2)/4] dy\, dx.$$

3838. *Proposed by J. R. Musselman, Western Reserve University.*

If I be the incenter of triangle $A_1A_2A_3$ with I_i the point of contact on the side A_jA_k, and I_i' the image of I in the side I_jI_k, then the circles $I_i'I_j'I_k$ pass through ϕ, the point of Feuerbach of $A_1A_2A_3$, also the circles $I_iI_jI_k'$ meet on $I_1'I_2'I_3'$ at a point ψ such that ϕ and ψ are symmetric as to the center of the common nine-point circle of $I_1I_2I_3$ and $I_1'I_2'I_3'$.

Note. The point of Feuerbach for triangle $A_1A_2A_3$ is the point of tangency of its inscribed circle with its nine-point circle. See Johnson's *Modern Geometry* for a discussion of this point.

3839. *Proposed by V. Thébault, Le Mans, France.*

A transversal Δ cuts the sides BC, CA, AB of a triangle in α, β, γ. Parallels, with arbitrary direction, through the vertices of the triangle cut Δ in α', β', γ'. Prove that the parallels to BC, CA, AB through α', β', γ' divide in the same ratio the straight line segments O_aH_a, O_bH_b, O_cH_c which join the circumcenters O_a, O_b, O_c to the orthocenters H_a, H_b, H_c of triangles $A\beta\gamma$, $B\gamma\alpha$, $C\alpha\beta$. See 3818 [1937, 111].

3840. *Proposed by V. Thébault, Le Mans, France.*

Parallel planes, with arbitrary direction, drawn through the vertices A, B, C, D of a tetrahedron cut a given straight line Δ in α, β, γ, δ. The planes through the latter four points parallel, respectively, to the faces BCD, CDA, DAB, ABC determine by the intersections of sets of three a tetrahedron $A_1B_1C_1D_1$ symmetrically equal to the given tetrahedron $ABCD$ (J. Neuberg, *Mathesis*, 1891, p. 50). Find the loci of the vertices of the tetrahedron $A_1B_1C_1D_1$ and of the center of symmetry of the two tetrahedrons when the direction of the parallel planes varies. This is an extension to space of 3817 [1937, 111].

SOLUTIONS

3638 [1933, 496]. *Proposed by Oystein Ore, Yale University.*

Let p be a prime and N the smallest exponent such that $p^N \equiv 1 \pmod{n}$ for a given number n. The irreducible factors of $x^n - 1 \pmod{p}$ then have degrees dividing N. Find the necessary and sufficient condition, that there exist prime divisors of $x^n - 1 \pmod{p}$ of degree N' for all divisors N' of N.

Solution by M. F. Becker, New York, N. Y.

According to the terminology of L. E. Dickson (*Introduction to the Theory of Numbers*, p. 16) N is the exponent to which p belongs (mod n).

The problem reduces to finding the necessary and sufficient condition that every divisor N' of N be the exponent to which p belongs for some divisor n' of n. In the proof two lemmas are used.

LEMMA 1: If p belongs to A (mod m_1) and B (mod m_2), where m_1 and m_2 are relatively prime, then p belongs to the least common multiple of A and B (mod m_1m_2).

Proof: Let p belong to t (mod m_1m_2). Then $p^t \equiv 1$ (mod m_1m_2); and $p^t \equiv 1$ (mod m_1). Therefore $t \equiv 0$ (mod A); and $p^t \equiv 1$ (mod m_2). Therefore $t \equiv 0$ (mod B).

Hence t must be divisible by the least common multiple of A and B. From the definition of t, it follows that t equals the least common multiple of A and B.

LEMMA 2: If $n = q^k$ (where q is a prime $\neq p$) and $N = q^\alpha A$ for every divisor N' of N of the form $q^i A$ ($i = 0, 1, \cdots, k$), then p belongs to N' (mod n'), where n' is a divisor of n.

Proof: By assuming the contrary it is easily shown that p belongs to A (mod q). Suppose $p^A - 1$ is exactly divisible by q^e ($e \neq 1$). Then

$$p^{Aq} - 1 = (p^A - 1)(p^{A(q-1)} + p^{A(q-2)} + \cdots + 1);$$
$$p^{A(q-i)} \equiv 1 \pmod{q^e};$$
$$p^{A(q-1)} + p^{A(q-2)} + \cdots + 1 \equiv q \pmod{q^e};$$
$$p^{Aq} - 1 \equiv 0 \pmod{q^{e+1}}; \qquad p^{Aq} - 1 \not\equiv 0 \pmod{q^{e+2}}.$$

Thus by mathematical induction we find that p belongs to Aq^r (mod q^{e+r}). This proves the lemma.

The decomposition of n into distinct prime factors is written

$$n = q_1^{k_1} q_2^{k_2} \cdots q_r^{k_r}.$$

Now p belongs to $q_i^{\alpha_i} A_i$ (mod q_i), ($i = 1, 2, \cdots, r$). Then N, the least common multiple of the $q_i^{\alpha_i} A_i$ terms decomposed into distinct prime factors, is

$$N = q_1^{\tau_1} \cdots q_r^{\tau_r} l_1^{\sigma_1} \cdots l_f^{\sigma_f}.$$

The conditions we are seeking are:

(1) For each q_i for which $A_i \neq 1$, there must be some $A_j = q_i^d$ for every integer d, where

$$1 \leqq d \leqq k_i \leqq \tau_i, \quad \text{or} \quad 1 \leqq d \leqq \tau_i < k_i.$$

(2) for $\tau_i > k_i$ there must be some $A_j = q_i^d$ for every integer d, where $k_i < d \leqq \tau_i$.

(3) For each l_i there must be some $A_j = l_i^d$ for each integer d, where $1 \leqq d \leqq \sigma_i$.

That the conditions are necessary is shown as follows:

1. If condition (1) is not satisfied, i.e., if there is no $A_j = q_i^d$, then there is no divisor of n for which p belongs to q_i^d.

2. If (2) is not satisfied then there is no n' corresponding to $N' = q_i^d$; $\tau_i \geqq d > k_i$.

3. If (3) is not fulfilled then there is no n' corresponding to $N' = l_i^d$ for some d, where $1 \leqq d \leqq \sigma_i$.

That the conditions are sufficient is shown as follows:

Let

$$N' = \prod_{i=1}^{r} q_i^{\gamma_i} \prod_{i=1}^{f} l_j^{\kappa_j}.$$

Consider $q_i^{\gamma_i}$; there will be two cases:

1. $\gamma_i \leqq \alpha_i$. If $A_i = 1$, then n' is divisible by q_i to a suitable power. (See Lemma 2). If $A_i \neq 1$, according to condition (1) there is $A_j = q_i^{\gamma_i}$. Hence n' must be divisible by a suitable power of q_j which is determined by the power of q_j contained in N'. (See Lemma 2).

2. $\gamma_i > \alpha_i$. Since $q_i^{\gamma_i}$ is a divisor of N, it must be a divisor of some A_k. By condition (2) there is an $A_j = q_i^{\gamma_i}$, and n' is then divisible by a suitable power of q_j. Consider $l_i^{k_i}$, $1 \leqq i \leqq f$. Since $l_i^{k_i}$ is a divisor of N, it is a divisor of some A_k. By condition (3) there is some $A_j = l_i^{k_i}$. Thus n' can be determined divisible by q_j. Consequently if the conditions are satisfied, when N' is given, the value of n' can be found by the method outlined above.

3707 [1934, 581]. *Proposed by Bernard Friedman, Brooklyn, New York.*

If p is a prime of the form $3n+1$, it can be expressed as

$$p = A^2 + 27B^2,$$

where A and B are positive integers, if and only if 2 is a cubic residue of p.

I. *Solution by the Proposer.*

By the cubic reciprocity law (see Bachmann, *Die Lehre von der Kreisteilung*, 14te Vorlesung)

$$\left[\frac{\omega}{\omega_1}\right] = \left[\frac{\omega_1}{\omega}\right], \tag{1}$$

where ω and ω_1 are primary complex primes in the realm $\sqrt{-3}$ and the expression on the left in (1) denotes the cubic character of ω with respect to ω_1. Also

$$\left[\frac{\alpha + \beta\rho}{a + b\rho}\right]^2 = \left[\frac{\alpha + \beta\rho^2}{a + b\rho^2}\right], \tag{2}$$

where ρ is a primitive cube root of unity and α, β, a, b are real positive integers. A complex prime $\omega = a + b\rho$ is primary if

$$a \equiv -1, \qquad b \equiv 0 \pmod 3). \tag{3}$$

If the norm of ω is p, a real prime, then

$$p = (a + b\rho)(a + b\rho^2) = a^2 - ab + b^2. \tag{4}$$

If we set $\omega_1 = 2$, then from (1) we see that 2 is a cubic residue of ω if and only if ω is a cubic residue of 2. But as is easily seen unity is the only cubic residue of 2.

Hence we have in turn

$$a + b\rho \equiv 1 \pmod 2, \qquad a \equiv 1 \pmod 2, \qquad b \equiv 0 \pmod 2, \tag{5}$$

$$\left[\frac{2}{a + b\rho^2}\right] = \left[\frac{2}{a + b\rho}\right]^2 = 1 = \left[\frac{2}{p}\right],$$

and 2 is a cubic residue of p. Set $b=6c$; then from (3), (5), and (4) we have

$$p = a^2 - 6ac + 36c^2 = (a - 3c)^2 + 27c^2 = A^2 + 27B^2.$$

The converse can be proved by reversing the steps.

II. *Solution by Marshall Hall, Institute for Advanced Study.*

Following Gauss, we may find another solution of this problem not depending upon the general law of cubic reciprocity, but intimately bound up with the nature of the pattern of cubic residues modulo p. Let $p=6k+1$, and let us divide the residue classes $1, 2, \cdots, p-1$ into three sets; A, the cubic residues modulo p, and two others $B\equiv\beta A$ and $C\equiv\beta^2 A$, where β is not in A. These sets exhaust the complete set of residues prime to p, and there are $(p-1)/3=2k$ members of each. Let α denote a typical member of A, β of B, and γ of C. Let us examine the arrangement of the α's, β's, and γ's in the sequence $1, 2, \cdots, p-1$. Suppose that there are a of the α's, each followed by an α, that is a solutions of

$$\alpha' \equiv \alpha + 1(p). \tag{1}$$

Considering in a similar manner the other possible sequences, we have the following equations with the number of solutions indicated

$$\begin{array}{lll} \alpha' \equiv \alpha + 1,\ a; & \alpha \equiv \beta + 1,\ d; & \alpha \equiv \gamma + 1,\ g; \\ \beta \equiv \alpha + 1,\ b; & \beta' \equiv \beta + 1,\ e; & \beta \equiv \gamma + 1,\ h; \\ \gamma \equiv \alpha + 1,\ c; & \gamma \equiv \beta + 1,\ f; & \gamma' \equiv \gamma + 1,\ i. \end{array} \tag{2}$$

Since every residue except $p-1$, which is an α, has a successor, we may immediately write down the following equations:

$$a + b + c = 2k - 1, \qquad d + e + f = 2k, \qquad g + h + i = 2k. \tag{3}$$

Moreover, since -1 is a cubic residue, we have $-\alpha=\alpha'$, $-\beta=\beta'$, $-\gamma=\gamma'$, and so b is the number of solutions of

$$\alpha + \beta + 1 \equiv 0,$$

and also d is. Hence $b=d$. And similarly $g=c$, $h=f$.

Now c is the number of solutions of

$$1 + \alpha + \gamma \equiv 0, \quad \text{or of} \quad \gamma^{-1} + \gamma^{-1}\alpha + 1 \equiv 0.$$

But γ^{-1} is a β and $\gamma^{-1}\alpha$ is another, say β'. Hence c is the number of solutions of

$$1 + \beta + \beta' \equiv 0.$$

But e is the number of solutions, and so $e=c$. Similarly $i=b$. We may now make a table

	A	B	C
A	a	b	c
B	b	c	f
C	c	f	b

showing the numbers of the different types of adjacencies in the sequence $1, 2, \cdots, p-1$. Equations (3) show that

$$c = 2k - 1 - a - b, \qquad f = a + 1,$$

giving all numbers in terms of a and b.

Let us now find the number of solutions of

$$1 + \alpha + \beta + \gamma \equiv 0. \tag{4}$$

As α runs over the entire set A we have from (2)

$$\begin{aligned}(1 + \alpha) + \beta + \gamma &= \alpha' + \beta + \gamma, && a \text{ times},\\ &= \beta' + \beta + \gamma, && b \text{ times},\\ &= \gamma' + \beta + \gamma, && c \text{ times}.\end{aligned}$$

For α' fixed we may write the equations, equivalent in turn,

$$\alpha' + \beta + \gamma \equiv 0, \qquad 1 + \beta\alpha'^{-1} + \gamma\alpha'^{-1} \equiv 0, \qquad 1 + \beta' + \gamma' \equiv 0,$$

and from the last we see that there are f solutions. In a similar manner we have

$$\begin{aligned}&\beta' \text{ fixed } \beta' + \beta + \gamma \equiv 0, \text{ or } 1 + \alpha + \beta \equiv 0, \ b \text{ solutions},\\ &\gamma' \text{ fixed } \gamma' + \beta + \gamma \equiv 0, \text{ or } 1 + \gamma + \alpha \equiv 0, \ c \text{ solutions}.\end{aligned}$$

Thus (4) has $af+b^2+c^2$ solutions. On the other hand, writing (4) as

$$(1 + \beta) + \alpha + \gamma \equiv 0,$$

we find that it has $bc+cf+bf$ solutions. Hence we have

$$af + b^2 + c^2 = bc + cf + bf. \tag{5}$$

Using the expressions for f and c, and $p=6k+1$, we may transform (5) to

$$4p = (6k - 9a - 7)^2 + 27(2k - a - 2b - 1)^2. \tag{6}$$

Now in any representation of a prime $p=6k+1$ in the form

$$4p = u^2 + 27v^2, \tag{7}$$

u and v are both even or both odd, and certainly $u\not\equiv 0 \pmod 3$. If u and v are both even

$$p = \left(\frac{u}{2}\right)^2 + 3\left(\frac{3v}{2}\right)^2 \tag{8}$$

is the unique representation of p in the form x^2+3y^2, where $y\equiv 0 \pmod 3$; while if both u and v are odd, choose their signs so that $(u-3v)/4$ is integral, and then

$$p = \left(\frac{u+9v}{4}\right)^2 + 3\left(\frac{u-3v}{4}\right)^2 \tag{9}$$

is the unique representation of p in the form x^2+3y^2, where $y\not\equiv 0 \pmod 3$. Hence p is representable in the form

$$p = A^2 + 27B^2 = A^2 + 3(3B)^2, \tag{10}$$

if and only if u, v of (7) are both even, or what is the same thing, if and only if a of (6) is odd. But on the other hand, a is odd if and only if 2 is a cubic residue of p. For a is the number of pairs of adjacent cubic residues in $1, 2, \cdots, p-1$, and if x, $x+1$ are both cubic residues so are $p-x-1$ and $p-x$. Hence we may count them by twos unless

$$(x, x+1) = (p-x-1, p-x) = \left(\frac{p-1}{2}, \frac{p+1}{2}\right)$$

and a is odd if and only if $(p-1)/2$, $(p+1)/2$ are cubic residues. But since

$$2\left(\frac{p\pm 1}{2}\right) \equiv \pm 1(p),$$

these are cubic residues if and only if 2 is a cubic residue of p.

It is interesting to note that quantitatively (6) tells us that every one of the numbers $a, b, \cdots, i$ differs from $p/9$ by an amount less than $\sqrt{p}$, actually less than $[(\sqrt{3}+1)\sqrt{p}+2]/9$.

3746. [1935, 454]. *Proposed by Paul Erdös, The University, Manchester, England.*

Given a triangle ABC, with the sides $a>b>c$, and any point O in its interior. Let AO, BO, CO cut the opposite sides in P, Q, R. Prove that

$$OP + OQ + OR < a.$$

Solution by the Proposer.

It is evident that a is greater than each of the segments AP, BQ, CR. Draw $OX\|AB$, $OY\|AC$, $XK\|OR$, $YL\|OQ$, where X and Y are on BC, K is on AB, L is on CA. From the pairs of similar triangles

$$OXY, ABC; \qquad BXK, BCR; \qquad CYL, CBQ,$$

we have

$$XY > OP; \qquad BX > XK = OR; \qquad YC > YL = OQ.$$

Hence

$$a = BX + XY + YC > OR + OP + OQ,$$

which is the desired inequality,

3747 [1935, 454]. *Proposed by Frank Irwin, University of California.*

Find the single condition that all the roots of the secular equation

$$\begin{vmatrix} a_{11}-x & a_{12} & \cdots a_{1n} \\ a_{21} & a_{22}-x & \cdots a_{2n} \\ \cdot & \cdot & \cdots \cdot \\ \cdot & \cdot & \cdots \cdot \\ \cdot & \cdot & \cdots \cdot \\ a_{n1} & a_{n2} & \cdots a_{nn}-x \end{vmatrix} = 0$$

should be equal, the a's being real and $a_{ji}=a_{ij}$; and hence determine the cases in which all the roots are equal.

I. *Solution by W. E. Roth, University of Wisconsin, Extension Division.*

If the given equation is written $|A-xI|=0$, where A is symmetric and has a unique characteristic value, a, then a is real and the traces of the matrices $B=A-aI, B^2, \cdots, B^n$ are all zero; for the characteristic equation satisfied by B is $B^n=0$. If we let $B=(b_{ij})$, $(i,j=1, 2, \cdots, n)$, where $b_{ij}=b_{ji}=a_{ij}$, if $i\neq j$ and $b_{ii}=a_{ii}-a$, then the trace of B^2 is

$$\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}^2 = 0.$$

Since b_{ij}, $(i, j=1, 2, \cdots, n)$, are all real, this equation can be satisfied if and only if $B=0$. That is $A=aI$ is the only real symmetric matrix having all characteristic roots equal to a.

II. *Solution by the Proposer.*

All the roots of the secular equation are real; but for any such equation, $x^n+a_1x^{n-1}+\cdots+a_n=0$, with roots $\alpha_1, \alpha_2, \cdots, \alpha_n$, the condition for all the roots to be equal is $\sum(\alpha_i-\alpha_j)^2=0$, or $(n-1)a_1^2-2na_2=0$. Applying this to the secular equation, we get as our condition

$$(n-1)\left(\sum a_{ii}\right)^2 - 2n\sum_{i<j}(a_{ii}a_{jj}-a_{ij}^2) = 0,$$

$$(n-1)\sum a_{ii}^2 - 2\sum_{i<j} a_{ii}a_{jj} + 2n\sum_{i<j} a_{ij}^2 = 0,$$

$$\sum (a_{ii}-a_{jj})^2 + 2n\sum_{i<j} a_{ij}^2 = 0.$$

Hence all the roots will be equal if and only if $a_{ij}=0$, $i\neq j$, and $a_{ii}=a_{jj}$.

Editorial Note. A solution may be obtained also from the following theorem:

If A is a symmetric square matrix of n^2 real elements with characteristic roots $c_1, c_2, \cdots, c_n$, there exists a real orthogonal matrix S with determinant of unit value such that the transform of A by S is a matrix all of whose elements are zero except those in the principal diagonal and they are the characteristic roots.

For the problem we have then

$$S^{-1}AS = aI, \qquad \text{or} \qquad A = aSIS^{-1} = aI.$$

Hence A must have all its elements zero except those in the principal diagonal and they are each a. This means that if we set the corresponding quadratic form equal to ar^2, $a \neq 0$, then

$$\sum a_{ij}x_ix_j \equiv a(x_1^2 + x_2^2 + \cdots + x_n^2) = ar^2,$$

and we have the equation of a sphere of radius r in n dimensional space, where the x's are rectangular co¨rdinates of a point. If $a=0$, the quadratic form is identically zero. The equivalent of the theorem may be found in Bôcher's *Higher Algebra*, p. 173, Exs. 1, 2, 3.

Solution II may be altered slightly to make it independent of the theorem as to the reality of the roots by omitting the use of the first identity. Since the characteristic equation under the given conditions is $(x-a)^n=0$, we have

$$-na = a_1, \qquad n(n-1)a^2 = 2a_2, \qquad (n-1)a_1^2 - 2na_2 = 0.$$

The proof then follows from this point as in the solution showing a necessary condition; the condition is then easily seen to be sufficient.

3752 [1935, 515]. *Proposed by V. Thébault, Le Mans, France.*

Let S_1, S_2, S_3, S_4 be the centers of four spheres; let O be the circumcenter of the tetrahedron whose vertices are the four centers; and let ω_1, ω_2 be, respectively, the radical centers of the given spheres and of their orthoptic spheres. Show that the points O, ω_1, ω_2 are collinear, and that $O\omega_2/O\omega_1=3$. Generalize. (Correction by the proposer.)

I. *Solution by N. A. Court, University of Oklahoma.*

A. If from the point L three mutually orthogonal planes may be drawn tangent to the given sphere (M), center M, the points of contact U, V, W form an equilateral triangle, the tangent planes cutting the plane UVW along the tangential triangle $U'V'W'$ of UVW, the line ML is perpendicular to the plane UVW and passes through the circumcenter H of the triangle UVW.

From the equilateral triangle $U'V'W'$ we have

$$UH:HU' = 1:2,$$

and from the similar right triangles LHU', MHU we have

$$MH:HL = HU:HU' = 1:2;$$

hence, considering the right triangle MLU,

$$MU^2 = ML\cdot MH, \qquad \text{or} \qquad 3MU^2 = ML^2.$$

Thus the locus of the point L is a sphere (L), the *orthoptic sphere* of the given sphere (M), the ratio of the squares of the radii of the spheres (L), (M) being equal to 3.

B. Let (A), (B), (C), (D); (A'), (B'), (C'), (D') be two groups of respectively concentric spheres; $a, b, c, d; a', \cdots$ the squares of their respective radii; (R), (R') their orthogonal spheres, R, R' their radical centers. If we suppose that

$$a':a = b':b = \cdots = k,$$

where k is a given constant, the three spheres (R), (R'), $(O)=(S_1S_2S_3S_4)$ are coaxal,* and if p is the distance of their radical plane from the common center of the two spheres (A), (A'), and O is the center of the sphere (O), we have, both in magnitude and in sign (ibid, p. 184, art. 581),

$$a = 2p\cdot OR, \qquad a' = 2p\cdot OR',$$

hence

$$OR':OR = a':a = k,$$

both in magnitude and in sign.

If the spheres (A'), $\cdots$ are the orthoptic spheres of the four given spheres (A), $\cdots$, we have $k=3$, hence

$$OR'/OR = 3.$$

Remarks. i. If we consider three circles in the plane and their three orthoptic circles, it may be shown in a manner similar to the above that we have

$$OR'/OR = 2.$$

ii. The formulas for two and three dimensions seem to suggest the formula $OR'/OR=n$ for n-dimensional space. Some interested reader may find it worth while to verify this formula.

II. *Solution by the Proposer.*

This problem is a particular case of a general theorem which we have given (*Mathesis*, 1932, Supplement) and which may be stated thus: Consider a sphere (ω), with center ω, for which the powers of the vertices S_1, S_2, S_3, S_4 of a tetrahedron have the form kl^2, km^2, kn^2, kp^2, where k is arbitrary and l^2, m^2, n^2, p^2 are given. The locus of the center ω, when k varies, is the straight line Δ passing through O, the circumcenter of the tetrahedron and perpendicular to the plane with barycentric coördinates l^2, m^2, n^2, p^2 with respect to the tetrahedron. To two values k_1, k_2 of the parameter k correspond two spheres (ω_1), (ω_2), and we have the following relation between their centers

$$O\omega_2/O\omega_1 = k_2/k_1.$$

Thus, if R_i is the radius of the sphere (S_i), its orthoptic sphere has the radius $\sqrt{3}R_i$. We then have

$$(S_i\omega_1)^2 - \rho_1^2 = R_i^2, \qquad (S_i\omega_2)^2 - \rho_2^2 = 3R_i^2, \qquad i = 1, 2, 3, 4,$$

* N. A. Court, Modern pure solid geometry, pp. 205–206, art. 641, The Macmillan Company, 1935. Or this MONTHLY, vol. 40, 1933, p. 181.

where ρ_1 and ρ_2 are the radii of the spheres orthogonal, respectively, to the four spheres (S_i) and to their orthoptic spheres. Hence

$$O\omega_2/O\omega_1 = 3.$$

More generally, if we consider the isoptic spheres of (S_i) with the radii λR_i, and denote by $\boldsymbol{\omega}_n$ the center of the sphere with radius ρ_n orthogonal to the isoptic system, we have in the same way

$$O\omega_n/O\omega_1 = \lambda^2.$$

Editorial Note. With the aid of vectors these theorems are easily proved for euclidean space of n dimensions. It is desirable, however, to state first some preliminary theorems which are needed here and also for certain problems which will appear later. In order for $n+1$ points in euclidean space of n dimensions to give a figure, the simplex, which may be regarded as a generalization of the tetrahedron, the points must be so located that, if any one is chosen as origin, the vectors to the remaining n points must be linearly independent. Let the vectors from the chosen vertex be denoted by $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n$, and form the symmetric determinant of the nth order having $\mathbf{b}_i\cdot\mathbf{b}_j$ as the element in the ith row and jth column

$$V^2 = |\,\mathbf{b}_i\cdot\mathbf{b}_j\,|. \tag{1}$$

This determinant is the square of V the n dimensional content of the parallelepiped having $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n$ as n of its edges. A necessary and sufficient condition that these vectors be linearly independent is that the determinant (1) be greater than zero. We assume that this is true, and in this case there is an associated system of n vectors which may be defined as follows: Replace the ith row in the determinant (1) by $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n$ and set this determinant equal to $V^2\mathbf{b}_i'$, $i=1, 2, \cdots, n$. We then have for the new system of associated vectors

$$\begin{aligned} \mathbf{b}_i'\cdot\mathbf{b}_j &= 0, \quad \text{if} \quad i\neq j,\\ &= 1, \quad \text{if} \quad i = j. \end{aligned} \tag{2}$$

It easily follows that

$$|\,\mathbf{b}_i'\cdot\mathbf{b}_j'\,| = V^{-2}, \tag{3}$$

and that the cofactor of $\mathbf{b}_i\cdot\mathbf{b}_j$ in (1) is $V^2\mathbf{b}_i'\cdot\mathbf{b}_j'$, while the cofactor of $\mathbf{b}_i'\cdot\mathbf{b}_j'$ in (3) is $V^{-2}\mathbf{b}_i\cdot\mathbf{b}_j$. The associated system is therefore linearly independent; and, if $\mathbf{x}$ is any vector in the given space, there is one and only one way of expressing $\mathbf{x}$ linearly in terms of the associated vectors

$$\mathbf{x} = \sum_{i=1}^{n}(\mathbf{x}\cdot\mathbf{b}_i)\mathbf{b}_i'. \tag{4}$$

We shall now show that there is one and only one point C in the given space which is at the same distance from the $n+1$ vertices of the simplex. There are

$(n+1)n/2$ planes which are perpendicular bisectors of the edges of the simplex, and the equations of these planes are linearly dependent upon n whose equations are

$$\mathbf{x}\cdot\mathbf{b}_i = \mathbf{b}_i^2/2, \qquad i = 1, 2, \cdots, n. \tag{5}$$

If the system (5) has a solution $\mathbf{c}$, we have by (4) and (5)

$$\mathbf{c} = \frac{1}{2}\sum_{i=1}^{n} \mathbf{b}_i^2\mathbf{b}_i'; \tag{6}$$

it easily follows that (6) does give a solution. This completes the preliminary theorems and we now turn to the theorems of the problem.

Let the vertices B_i of the simplex be the centers of spheres with radii $\sqrt{k}R_0$, $\sqrt{k}R_1, \cdots, \sqrt{k}R_n$, where k is arbitrary and the R_i's are given. There are $(n+1)n/2$ radical planes of pairs of these spheres whose equations are linearly dependent upon the following n equations,

$$2\mathbf{b}_i\cdot\mathbf{x} = \mathbf{b}_i^2 + k(R_0^2 - R_i^2), \qquad i = 1, 2, \cdots, n. \tag{7}$$

These equations have one and only one solution, which is the vector to the radical center of the sphere system for the given k. This solution is

$$\mathbf{x} = \mathbf{c} + \frac{k}{2}\mathbf{d}, \qquad \mathbf{d} = \sum_{i=1}^{n}(R_0^2 - R_i^2)\mathbf{b}_i'. \tag{8}$$

If all the R's are equal, the radical center for any value of k is C; this is obvious without this analysis. Excluding this trivial case we have $\mathbf{d}\neq 0$, and we may then assume that $R_0\neq 0$. As k varies, it is clear that the radical centers move on the straight line through C parallel to the vector $\mathbf{d}$. For $k=2$ the radical center is at D, where CD is the vector $\mathbf{d}$. This vector $\mathbf{d}$ has a special property which we now consider. Suppose that the vector $\mathbf{d}$ is placed so that its initial point is at a vertex, say B_i; then its other end is the point P_i with the vector $\mathbf{b}_i+\mathbf{d}$. The equation of the polar of P_i with respect to the sphere with center B_i and radius R_i is easily found to be

$$\mathbf{x}\cdot\mathbf{d} = R_i^2 + \mathbf{b}_i\cdot\mathbf{d} = R_i^2 + R_0^2 - R_i^2 = R_0^2. \tag{9}$$

Hence these $n+1$ polar planes coincide in the plane (9); and we now show that this plane has an equation in barycentric coördinates with respect to the simplex whose coefficients are respectively the R_i^2's. If $R_i=0$, we see from (8) that (9) is satisfied by $\mathbf{x}=\mathbf{b}_i$, and hence the plane passes through B_i: if $R_iR_j\neq 0$, and if N_{ij} denotes the point of intersection of (9) with the edge B_iB_j, we easily find that

$$(B_iN_{ij})R_j^2 + (N_{ij}B_j)R_i^2 = 0, \tag{10}$$

using for the equation of B_iB_j, $\mathbf{x}=(1-t)\mathbf{b}_i+t\mathbf{b}_j$. If $R_i=R_j\neq 0$, then the plane (9) is parallel to the edge B_iB_j. Now consider a system of homogeneous coördi-

nates $x_0, x_1, \cdots, x_n$ of a point P such that x_i is equal to any constant $k\neq 0$ times the content of the simplex formed by P and all the vertices except B_i. Then P is the centroid of the masses $x_0, x_1, \cdots, x_n$ placed at the respective vertices B_i. We now show that the plane (9) has the equation

$$\sum_{i=0}^{n} R_i^2 x_i = 0 \tag{11}$$

in this system of coördinates. The intersection of the plane (11) with B_iB_j is obtained by setting all the x's in (11) equal to zero except x_i and x_j; and we obtain for this intersection

$$R_i^2 x_i + R_j^2 x_j = 0.$$

Hence the intersection divides B_iB_j in the ratio $-R_i^2/R_j^2$, and it must be the point N_{ij}. The special cases noted above are easily handled, and it results that the two planes are the same. The coefficients R_i^2 in (11) are respectively the distances of the plane from the vertices B_i multiplied by the length of $\mathbf{d}$, as is clear from the polar property above; these coefficients may then be regarded as the coördinates of the plane in the barycentric system of coördinates. These coordinates give simple proofs of a generalization of the theorems of Ceva and Menelaus. We may consider the plane (11) as a polar plane in another manner. Suppose that none of the R_i's are zero, and consider the degenerate surface

$$x_0 x_1 \cdots x_n = 0.$$

The polar plane of the point $R_0^{-2}, R_1^{-2}, \cdots, R_n^{-2}$ with respect to this surface is the plane (11).

We consider now the orthoptic and isoptic spheres of the given spheres of radii R_i. If R is the radius of a sphere with its center at the origin O, and if from a point P there are n mutually orthogonal planes tangent to the sphere, then obviously OP is the diagonal of a rectangular parallelepiped all of whose edges are R. Hence the locus of P is a concentric sphere with the radius $\sqrt{n}R$. If through P there are n planes tangent to the sphere such that the radii to the points of contact make with each other an angle whose cosine is $c > -1/(n-1)$, then

$$OP = R\sqrt{\frac{n}{1+(n-1)c}},$$

and the locus of the point P is a concentric sphere with the radius OP. For $c=0$, we have the orthoptic sphere. Thus for k in (8) we may take the quantity under the radical, and the theorems of the problem for the orthoptic and isoptic spheres easily follow.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items of interest to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

The Missouri Section of the Mathematical Association of America, the Mathematics and Astronomy Sections of the Missouri Academy of Science, and the Astronomy Section of the Academy of Science of St. Louis met jointly at Washington University, St. Louis, on April 23–24, 1937. The following papers of a mathematical nature were presented: 1. "Sets of conjugate operators in the groups of order 32" by Professor D. T. Sigley, University of Kansas City. 2. "Certain diophantine equations of degree two" by A. E. Ross, St. Louis University. 3. "Theory and construction of sun-dials" by Professor W. H. Roever, Washington University. 4. "Note on a theorem characterizing geodesic arcs in complete, convex metric spaces" by Professor L. M. Blumenthal, University of Missouri. 5. "Poristic systems of doubly quadratic equations" by H. S. Murray, Fulton, Mo. 6. "Integration of linear differential equations in series by the operational method" by Professor Eugene Stephens, Washington University. 7. "On certain qualitative properties of the solutions of second order linear differential equations" by Professor Gabriel Szegö, Washington University. 8. "The orbit of the visual binary star, Omicron Sigma 298" by Professor Jessica Y. Stephens, Washington University. 9. "Cauchy's method of forming normal equations from a set of linear observation equations" by Professor H. R. Grummann, Washington University. 10. "Quartic surfaces invariant under the symmetric group, G_{24}" by Professor H. E. Crull, Park College. Professor G. E. Wahlin of the University of Missouri was elected chairman of the Mathematics Section of the Missouri Academy of Science for the next year.

The American Documentation Institute has been incorporated on behalf of leading national scholarly, scientific and informational societies to develop and operate facilities that are expected to promote research and knowledge in various intellectual fields. A first objective of the new organization will be to develop and apply the new technique of microphotography to library, scholarly, scientific and other material. It will be able to conduct scholarly publication by various methods as required by cooperating organizations. Organized as a Delaware corporation "not for profit" but for educational, literary and scientific purposes, the new organization resulted from a meeting attended by delegates from national councils, societies, and other organizations in Washington on March 13. The board of trustees elected consists of: Dr. Robert C. Binkley, Western Reserve University; Dr. Solon J. Buck, Director of Publications, National Archives; Watson Davis, Director, Science Service; Dr. James Thayer Gerould, Librarian, Princeton University Library; Dr. Ludvig Hektoen, Chairman, National Research Council.

At the meeting of the American Association for the Advancement of Science in Denver, the following papers in mathematics were presented on June 24: Incidence of taxation in a simplified general equilibrium, by R. W. Shephard; On mechanical quadratures, by Professor J. A. Shohat; Stability of regimes of coöperation and competition, by Professor G. C. Evans and Kenneth May; A certain index number as a mean $f(x_1, x_2, \cdots, x_n)$ with $f(c, c, \cdots, c)$ defined only when $c=1$, by Professor E. L. Dodd; Applications of index numbers to the general economic equilibrium, by F. W. Dresch.

Professor A. J. Lewis of the University of Denver served as representative of the Mathematical Association at the semi-centennial celebration of the University of Wyoming on June 5–7, 1937.

Miss Mabel M. Heren, in honor of thirty years' service as a member of the department of mathematics at Knox College, has been appointed to the Henry M. Hitchcock chair of mathematics.

Assistant Professor Marguerite Lehr of Bryn Mawr College has been promoted to an associate professorship.

Assistant Professor Rufus Oldenburger of Armour Institute has been granted a leave of absence to spend the next academic year at the Institute for Advanced Study.

G. W. Petrie III of the South Dakota State School of Mines has been promoted to an assistant professorship.

Dr. H. Y. Benedict, President of the University of Texas, died May 10, 1937, at the age of sixty-seven. He went to the University of Texas as instructor in mathematics in 1899, became dean of the College of Arts and Sciences in 1911, and president of the university in 1927. He was a charter member of the Mathematical Association.

H. O. Hanson, of the Mutual Life Insurance Company of New York, died at his home in West Brattleboro, Vermont, May 3, 1937. He was a charter member of the Mathematical Association.

The following eighty-four doctorates with mathematics as major subject were conferred during 1936 in universities in the United States. The university, month in which the degree was conferred, minor subject (other than mathematics), and title of dissertation are given in each case if available.

Mae R. Anderson, University of Chicago, August, *Representation as a sum of multiples of polygonal numbers.*

H. C. Ayres, California (Berkeley), December, *Existence and embedding theorems for a hyperbolic system of partial differential equations.*

A. H. Bailey, Ohio State, June, *An approach to the study of conic sections, based on a group of projective transformations.*

John Bardeen, Princeton, January, *Quantum theory of the work function.*

R. H. Bardell, University of Chicago, August, *The inequalities of Morse for a parametric problem of the calculus of variations.*

P. O. Bell, California (Berkeley), August, *Certain covariant configurations associated with a general curved surface.*

J. W. Blincoe, Virginia, June, *The reduction of plane quartic curves to canonical forms by means of their euclidean concomitants.*

L. H. Bowen, Cornell, June, *Composite double curves on rational ruled surfaces.*

Joel Brenner, Harvard, February, *The linear homogeneous group modulo p.*

J. R. Britton, University of Colorado, *Tchebychef orthogonal polynomials in a single real variable.*

J. C. Brixey, University of Chicago, June, *The null forms $Ax^2+By^2+Cz^2+Du^2$ which represent all integers.*

A. C. Burdette, University of Illinois, June, *On simultaneous expansions of analytic functions in composite power series.*

F. A. Butter, Jr., Stanford, June, minor in physics, *A contribution to the theory of the arithmetic-geometric mean.*

J. F. Calvert, Pittsburgh, June, minor in electrical engineering, *Insulation problems in high voltage rotating machines.*

E. A. Cameron, North Carolina, September, *On loci associated with certain osculants of a plane curve.*

J. E. Case, S. J., University of Chicago, March, *The behavior of the Hessian at a multiple point of a curve.*

George Comenetz, Columbia, January, *Curvature trajectories.*

E. G. H. Comfort, Brown, June, *On the preservation of Hölder properties of initial conditions in the solution of wave equations.*

D. C. Dearborn, Duke, June, *Inequalities among the invariants of Pfaffian systems.*

D. B. De Lury, Toronto, June, *On the representation of numbers by certain indefinite quadratic forms.*

D. M. Dribin, University of Chicago, March, *Representation of binary forms by sets of ternary forms.*

Nelson Dunford, Brown, June, *Part I. Integration in general analysis. Part II. On a theorem of Plessner. Part III. A particular sequence of step functions.*

P. S. Dwyer, University of Michigan, June, *Combined expansions of products of symmetric power sums and of sums of symmetric power products with application to sampling.*

A. D. Fialkow, Columbia, June, *Trajectories and lines of force.*

Sidney Frankel, Rensselaer Polytechnic Institute, June, *On the expansion of functions in series of functions.*

Bernard Friedman, Massachusetts Institute of Technology, May, minor in physics, *Analyticity of equilibrium figures of rotation.*

J. W. Givens, Jr., Princeton, June, *Tensor coördinates of linear spaces.*

Andre Gleyzal, Ohio State, June, *On transfinite real numbers, general orders, Riemannian and Finsler spaces.*

H. H. Goldstine, University of Chicago, August, *Conditions for a minimum of a functional.*

J. W. Hahn, Rice, June, *Projective transformations in two complex variables.*

Marshall Hall, Yale, June, *Isomorphism between linear recurring sequences and algebraic rings.*

Israel Halperin, Princeton, April, *Adjoints and closures of linear differential operators.*

H. J. Hamilton, Brown, June, *Part I. On transformations of double series. Part II. Transformations of multiple sequences.*

R. A. Harrison, Cornell, June, *Cremona webs in S_3 without base curves.*

O. G. Harrold, Jr., Stanford, June, minor in physics, *On the expansion of the remainder in the open-type Newton-Cotes quadrature formula.*

R. A. Higdon, Iowa State College, minor in physics, *Stresses in moderately thick rectangular plates.*

I. E. Highberg, California Institute of Technology, June, minor in physics. *Polynomials in abstract spaces.*

J. D. Hill, Brown, October, *Part I. Some theorems on double limits. Part II. A theorem in the theory of summability. Part III. On perfect methods of summability.*

C. C. Hurd, University of Illinois, June, *Properties of solutions of linear differential equations containing a parameter.*

L. P. Hutchison, University of Kentucky, August, *On implicit function and Lagrange multiplier theorems.*

V. P. Jensen, Iowa State College, minor in theoretical physics, *The application of conformal transformation theory to the determination of stress problems.*

M. L. Kales, Brown, October, *Tauberian theorems related to Borel and Abel summability.*

Morris Kline, New York, June, minor in physics, *Homomorphism and isomorphism of rings and fields of point sets.*

Sister Mary Thomas à Kempis Kloyda, University of Michigan, June, *Linear and quadratic equations, 1550–1660.*

G. B. Lang, University of Illinois, February, *On finite systems of linear differential equations of infinite order with constant coefficients.*

H. D. Larsen, Wisconsin, June, *On the bias in the simple arithmetical index number.*

Madeline Levin, Bryn Mawr, June, *An extension of the Lefschetz intersection theory.*

Harry Matison, Princeton, June, *On certain classes of integral functions.*

A. E. May, Wisconsin, June, *On the equivalence of pairs of Hermitian matrices in $R(\sqrt{k})$.*

Dora McFarland, University of Chicago, August, *Division algebras defined by non-Abelian groups.*

J. C. C. McKinsey, California (Berkeley), May, *On Boolean functions of many variables.*

L. E. Mehlenbacher, University of Michigan, June, *The interrelations of the fundamental solutions of the hypergeometric equation.*

L. L. Merrill, Rensselaer Polytechnic Institute, June, minor in physics, *The direct determination of a function knowing the infinite sum of its successive integrals.*

W. A. Mersman, California Institute of Technology, June, minor in physics, *Abstract integration.*

A. H. Odoms, Cincinnati, June, *On the summability of double Fourier series.*

W. A. Patterson, Ohio State, June, *Inverse problems of the calculus of variations for multiple integrals.*

E. L. Peterson, Purdue, June, major in mathematical physics, minors in mathematics, physics, and chemistry, *A wave mechanical computation of the resistivity of metallic sodium.*

Walter Prenowitz, Columbia, June, *Characterization of plane collineations in terms of homologous families of lines.*

E. S. Quade, Brown, June, *Part I. The category of the class Lip* (α, p). *Part II. A generalized Parseval's relation. Part III. A note on Lipschitz classes. Part IV. Trigonometric approximation in the mean for functions in the class Lip* (α, p).

Ruth B. Rasmusen, University of Chicago, December, *Conjugate osculating quadrics associated with the lines of curvature.*

Sister Mary Henrietta Reilly, Catholic University, minor in physics, *Self-symmetric quadrilaterals in-and-circumscribed to the plane rational quartic curve with a line of symmetry.*

Moses Richardson, Columbia, October, *On the homology characters of symmetric products.*

J. S. Rosen, Washington University, minor in physics, *Some generalizations of Bessel functions.*

Benjamin Rosenbaum, Yale, June, *On divisibility and irreducibility.*

Arthur Sard, Harvard, February, *The measure of the critical values of functions.*

A. C. Schaeffer, Massachusetts Institute of Technology, May, minor in physics, *Existence theorem for the flow of an ideal incompressible fluid in two dimensions.*

Brother Louis de La Salle Seiller, Catholic University, minor in physics, *Investigation on the basis numbers and class number of higher algebraic domains.*

W. E. Sewell, Harvard, February, *Generalized derivatives and approximation by polynomials.*

M. E. Shanks, University of Iowa, August, minor in physics, *Properties of analytic functions on regions bounded by irregular curves.*

O. T. Snodgrass, University of Missouri, August, *A multiplicative representation of the elements of a ring.*

T. A. Southard, Ohio State, September, *On certain projective geometries and their relation to algebra.*

Vivian Spencer, University of Pennsylvania, June, *Persymmetric determinant and Jacobi matrix expressions for orthogonal Tchebycheff polynomials.*

N. E. Steenrod, Princeton, June, *Universal homology groups.*

Alvin Sugar, California (Berkeley), December, *Researches on Waring's problem for cubic polynomials.*

A. G. Swenson, University of Michigan, June, *Factorial moments.*

A. E. Taylor, California Institute of Technology, June, minor in physics. *Analytic functions in general analysis.*

C. B. Tompkins, University of Michigan, February, *A type of integral invariant associated with a defined class of n-dimensional variety in euclidean* $(2n-1)$*-space.*

S. B. Townes, University of Chicago, August, *Reduced positive quaternary quadratic forms.*

W. S. Turpin, Johns Hopkins, June, *On the fundamental group of a certain class of algebraic curves.*

D. L. Webb, California Institute of Technology, June, minor in physics. *Many-valued logics.*

Evelyn P. Wiggin, University of Chicago, June, *A boundary value problem of the calculus of variations.*

W. L. Williams, University of Chicago, August, *Permanent configurations in the problem of five bodies.*

H. S. Zuckerman, California (Berkeley), May, *New results for the number* $g(n)$ *in Waring's problem.*

The following four doctorates were conferred in 1935, but not included in the list in the preceding volume of the MONTHLY (vol. XLII, pp. 446–450).

R. W. Cowan, California (Berkeley), December, *The solution of the linear homogeneous difference equation of the second order with quadratic coefficients.*

A. R. Noble, California (Berkeley), September, *On the enumeration of uniform squares.*

J. H. D. Teller, Kentucky, August, *On a class of quaternion algebras.*

J. M. Thompson, California (Berkeley), December, *Mathematical theory of production stages in economics.*

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DIRECTORY

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BOOKS FOR REVIEW should be addressed to REVIEW EDITOR, American Mathematical Monthly, 531 West 116th Street, New York, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of addresss to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

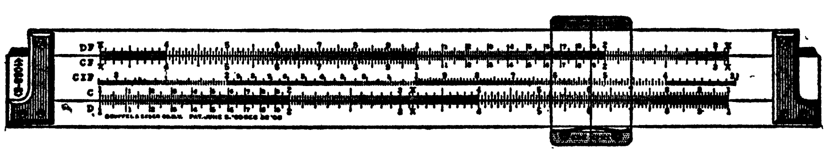
MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-first Summer Meeting, Pennsylvania State College, Sept. 6–7, 1937.
Twenty-second Annual Meeting, Indianapolis, Ind., December 30–31, 1937.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1937 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Waynesburg, Pa., May 1; Pittsburgh, October 23.
ILLINOIS, DeKalb, May 14–15.
INDIANA, Greencastle, April 30–May 1.
IOWA, Dubuque, April 16–17.
KANSAS, Wichita, April 3.
KENTUCKY, Louisville, May 1.
LOUISIANA-MISSISSIPPI, Hammond, La., March 5–6.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Lynchburg, Va., May 8.
MICHIGAN, Ann Arbor, March 20.
MINNESOTA, St. Paul, May 15.
MISSOURI.
NEBRASKA, Lincoln, May 7.
OHIO, Columbus, April 1.
OKLAHOMA, Tulsa, February 5.
PHILADELPHIA, Haverford, Nov. 27.
ROCKY MOUNTAIN, Greeley, Colo., April 16–17.
SOUTHEASTERN, Nashville, Tenn., April 16–17.
SOUTHERN CALIFORNIA, Los Angeles, March 6.
SOUTHWESTERN, State College, N.M., April 2–3.
TEXAS, Houston, April 23–24.
WISCONSIN, Milwaukee, May 8.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS



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DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

EDITED BY
ELTON JAMES MOULTON, Editor-in-Chief
HERBERT ELLSWORTH SLAUGHT†
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THE APRIL MEETING OF THE ROCKY MOUNTAIN SECTION

The twenty-first annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at Colorado State College of Education, Greeley, Colorado, on Friday and Saturday, April 16 and 17, 1937.

There were three sessions. Professor A. E. Mallory presided at each of the Friday sessions and at the business meeting Saturday morning. Professor C. A. Hutchinson presided during the Saturday morning program. Professor W. L. Hart of the University of Minnesota was the guest speaker.

The attendance was sixty-six, including the following twenty-four members of the Association: L. A. Aroian, C. F. Barr, J. R. Britton, J. R. Everett, J. C. Fitterer, G. W. Gorrell, D. F. Gunder, W. L. Hart, I. L. Hebel, C. A. Hutchinson, L. Louise Johnson, A. J. Kempner, Claribel Kendall, A. J. Lewis, S. L. Macdonald, A. E. Mallory, A. S. McMaster, W. K. Nelson, Greta Neubauer, E. D. Rainville, O. H. Rechard, A. W. Recht, C. H. Sisam, and W. M. Stewart.

The following officers were elected for the coming year: Chairman, C. A. Hutchinson, University of Colorado; Vice-Chairman, C. F. Barr, University of Wyoming.

The members and friends of the Association were guests of Colorado State College of Education at a dinner Friday evening. The Saturday morning session was a joint meeting with the National Council of Teachers of Mathematics.

The following thirteen papers were read:

1. "A method of finding a solution of a system of three linear matrix equations in three unknowns" by Professor O. H. Rechard, University of Wyoming.
2. "Mechanical and graphical calendars" by Professor W. K. Nelson, University of Colorado.
3. "Note on an operational formula" by Professor C. A. Hutchinson, University of Colorado.
4. "Concyclic points and some allied configurations" by Professor C. H. Sisam, Colorado College.
5. "Certain distributions with binomial series as components" by Professor A. G. Clark, read by Professor L. A. Aroian, Colorado State College.
6. "Elementary remarks on Moebius's barycentric calculus" by Professor A. J. Kempner, University of Colorado.
7. "Inexact mathematics" by Professor W. L. Hart, University of Minnesota, by invitation of the Section.
8. "Statement of problems regarding high school mathematics—a report" by Professor C. A. Hutchinson, University of Colorado, and H. W. Charlesworth, East High School, Denver.
9. "Can mathematical values be measured?" by Professor O. H. Rechard, University of Wyoming.
10. "Who should study mathematics? What should be studied?" by Professor A. E. Mallory, Colorado State College of Education, and Ethelyn W. Rhiner, Greeley Public Schools.

11. "Some curriculum questions" by Professor C. H. Sisam, Colorado College.

12. "Remarks on the calculation of π" by Marjorie H. Beaty, University of Colorado, introduced by Professor Hutchinson.

13. "The trend in secondary mathematics and associated collegiate phenomena" by Professor W. L. Hart, University of Minnesota.

Abstracts of some of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. In considering the system $A_iXB+C_iYD+E_iZF=C_i(i=1, 2, 3)$ in which all the matrices are square and the constant matrices are non-singular, it is found that a solution in terms of the constant matrices involves the solution of a k-termed linear matrix equation in a single unknown, k, a function of the order of the matrices. Since, so far as Professor Rechard has been able to determine, the value of the unknown in such an equation has been described in terms of the constant matrices in the equation only for the cases $k=1, 2$, it is necessary in order to solve the system under consideration to transform the problem of solving this latter equation to that of solving an ordinary algebraic system of n^2 linear equations in n^2 unknowns. Professor Rechard showed how this could be accomplished and illustrated the method for a system in which the matrices were all of second order.

2. Any mechanical or graphical device for adding numbers might be used in making a perpetual calendar. Two calendars designed by Professor Nelson, one a desk calendar and the other a slide rule calendar of form to carry in a notebook, were demonstrated. The desk calendar may be set to show the year, name of month, and calendar for the month for any month in the twenty-four hundred year period beginning with 1 A.D. The notebook calendar may be set to show the calendar for a year at a time for any year of the same period.

3. Professor Hutchinson's note appears in this MONTHLY, pages 371–2.

4. Professor Sisam gave a simple method of deriving a series of formulas and identities arising from certain configurations of points in the plane and in space.

5. Professor Clark considered the distribution of total successful occurrences of a set of trials of an event, where, although the individual trials have an equal constant probability of successful outcome, such an outcome in an individual trial is counted as of a certain multiplicity preassociated with that trial. It was found that the resulting distribution of total successes could be broken into components which are simple binomial series. Relations were derived for the movements of such a distribution in terms of p, the constant probability of success, and r_j $(j=1, 2, \cdots, s)$, the multipicities associated with the s trials of the event.

7. Professor Hart emphasized the fact that, although mathematics has important uses for formulas and equalities, on the other hand there is a major section of analysis where, in contrast, features such as inequalities and successive approximations are met, and he suggested the importance of this contrast

in overcoming the restricted formal viewpoint of more elementary mathematics.

12. Miss Beaty gave a comparison between the method of isoperimeters and that of equal areas.

13. In this paper Professor Hart discussed proposals about secondary mathematics connected with existing tendencies to socialize the secondary curriculum. He labeled any approach to this problem as illogical if it involves a simultaneous attack on college entrance requirements. Regardless of the pertinence of such an attack, he advanced criticisms of the technique of some of its supporting educational research and suggested that it merely establishes the consistency of established entrance requirements and course prerequisites. Finally, he rested the case for secondary mathematics largely on the existing prerequisites, either tangible or intangible, for college courses and curricula, rather than on inflexible entrance requirements.

A. J. Lewis, *Secretary*

THE APRIL MEETING OF THE SOUTHEASTERN SECTION

The fifteenth annual meeting of the Southeastern Section of the Mathematical Association of America was held in Nashville, Tennessee, on Friday and Saturday, April 16–17, 1937. There were in attendance about one hundred fifty persons from thirty-six institutions, including the following forty-eight members of the Association: H. G. Ayre, D. H. Ballou, W. S. Beckwith, R. V. Blair, Iris Callaway, M. G. Carman, R. D. Carmichael, T. C. Carson, Edna J. Cofield, W. A. Cordrey, H. M. Cox, Forrest Cumming, L. A. Dye, W. W. Elliott, D. C. Harkin, M. A. Hill, Jr., P. R. Hill, P. M. Hummel, W. R. Hutcherson, R. O. Hutchinson, J. A. Hyden, J. B. Jackson, Rosa L. Jackson, H. T. Karnes, G. B. Lang, F. A. Lewis, J. F. Locke, A. N. McPherson, Nellie P. Miser, W. L. Miser, W. A. Moore, J. S. Morrel, Mabel I. Nowlan, W. P. Ott, K. B. Patterson, D. D. Peele, W. W. Rankin, H. A. Robinson, J. A. L. Saunders, W. E. Sewell, A. R. Sloan, F. H. Steen, R. P. Stephens, Ruth W. Stokes, D. L. Webb, W. L. Williams, F. L. Wren, J. T. C. Wright, E. Kathryn Wyant.

Sessions were held the afternoon of the 16th at George Peabody College, and the evening of the 16th and the morning of the 17th at Vanderbilt University. Chairman W. W. Rankin presided, except Friday evening and part of Saturday morning when the Section was divided into subgroups according to the nature of the papers presented. Subgroups were presided over by Vice-Chairman J. B. Jackson, Dean C. M. Sarratt and Professor W. P. Ott. On the evening of the 16th a dinner was held in honor of the visiting speaker, Dean R. D. Carmichael of the University of Illinois. At this time Professor F. L. Wren presided.

At the business session on the 17th the following officers were chosen for 1937–38: Chairman, J. B. Jackson, University of South Carolina; Vice-Chair-

man, J. A. Hyden, Vanderbilt University; Secretary-Treasurer, H. A. Robinson, Agnes Scott College. The next meeting was scheduled for April, 1938, at Georgia School of Technology. A resolution was passed relative to the loss sustained by the Section in the passing of Professor T. R. Eagles, Mr. S. I. Jones and the Reverend J. H. Meyer. A committee was appointed to continue the study on the A.M. degree requirements in mathematics.

The following twenty-eight papers were read. In the absence of the authors, Nos. 4 and 25 were read by Professor Forrest Cumming and the Secretary.

1. "A proof of the fundamental theorem of algebra" by Professor J. S. Morrel, Vanderbilt University.

2. "The freshman course, what shall we do with it?" by Professor M. A. Hill, Jr., University of North Carolina.

3. "Concerning an elementary algebraic application to errors in accounting" by Professor W. R. Hutcherson, Berea College.

4. "Our fifteenth anniversary" by Professor R. P. Stephens, University of Georgia.

5. "Mathematics in the 17th century" by Professor W. W. Rankin, Duke University, retiring chairman.

6. "Examining an examination" by H. M. Cox, secretary to the Examiners, University System of Georgia.

7. "A.M. degree requirements in mathematics" by Professor W. W. Elliott, Duke University.

8. "Discovery of the freedom to inquire" by Dean R. D. Carmichael, University of Illinois.

9. "Convex permanent configuration in the problem of five bodies" by Professor W. L. Williams, University of South Carolina.

10. "A new method of determining the fundamental unit of a real quadratic field" by Dr. P. M. Hummel, University of Alabama.

11. "Sensus algebra" by Professor D. C. Harkin, Alabama Polytechnic Institute.

12. "Abelian and Tauberian theorems" by Dean R. D. Carmichael, University of Illinois.

13. "Number thirteen" by Professor P. R. Hill, University of Georgia.

14. "An involutorial Cremona transformation in S_3 determined by a pencil of quadrics and two null systems" by Professor L. A. Dye, The Citadel.

15. "A straight-edge construction of a plane quartic Cremona transformation" by Professors J. A. L. Saunders and L. A. Dye, The Citadel.

16. "On collineation representation of a class of simple groups" by Professor F. A. Lewis, University of Alabama.

17. "An elementary proof for a criterion of irreducibility of a polynomial" by Dr. D. L. Webb, Georgia School of Technology.

18. "Linear mixed systems of infinite order with constant coefficients" by Professor G. B. Lang, West Georgia College.

19. "Neo-Sylvester contractions and the calculation of partial and multiple coefficients of correlation" by Professor F. L. Wren, Peabody College.

20. "On interpolation" by Professor D. C. Harkin, Alabama Polytechnic Institute.

21. "Oughtred-Leibniz mark of power" by Professor D. C. Harkin.

22. "On the polynomial derivative constant for an ellipse" by Dr. W. E. Sewell, Georgia School of Technology.

23. "The tetrahedron and its Cevian" by E. J. Scott, Vanderbilt University, introduced by the Secretary.

24. "Problem material for freshman mathematics" by Professor K. B. Patterson, Duke University.

25. "An inequality concerning the plane triangle" by Professor D. F. Barrow, University of Georgia.

26. "A geometrical property of the cubic and quartic with double roots" by Professor D. H. Ballou, Georgia School of Technology.

27. "Remarks on an angle trisection" by J. D. Rommel, Vanderbilt University, introduced by the Secretary.

28. "A case of planar motion" by Professor H. A. Robinson, Agnes Scott College.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Dr. Morrel presented a simplified proof, of the Gordan type, of the fundamental theorem.

2. Professor Hill gave advantages and limitations of the freshman course made up of the usual separate units, algebra, trigonometry, and analytics, in comparison with the new survey course. He asserted that far more important than the type of course was the manner of presentation and the ability of the teacher.

3. Professor Hutcherson presented a study of the arithmetic properties of the errors "usually" made in accounting. When an error is a multiple of both 9 and 111, it is quite possible that the decimal has been shifted three places. Other examples where elementary algebra was employed in accounting were given.

4. In 1916 Dean Stephens conceived the idea of organizing the Southeastern Section of the Mathematical Association of America. In 1922 the formal organization was perfected. In his paper he traced the history and achievements of the Section.

5. Professor Rankin discussed the type of mathematics which was developed during the Newtonian period. With Descartes's new method for discovering mathematical truths, mathematics builders of the seventeenth century perhaps discovered more mathematics in the next seventy-five years than had been discovered in the preceding three thousand years. Personal touches of the lives of the men of this period were given.

6. Mr. Cox discussed the technical consideration of an objective examination, and gave the results of a study of a series of examinations upon the experimental curriculum of his University System.

7. In the discussion led by Professor Elliott reasons for strengthening the requirements for the master's degree were given.

8. In this general address, Dean Carmichael gave a brief historical account of the development of the ideal of the freedom of inquiry, indicating how it arose in the first case. He analyzed the Constitutional provisions which guarantee the freedom of inquiry in the United States. His address was centered around the matter of the freedom of the investigator to inquire and to publish his results.

9. Professor Williams determined the necessary and sufficient conditions for permanency of configuration in the problem of five bodies, and proved the existence of solutions for the convex case.

10. Dr. Hummel described a direct method of determining the fundamental unit of a real quadratic field.

11. Professor Harkin illustrated the Diophantus-Lagrange method of solving quadratics. He stated that the method deserved wider attention and use.

12. This paper by Dean Carmichael contained an expository introduction to the problem of Abelian and Tauberian theorems and stated new results in each of these two closely related fields.

13. Professor Hill gave the results of a statistical study in which the number thirteen played a prominent role.

14. The transformations discussed by Professor Dye were of order $4n+5$ and had as fundamental elements two space curves of order 4 and $2n+1$, and $4n+8$ parasitic lines.

15. A straight-edge construction of the images of points and lines in a quadratic correlation was discussed by Professors Saunders and Dye as a preliminary to the construction of the product of two such correlations which was a quartic transformation having three double and three simple fundamental points.

16. Professor Lewis determined collineation representations of degree $\frac{1}{2}(n-1)$ and $\frac{1}{2}(n+1)$ of each group of the class of simple groups.

17. Dr. Webb found a criterion of irreducibility of a polynomial by studying its zeros.

18. Professor Lang applied the Pincherle transformation to certain equations and obtained a solution subject to certain conditions. His results were extended to the corresponding system of equations.

19. Professor Wren applied Neo-Sylvester contractions to the calculation of partial coefficients of regression and correlation. His method simplified the process of obtaining such coefficients and also provided a systematic check on certain other calculations.

20. Professor Harkin's simplified definition and notation for divided differences enabled him to derive certain interpolation formulas by elementary algebra.

21. Professor Harkin stated that the present post-positive power mark is the exception among the usual pre-positive operators. The pre-positive notation used by Oughtred and Leibniz would lead to more uniformity and simpler printing.

22. Dr. Sewell proved that if a polynomial of degree n in z is in modulus not greater than M on an ellipse C whose semi-axes are a and b, then the modulus of the derivative of this polynomial on C is not greater than Mn/b.

23. Mr. Scott gave an analytic proof of the perspective relationship of the tetrahedron to its Cevian with respect to a point. The plane of perspectivity was shown to be the harmonic polar of the point with respect to the tetrahedron.

24. As an effective method of teaching, Professor Patterson discussed the possibilities of getting freshmen to invent, solve, and verify the solution of problems.

25. Professor Barrow solved problem 3740 (1935, p. 396 of this MONTHLY) and established several more general inequalities.

26. Professor Ballou showed that when the cubic or quartic equation has a double root, the other roots of that equation are at real foci of the curve which is the locus of the complex roots of the equation, the constant term being considered as a parameter. (See J. A. Ward, this MONTHLY, vol. 43, p. 529.)

27. Mr. Rommel presented a ruler and compasses construction for an approximate trisection of small angles.

28. Professor Robinson discussed the case of planar motion when one centrode was a circle and the other was to be determined so that a path of a point would be a circle. A mechanical description of the locus was given.

H. A. ROBINSON, *Secretary*

THE ANNUAL MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at the College of Saint Catherine, St. Paul, Minnesota, on Saturday, May 15, 1937. A morning session was held at 10:30 o'clock and was followed by luncheon and an afternoon session at 2:15 o'clock.

Professor A. L. O'Toole of the College of St. Catherine, chairman of the Section, presided at the morning session, and Professor H. P. Thielman of the College of St. Thomas took the chair for the afternoon. Eighty-seven persons attended the meeting including the following thirty-two members of the Association: Mae R. Andersen, C. J. Blackall, R. W. Brink, W. E. Brooke, L. E. Bush, W. H. Bussey, C. S. Carlson, S. Elizabeth Carlson, Sister M. Claudette, R. W. Cowan, Arthur Danzl, J. H. Daoust, Bernard Dimsdale, Margaret C. Eide, W. L. Hart, Clara L. Hancock, J. S. Hickman, Dunham Jackson, W. H. Kirchner, E. N. Oberg, A. L. O'Toole, J. M. Rysgaard, R. B. Saunders, M. G. Scherberg. Ole Schey, E. N. Shawhan, A. J. Strane, F. J. Taylor, H. P. Thielman, Ella Thorp, K. W. Wegner, G. L. Winkelmann.

At the business session officers were elected for the coming year as follows: Chairman, G. L. Winkelmann, St. John's University; Secretary, A. L. Underhill, University of Minnesota; Members of the Executive Committee: L. E. Bush, College of St. Thomas; C. S. Carlson, St. Olaf College; Sister M. Claudette, College of St. Benedict. A resolution was adopted as follows: "Resolved that the Secretary of the Section be instructed to investigate the practicability

of holding two meetings each year, one to be held in the Twin Cities, the other in some other section of the State; (2) that he be further instructed to investigate ways and means for financing the appearance of an invited speaker for at least one of these meetings each year; (3) that he be instructed to investigate the advisability of joining with some organization of high school teachers in at least one meeting; (4) that he be instructed to make recommendations with regard to these matters at the next meeting of the Section." A resolution was adopted expressing the appreciation of the Section for the gracious hospitality of the College of St. Catherine and its staff in connection with the excellent arrangements for the meeting.

The following twelve papers were presented:

1. "A few supplementary exercises for mathematical induction" by Professor C. S. Carlson, St. Olaf College.

2. "Determination of the segments in the theorem of Menelaus" by Professor F. J. Taylor, College of St. Thomas.

3. "Applications of the Heine-Borel theorem" by Dr. M. G. Scherberg, University of Minnesota.

4. "Uniqueness of the weight function of a system of orthogonal polynomials" by Professor Dunham Jackson, University of Minnesota.

5. "Problems of time-series analysis" by Professor E. A. Gaumnitz, College of St. Thomas, introduced by Professor Bush.

6. "An original trigonometric mnemicon" by Father G. L. Winkelmann, St. John's University.

7. "On the use of the Laplace transformation for the expansion of functions in terms of Bessel functions" by Professor H. P. Thielman, College of St. Thomas.

8. "Elementary geometry applied to a plane area and its projection" by Professor H. C. T. Eggers, University of Minnesota, introduced by the Secretary.

9. "A rule for calculating the coefficients of an equation with given roots" by Dr. K. W. Wegner, University of Minnesota.

10. "The solution of a certain linear non-homogeneous difference equation of second order" by Dr. R. W. Cowan, College of St. Scholastica.

11. "Classes of ideals in algebraic domains" by Brother Louis De La Salle, St. Mary's College, introduced by the Secretary.

12. "Examination technique in higher algebra" by J. H. Daoust, University of Minnesota.

Abstracts of these papers follow, the numbers corresponding to the numbers in the list of titles:

1. Professor Carlson gave six exercises in the form of equations, for use in teaching mathematical induction in a course in College Algebra. The equations were such that they checked for $n=1, 2, 3, 4$, but were found untrue for $n=5$.

2. Professor Taylor explained a scheme for determining the alternate segments into which the sides of a triangle are divided, in applying the theorem of

Menelaus. Write the letters representing the vertices of the triangle in a column (or line), repeating the first vertex. Opposite the intervals between letters write the letters for the three points of division. The letters on the diagonals in one direction give one set of segments and the letters on the diagonals in the other direction give the other set.

3. Dr. Scherberg indicated the manner in which the fundamental properties of a function $f(x)$, continuous in a closed interval (a, b), may be easily demonstrated by direct application of the Heine-Borel theorem. For example, after boundedness has been proved, the values $f(x)$ must have a greatest dense point M which is the largest value of $f(x)$ in (a, b). Otherwise we could associate with each point $x=p$ an open interval I_p containing p such that $f(x)-f(p) < \frac{1}{2}[M-f(p)]$ so that $f(x) < M-\frac{1}{2}[M-f(p)]$ in I_p, and, by the Heine-Borel theorem, M could not have been a greatest dense point. The demonstration structure is unified by this procedure.

4. In this paper Professor Jackson gave a concise proof of the fact that if a complete system of orthogonal polynomials possesses the property of orthogonality with respect to each of two weight functions, one of these functions must be essentially a constant multiple of the other.

5. Professor Gaumnitz classified the forces tending to bring variation in a time-series as secular trend, seasonal variation, cyclical movements, and irregular fluctuations. The problem is to separate the observed data into its component parts. Periodogram analysis represents one type of departure. There are many criticisms of this technique as applied to economic time-series, one of the most important of which is that periodogram analysis cannot be reliably applied for short time periods. Furthermore, difficulty arises in the extension of the results of periodogram analysis into periods not covered by the data, and this type of extrapolation is necessary in the computation of indexes of business conditions. The present state of time-series analysis technique presents a challenge to mathematical statisticians to discover a method of direct solution for the cyclical component.

6. Father Winkelmann exhibited a device of his invention to aid the memory in the use of trigonometric functions and their inverses as applied in plane trigonometry and in differential and integral calculus. The three primary colors with the three complementary colors are used in place of the logarithms of these functions. An indicator is introduced to facilitate the determination of the sign in the four quadrants. Each representation of the function has a peg. On these pegs a cord may travel in single, double, or triple steps in the fashion of the six sixth roots of unity, giving a single, or a double, or a triple periodicity in the geometry, resulting always in a product of trigonometric functions equivalent to unity.

7. On the assumption that a function $F(z)$ can be expanded in a series of the form

$$F(z) = \sum_{n=0}^{\infty} a_n j_{\nu+n\mu}(z), \qquad \nu > 0, \mu > 0, \tag{1}$$

where $j_\nu(z)=z^{-1}\nu J_\nu(z)$, and $J_\nu(z)$ is the Bessel function of the first kind of order ν, Professor Thielman gave a determination of the coefficients a_n in the following manner. The Laplace transformation of both sides is taken. There results the series $f(z)=\sum_{n=0}^{\infty}a_n(\sqrt{z^2+1}-z)^{\nu+n\mu}$, where $f(z)$ is the Laplace transform of $F(z)$. Setting $t=\sqrt{z^2+1}-z$, $f(z)$ becomes $f(1-t^2/2t)$. Expanding this function in powers of t, the coefficients a_n are determined. It follows from a well known theorem of Volterra that if $\sum_{n=0}^{\infty}a_nt^{\nu+n\mu}$ is convergent in any circle about the origin of radius greater than zero, then the series (1) will be convergent for all values of z for which the $j_\nu(z)$ remain finite. Illustrations were given by applying this method of elementary functions to obtain their well known expansions in terms of Bessel functions. It was pointed out that, by taking certain fractional integrals of the resulting expansions, developments in terms of squares of Bessel functions could be obtained.

8. Professor Eggers derived synthetically the relation between a plane area and its projection on a second plane by employing a method frequently used in plane geometry for finding the area of a triangle. This relation lends itself naturally and simply to the derivation of the expression for the area of a triangle in three-dimensional space in terms of the coördinates of the vertices.

9. Dr. Wegner explained a simple and convenient, but apparently little known rule which was worked out by Professor M. H. Ingraham of the University of Wisconsin.

10. Dr. Cowan took the form of the solution as an expansion in ascending powers of a parameter with undetermined coefficients. Substitution of this series in the given difference equation leads to a set of non-homogeneous linear difference equations of the second order. These are solved by the use of Appell's integral. The original series is shown to converge for sufficiently large values of the unknown by setting up a dominant series which is known to converge.

11. Brother Louis De La Salle proved several theorems on class number of algebraic fields, of which the following is typical. Let θ be a root of the polynomial equation of odd prime degree, $f(x)\equiv x^p-a_1x^{p-1}-\cdots-a_p-0$, the a_i being rational integers, and let p_1, p_2 be two rational primes such that $a_i\equiv 0$ (mod p_1p_2), ($i=1,\cdots,p$) and $a_p\equiv 0$ (mod p_1^2 or p_2^2). Let $\Gamma(\theta)$ be the algebraic field generated by "adjoining" θ to the rational field Γ. Then if the congruence $x^p-p_1\equiv 0$ (mod p_2) has no rational integral solution, the class number of $\Gamma(\theta)$ is a multiple of p. For any p, an infinite number of pairs p_1, p_2 can be found satisfying the condition. Numerical examples are easily obtained.

12. Mr. Daoust presented some of the results of an examination study in higher algebra conducted at the University of Minnesota. Substantial evidence was given to show that the multiple choice type of test question has little value in higher algebra testing. It was further pointed out that even simple problems, if stated in sentence form, sometimes requiring the application of fundamental rules and principles in their interpretation, will tend to differentiate the students according to their abilities *in the fundamental skills* more than will essentially the same problems presented in symbolic form.

R. W. Brink, *Acting Secretary*

THE SPRING MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at Randolph-Macon Woman's College, Lynchburg, Virginia, on Saturday, May 8, 1937. The chairman, Dr. John Williamson of Johns Hopkins University, presided at both sessions, morning and afternoon. Five papers were read at the morning session, and in the afternoon, at the invitation of the Section, Professor John von Neumann of the Institute for Advanced Study delivered a lecture on "Axiomatic treatment of projective geometry."

Following the afternoon session those in attendance were entertained by the students of the Greek Department in a play by Aristophanes.

At a business meeting the following officers were elected for next year: Chairman, Gillie A. Larew, Randolph-Macon Woman's College; Secretary, Michael Goldberg, Bureau of Ordnance, Navy Department; additional members of the Executive Committee, G. A. Bingley, St. John's College, and Oscar Zariski, Johns Hopkins University.

The attendance was forty-one, including the following thirty members of the Association: O. S. Adams, M. W. Aylor, N. H. Ball, J. W. Blincoe, C. C. Bramble, W. E. Byrne, Eleanor Calkins, Alexander Dillingham, J. A. Duerksen, Almeda J. Garland, Michael Goldberg, C. D. Gregory, Isabel Harris, L. M. Kells, R. H. Knox, Jr., W. D. Lambert, Gillie A. Larew, Eugenie M. Morenus, F. D. Murnaghan, E. K. Paxton, W. T. Puckett, Jr., R. E. Root, L. W. Smith, John Tyler, John von Neumann, C. H. Wheeler, G. T. Whyburn, Evelyn P. Wiggin, John Williamson, R. C. Yates.

The fall meeting will be held on December 4, 1937 at Johns Hopkins University, Baltimore, Md.

The following six papers were read:

1. "A classification of bicircular quartics by means of their euclidean concomitants" by Dr. J. W. Blincoe, University of Virginia.

2. "A projective invariant in the linear transformation $B=C\times A$ where A, B, C are three-dimensional square matrices" by M. W. Aylor, University of Virginia.

3. "Solvable cases of the equations for wind velocity" by Professor R. E. Root, Postgraduate School, U. S. Naval Academy.

4. "A conformal map of the world in a square" by Dr. O. S. Adams, U. S. Coast and Geodetic Survey.

5. "Asymptotic expansions" by Dr. N. H. Ball, U. S. Naval Academy.

6. "Axiomatic treatment of projective geometry" by Professor John von Neumann, Institute for Advanced Study.

Abstracts of these papers follow, the numbers corresponding to the numbers in the list of titles:

1. An algebraically complete system of euclidean invariants and covariants

for the bicircular quartic consists of five invariants and two covariants. By means of such a system, Dr. Blincoe showed how to set up necessary and sufficient conditions for the bicircular quartic to assume any one of a number of special forms; for example, the cardioid or lemniscate. Furthermore, assuming that the conditions for a particular form are satisfied, he demonstrated how the quartic may be expressed as a function of its invariants and covariants.

2. Interpreting the vectors in the matrices A and B as the homogeneous coördinates of six points in a plane, Mr. Aylor showed that there is an invariant made up of the elements of C such that these six points lie on a conic. If the products of the elements of C are defined as $(C_{11}C_{22}C_{33},\ C_{12}C_{23}C_{31},\ C_{13}C_{32}C_{21},\ -C_{13}C_{22}C_{31},\ -C_{21}C_{33}C_{12},\ -C_{11}C_{23}C_{32}) \equiv (I_1,\ I_2,\ I_3,\ J_1,\ J_2,\ J_3)$, then the points of A and B are on a conic if, and only if, $I_1I_2+I_1I_3+I_2I_3=J_1J_2+J_1J_3+J_2J_3$.

3. Professor Root discussed the differential equations for horizontal motion of the air with particular reference to the character of assumptions, as to pressure and density distribution, which make the equations readily solvable. Among others, he showed particular solutions for cases of elliptical isobars in which the paths of motion are ellipses differing from the isobars. If parameters are so chosen that the isobars are circles, then the paths of motion coincide with the isobars. These elliptical paths arise when particular initial conditions are imposed on general solutions consisting of two superimposed elliptical motions of different periods. The motion may be regarded as superimposed on a general translation.

4. Dr. Adams was asked by Mr. J. C. Carpenter to supply a map of the world to be painted in a square on the wall of a new airport building in Fort Worth, Texas. The map has the poles situated at the midpoints of two opposite sides. The method of attack and the plan for the computation have special interest in a mathematical way. It is an interesting application of elliptic integrals to conformal mapping. A complete account of the theory of the map, together with a table of the coördinates, is soon to appear in *Le Bulletin Géodésique*.

5. The paper of Dr. Ball reviewed the principal properties of asymptotic expansions of functions and presented some applications in electrical engineering. Several methods of obtaining such expansions were discussed, including a method for determining formulas for the zeros of a function from its asymptotic expansion. The nature and cause of the discrepancies in asymptotic formulas obtained by different methods were touched upon.

6. Professor von Neumann presented a generalized axiomatic treatment of geometry. The classical axiomatizations of geometry are all based on the three classes of "undefined entities": points, rings, and planes, the last class being sometimes omitted. It was first proposed by K. Menger (*Jahresbericht der Deutschen Mathematiker Vereinigung*, 1928) to use only one class of "undefined entities," the class of all linear subspaces $a, b, c, \cdots$, mainly with the n-dimensional case in view. He formulated a set of axioms on this basis. G. Birkhoff (*Annals of Mathematics*, 1936) solved the same problem with the help of the

modern "Theory of Lattices." In this setup one undefined relation may be used, the incidence relation written $a<b$, meaning that a is a proper subclass of b. The axioms must express that the system $L: a, b, c, \cdots$ is a "partially ordered set" with respect to the relation $a<b$; and also that for any two subspaces a, b, a "least upper bound" c exists, where c is such that the statement $x>c$ is equivalent to $x>a$ and $x>b$. This c will be denoted by $a+b$. There must also be a "greatest lower L" d such that the statement $x<d$ is equivalent to $x<a$ and $x<b$. This d will be denoted by ab. The formal properties of this addition and multiplication are those of Boolean algebras except that the distributive law does not hold.

From the geometrical point of view the existence of a numerical dimensional function $D(a)$ is essential, that is, one which will fulfill the additive functional equation $D(a+b)+D(ab)=D(a)+D(b)$. G. Birkhoff showed that the existence of such a D follows from the axiom: $a \leqq c$ implies $(a+b)c=a+bc$. This is Dedekind's modular axiom, a weakened form of the distributive law. Professor von Neumann succeeded in building up on the same foundations more general geometrical systems where the range of the dimensional function $D(a)$ is a continuous interval of real numbers (*Proceedings of the National Academy*, 1936 and 1937). These "continuous geometries" have many properties in common with the usual projective geometry, but no minimum linear subspaces (points) exist in them. They also permit applications to algebra which lead to a new and wider theory of semi-simple hypercomplex number systems.

MICHAEL GOLDBERG, *Secretary*

THE FIFTH ANNUAL MEETING OF THE WISCONSIN SECTION

The fifth annual meeting of the Wisconsin Section of the Mathematical Association of America was held at Milwaukee-Downer College on May 8, 1937. The chairman of the Section, the Rev. L. A. V. DeCleene of St. Norbert's College, presided.

The attendance was forty-eight, including the following twenty-five members of the Association: R. H. Bardell, Leon Battig, Ethelwynn R. Beckwith, W. W. Bigelow, H. H. Conwell, L. A. V. DeCleene, Henry Ericson, R. C. Huffer, Elizabeth E. Knight, Caroline A. Lester, C. C. MacDuffee, Morris Marden, Sister Mary Felice, J. S. McNair, R. E. Norris, G. A. Parkinson, H. P. Pettit, Irene Price, W. E. Roth, C. W. Smith, I. S. Sokolnikoff, P. L. Trump, J. I. Vass, J. A. Ward, Margarete C. Wolf.

Sessions were held in the morning and afternoon, with luncheon in McLaren Hall at 12:30. At the close of the luncheon, Doctor Lucia R. Briggs, President of Milwaukee-Downer College, graciously welcomed the visiting mathematicians and their guests. Following the afternoon meeting President Briggs and Professor Beckwith were hostesses to the visitors at tea. A business meeting was held at 2 P.M. at which the following were elected officers for the coming year: Chairman, Mrs. Ethelwynn R. Beckwith, Milwaukee-Downer College;

Secretary, G. A. Parkinson, University of Wisconsin Extension Division; Program Committee, R. C. Huffer, Beloit College, J. S. McNair, Sheboygan High School. An invitation to meet next year at St. Norbert's College was accepted by unanimous vote. Appreciation for the hospitality extended to the Section by Milwaukee-Downer College was expressed by a rising vote.

The following papers were presented:

1. "The computation of complex roots of algebraic equations with numerical coefficients" by Professor R. C. Huffer, Beloit College.

2. "The problem of Apollonius" by J. S. McNair, Sheboygan Senior High School.

3. "A theorem on determinants" by Professor W. E. Roth, University of Wisconsin Extension Division.

4. "Certain aspects of a teacher's training program as related to mathematics" by Dr. P. L. Trump, Wisconsin High School, Madison.

5. "Number fields" by Professor C. C. MacDuffee, University of Wisconsin.

Abstracts of the papers follow, numbered in accordance with their listing above:

1. Professor Huffer discussed the methods of Horner, Graeffe, and Bernoulli for solving numerical algebraic equations, and deduced criteria for the number of steps needed to obtain a desired number of significant figures. Illustrations were given of the adaptation of these methods to computation with calculating machines. Transformations to reduce the labor required in solving for both real and imaginary roots were discussed. In particular, a combination of the methods of Horner and Bernoulli was shown to be especially adaptable to machine computations.

2. Given three things (points, straight lines, or circles) in position, to describe a circle passing through the given points, and touching the given straight lines or circles. This was the problem of tangencies as stated by Apollonius. Mr. McNair discussed the ten cases of this general problem, and showed a straight-edge and compasses construction for each case. The problem of finding a circle tangent to three given circles is known as the problem of Apollonius.

3. The usual cyclic determinants and other forms, such as those studied by Drude, Noether, Scorza, and others, are here treated by Professor Roth in more general form. The elements c_{ij} of the determinant, $|c_{ij}|$, $(i, j=1, 2, \cdots, r)$ are replaced by square matrices of order n and succeeding rows may be other than mere permutations of the matrices, c_{ij}, which occur in (say) the first row of the given determinant. Such determinants are expressible as products of r factors.

4. Dr. Trump discussed several implications of some of the more recent developments in secondary school mathematics instruction for the training of teachers. Many objectives relating to appreciation, understanding, habits of thought, and transfer of training are being considered more systematically. This requires of the teacher a comprehensive subject matter preparation as well as a broad general background in many fields. The professional training of the teacher should be handled in close cooperation by the specialists in

education and the specialists in mathematics. The promising teacher should be encouraged to continue preparation beyond the usual four year college course.

5. This paper was an exposition by Professor MacDuffee of the simpler aspects of the modern theory of number fields. If a field F is extended to its polynomial domain, the process of forming the quotient field of this domain leads to a transcendental extension of F, while the process of taking residues modulo an irreducible polynomial leads to an algebraic extension of F. Algebraic fields are algebraic extensions of the rational field, while Galois fields are algebraic extensions of the field of residues modulo a prime integer. The method of Cauchy sequences by which the real field is customarily obtained from the rational field can be treated abstractly, and various types of perfect field thereby obtained, including the p-adic fields of Hensel.

G. A. PARKINSON, *Secretary*

THE APRIL MEETING OF THE IOWA SECTION

The twenty-sixth regular meeting of the Iowa Section was held at the University of Dubuque, Dubuque, Iowa, on Friday and Saturday, April 16–17, 1937 in conjunction with the fifty-first regular meeting of the Iowa Academy of Science. Due to absence of the chairman, the vice-chairman, Professor L. E. Ward presided at both sessions.

The attendance was about thirty, including the following seventeen members of the Association: John Breiland, J. O. Chellevold, E. W. Chittenden, L. M. Coffin, N. B. Conkwright, A. T. Craig, Cornelius Gouwens, O. C. Kreider, J. V. McKelvey, E. E. Moots, H. L. Rietz, W. J. Rusk, E. R. Smith, John Theobald, L. E. Ward, C. W. Wester, Roscoe Woods.

On Friday evening the members of the Association and the Iowa Academy of Science had a joint dinner in Peters Commons of the University of Dubuque. The officers of the section elected for 1937–38 are as follows: Chairman, L. E. Ward, University of Iowa; Vice-Chairman, E. E. Moots, Cornell College; Secretary-Treasurer, Cornelius Gouwens, Iowa State College. A resolution expressing the appreciation of the members of the Section for the hospitality and courtesy extended to them by their host, the University of Dubuque, was adopted at the business session. The invited address was given by Professor E. W. Anderson of Iowa State College. The following nine papers were read:

1. "A proof of Budan's Theorem" by Professor N. B. Conkwright, University of Iowa.

2. "Descartes's rule of signs" by Professor N. B. Conkwright, University of Iowa.

3. "On the decomposition of 4 $(x^p-1)/(x-1)$, for a prime p, $200<p<225$" by Professor Cornelius Gouwens, Iowa State College.

4. "Airfoil theory" by Professor E. W. Anderson, Iowa State College, by invitation.

5. "General formulae for homozygosis" by H. W. Norton, III, Iowa State College, introduced by the Secretary.

6. "Tests for homogeneity of variances" by Professor A. E. Brandt, Iowa State College, introduced by the Secretary.

7. "On the complete independence of Huntington's postulates for Boolean algebras" by F. D. Rigby, University of Iowa, introduced by Professor E. W. Chittenden.

8. "Volterra's biological dynamics" by Professor E. S. Allen, Iowa State College, introduced by the Secretary.

9. "Correlations between grades in freshman mathematics and pre-test scores" by Professor E. W. Chittenden, University of Iowa.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles.

1. Professor Conkwright gave a proof of Budan's Theorem in an elementary manner by the use of Rolle's theorem and mathematical induction.

2. In this paper Professor Conkwright established the validity of Descartes's rule of signs by mathematical induction. The argument was based upon Rolle's theorem.

3. Professor Gouwens continued a paper of last year on the decomposition of $4(x^p-1)/(x-1)$ for p a prime not greater than 225.

4. In an historical introduction, Professor Anderson dealt with some of the important events and persons connected with the development of aerodynamic theory. In the next section he reviewed briefly the hydrodynamics of irrotational two-dimensional flow of a perfect fluid as interpreted by a complex potential flow-function. Then the circulation theory of lift on an infinitely long cylinder was discussed, and the general complex transformation which will carry the flow about any airfoil section into flow about any cylinder, preserving the direction and magnitude of the flow at infinity, was presented. Theoretically, this allows the determination of the lift on and the pressure distribution about the airfoil. A special case, the Joukowsky transformation, was given. Then the new theories for studying flow about arbitrary monoplane (T. Theodorsen) and biplane (I. E. Garrick) wing sections were outlined.

5. Mr. Norton solved a general linear fractional recursion formula and gave the application of the solution to certain problems in genetics.

6. Professor Brandt gave a test of significance of the difference between two variances using a known distribution, Chi-square, for which values have been tabled. This test was extended to a test of homogeneity of n variances from samples of $(k+1)$ observations.

7. A set of postulates consists of two groups, a primary set defining a system and a secondary set conditioning the operations postulated in the primary group. Mr. Rigby discussed the complete independence of a secondary group of three postulates included in Huntington's "fourth set" of postulates for Boolean algebra. It was found that for a two element Boolean algebra the associative law of addition need not be postulated. All examples but one exist on a space of two elements. The eight existences required can be constructed for algebras of n elements for all values of $n \geqq 2$.

8. After devoting many years to the study of differential, integral, and integro-differential equations and of mechanics, Vito Volterra has recently investigated the interaction of biological species, finding many applications for theories he had previously developed. In the present report Professor Allen was concerned chiefly with the publications of 1936. These deal with the differential equations of competing species and show the close analogy between the treatment to which they are susceptible and the methods of classical mechanics. Thus the equations themselves are the Euler equations of a variational problem. Variables yielding a Hamiltonian function and the resulting canonical equations are named. There is a theorem of conservation of energy and, under certain circumstances, one of least action.

9. Professor Chittenden reported that the pre-tests in algebra prepared by the committee on tests of the Mathematical Association of America have a median correlation with grades in course of $r=.6$. This result was obtained from scores and grades made by about fifteen hundred students in five institutions and corresponded closely with results obtained by a number of institutions using mathematics training tests in the series of placement tests published by the University of Iowa.

CORNELIUS GOUWENS, *Secretary*

FIRST ANNUAL MEETING OF THE SOUTHWESTERN SECTION

The first annual meeting of the Southwestern Section of the Mathematical Association of America was held at the New Mexico State College of Agriculture E. J. and Mechanic Arts, on Friday and Saturday, April 2–3, 1937.

The attendance was thirty-four, including the following thirteen members of the Association: C. A. Barnhart, J. W. Branson, R. F. Graesser, E. L. Harp, Jr., E. A. Hazlewood, H. D. Larsen, H. B. Leonard, C. V. Newsom, E. J. Purcell, P. K. Rees, W. C. Risselman, F. W. Sparks, R. S. Underwood.

On Friday evening there was a banquet for mathematicians and their guests. The principal address after the dinner was given by Professor Emeritus J. B. Shaw of the University of Illinois.

At the business meeting a resolution was passed expressing the appreciation of the group to Professor Newsom for his efforts in forming the Section. A second resolution expressed the thanks of the members of the Section to the faculty of the New Mexico State College for their hospitality. The following officers were elected for next year: Chairman, R. F. Graesser, University of Arizona; Vice-Chairman, R. S. Underwood, Texas Technological College.

With Professor P. K. Rees, Chairman of the Section, presiding, the following papers were presented:

1. "Some mathematical computations involved in a determination of the oxygen parameters of sodium periodate" by Professor E. A. Hazlewood, New Mexico State College.

2. "Hedging as a mathematical art" by Dr. H. D. Larsen, University of New Mexico.

3. "Involutions" by Dr. E. J. Purcell, University of Arizona.

4. "The characteristic equation of a certain type of continued fraction expansion" by Professor P. M. Singer, New Mexico State Teachers College, introduced by the Secretary.

5. "A study of a recent theorem of W. B. Ford" by D. A. Lawson (introduced by Professor Newsom) and Professor C. V. Newsom, University of New Mexico.

6. "Algebras defined by abstract groups whose operators are all of the form A^xB^y" by Professor J. B. Shaw, University of Illinois, introduced by the Secretary.

7. "Some corollaries of the Fourier-Budan theorem" by Professor W. C. Risselman, Arizona State Teachers College at Flagstaff.

8. "Relative delicacy of certain convergence tests" by Professor R. S. Underwood, Texas Technological College.

Abstracts of some of the papers follow. The numbers correspond to the numbers in the list of titles.

2. When different odds are quoted on an event which must occur in one of two ways, a person can so hedge his bets that he will win however the event takes place. Dr. Larsen derived formulas for determining the amount of each bet so that there will be a maximum gain. He generalized the theory to the case of an event which can occur in N ways, and derived the condition that a gain be possible.

4. Professor Singer showed that in a periodic continued fraction expansion of any number of complete quotient terms if the partial quotients at each step are given by $q_n+1, q_n, q_n, \cdots, q_n, q_n-1$, the characteristic equation of the expansion reduces to a quadratic equation and an equation whose roots are powers of the roots, other than one, of unity. The exponent of the power is equal to the number of complete quotient sets in the period. The quadratic equation is the characteristic equation of the binary expansion whose partial quotients are $q_1, q_2, \cdots, q_n, \cdots$. Therefore, in general, the expansion represents numbers in the realm of a quadratic irrationality and in the realm of the roots of unity.

5. In a study of the behavior of certain types of power series when the modulus of the variable becomes large, Ford proved a theorem (*Asymptotic Developments* by W. B. Ford, University of Michigan Press, 1936, p. 5) in which it was necessary to employ an arbitrary periodic function of a specified type. Lawson and Newsom discussed the merits of a variety of choices for such a function. In particular, it was shown how the resulting analysis is changed and, at times, simplified by the choice of the function, $Q(w)=e^{2\pi iw}-1$.

6. Professor Shaw proved that the general defining equation for abstract groups all of whose operators are of the form A^xB^y, where $A^a=1-B^b$, and $A^\delta=B^\epsilon$, is $B^xA^y=A^{yq^x}B^{xr^y}$ where q is prime to a, and $q^c\equiv 1 \pmod{a}$, $r^d\equiv 1 \pmod{b}$, $q=1+\lambda d$, $r=1+\mu c$, and of course c is a divisor of $\lambda(a)$ and d is a divisor of $\lambda(b)$. Also $a=dg$, $b=ch$. These groups include among them the dihedral groups,

dicyclic groups, and several unnamed but important classes. In the algebra we may separate A into a form linear in idempotent, mutually nilfactorial, parts: $A=\sum_{i=1}^{a}\sigma^{i}k_{i}$, and likewise $B=\sum_{j=1}^{b}\tau^{j}\pi_{j}$, where σ is a primitive a root of unity, and τ is a primitive b root of unity. The algebra is reducible into matric algebras, whose partial moduli will have forms $(K_s+K_t+\cdots)(\pi_u+\pi_v+\cdots)$. These forms multiplied by various of the idempotent K's and π's will give the individual units of the algebra.

7. It is known that the methods of the Fourier-Budan Theorem yield the exact number of roots of a real polynomial on an interval if the roots are all real. Descartes' Rule of Signs gives the exact number of positive roots if the roots are all real. Proofs of these facts are given in Serret's *Algèbre Supérieure.* Professor Risselman stated that these facts are corollaries of Budan's Theorem as given, for example, in Dickson's *First Course in the Theory of Equations.*

8. Two theorems, the first due to Mrs. L. Bahm, establish the fact that the positive-term test: "(a) $nu_n>k>0$ implies divergence; (b) $n^p u_n<K<\infty$ $(p>1)$ implies convergence" is more delicate than the nth root test and also Raabe's refinement of the ratio test. A third theorem proves that the positive-term test: "(a) $n(\log n)u_n>k>0$ implies divergence; (b) $n(\log n)^p u_n<K<\infty$ $(p>1)$ implies convergence" is more delicate than the test: "If $u_{n+1}/u_n=1/(1+1/n+\theta_n/n^2)$, (c) $\theta_n \log n<1$ implies divergence; (d) $\theta_n \log n>k>1$ implies convergence."

On Saturday afternoon there was a symposium on Teaching problems in mathematics. With Professor R. F. Graesser presiding, the following papers were presented:

1. "The teaching of a unit in hyperbolic functions" by Professor Edna Graham, West Texas State Teachers College, introduced by Professor Graesser.

2. "College mathematics and the new curriculum" by Professor F. W. Sparks, Texas Technological College.

3. "Suggestions for research in teaching procedure" by Professor J. W. Branson, New Mexico State College.

4. "Meeting the problem in freshman mathematics" by Professor J. L. Olpin, Gila Junior College, Thatcher, Ariz., introduced by Professor Graesser.

5. "What is to be done about high school mathematics" by Professor Charles Wexler, Arizona State Teachers College at Tempe, introduced by Professor Graesser.

Abstracts of these papers follow:

1. Professor Graham reported that in one of her classes the unit in hyperbolic functions of twelve lessons followed immediately the work in infinite series. No textbook was used. The definitions were built out of the infinite series for e^x, e^{-x}, $\sin x$, $\cos x$, $(1/i)\sin ix$, and $\cos ix$. Euler's formulas were developed and used for their analogous relations to hyperbolic functions. Four guide sheets were distributed to each member of the class. Eight reports were made including: (1) Graphs, (2) Relations of hyperbolic functions to circular functions, (3) Calculus of hyperbolic functions, (4) The Gudermannian, (5) Double inter-

polation, (6) Line representation of hyperbolic functions, (7) Areas of corresponding triangles and corresponding sectors, (8) Applications: catenary, alternating currents.

2. Professor Sparks stated that the high school course in mathematics that seems to be generally followed under the new curriculum is arithmetic in the eighth grade, general mathematics in the ninth, advanced algebra in the tenth, general geometry in the eleventh, and trigonometry in the twelfth. Generally speaking, no mathematics beyond that given in the ninth grade is required for graduation. The indications are that only a small percent of the students will take more than the required course. In the colleges the mathematics requirements are increasing rather than decreasing, especially in the technical and professional courses. This creates a problem that will become more and more serious until a solution is found.

3. Professor Branson suggested that the following questions be answered on an experimental rather than an "I think" basis: (a) To what extent should the traditional language be simplified in elementary courses in college mathematics? (b) What are the minimum essentials that should determine requirements for a passing grade? (c) Should the traditional freshman courses in algebra, trigonometry, and analytics be given separately or should algebra run simultaneously with each of the other two subjects? (d) Would it be profitable to omit much of the ordinary work of formal integration and treat that topic somewhat as logarithms are treated? (e) Can the present practice of offering specialized courses for beginning college students be justified or should freshman students all take the same courses in mathematics?

4. Professor Olpin outlined the general procedure which he has found to be most successful in teaching freshmen. "Students who register for freshman mathematics cannot be expected to have very much background," he stated. Generally the college entrance credits in algebra and geometry do not represent their knowledge of these subjects. It is the task of the freshman mathematics teacher to take these young people as they are and teach them mathematics, to dispel the fear that many of them have for the subject, and to convince them that it is easy. Every effort must be put forth to develop an interest in and an appreciation for mathematics.

5. Professor Wexler proposed four things: (a) Geometry, not algebra, should be the first mathematics studied in high school; (b) No formal course in algebra should be given, but algebra should be woven into the study of later work in mathematics as needed; (c) This later work should include trigonometry, elementary analytic geometry, and calculus; (d) High I.Q., or prognosis tests, or the student's showing in plane geometry should decide whether the student should be allowed to go on in mathematics. To effect these changes, administrators should engage only well qualified teachers in mathematics, and college mathematics departments should ruthlessly weed out from among their majors the incompetents who just get by in each course.

W. C. RISSELMAN, *Secretary*

THE PRINCIPIA AND THE MODERN AGE*

By C. S. SLICHTER, University of Wisconsin

The beginning of a unified conception of the universe was not made by mathematicians. It was the discovery of poets. That is to say, the first approach to nature was aesthetic and not scientific. Over two thousand years ago Solomon is reported to have given acknowledgment to his God in these words:

> "He hath given me the certain knowledge of the things that are, namely to know how the world was made, and the operation of the elements; the beginning, ending, and the midst of the times; the alterations of the turning of the sun, and the change of seasons; the circuit of the years and the position of the stars: . . . And all such things as are either secret or manifest, them I know."

These are strong words from Solomon. His enthusiasm seems to have outrun his modesty. Only poets can indulge in such overwhelming self-confidence. They call it "inspiration." Sometimes I think it is as sound an approach to reality as any other.

One would conclude from the words attributed to Solomon that he and not Newton had written the *Principia*. But Newton in his modesty made no claim similar to Solomon's. He was only a child gathering a few pebbles on the shore of the vast sea. Solomon's words, sympathetically interpreted, mean that his approach to nature was aesthetic—an approach by faith which is open to everyone, whether a person of little or of much sagacity. Solomon was expressing his belief in the wisdom of a single author of the world and was giving vent to his ecstasy in viewing the harmonies and beauty of created things. He was setting himself against those of his day who believed in a multitude of gods or demons, who saw an individual god back of each manifestation of nature. "Surely vain are all men," he said, " . . . who could not, out of the good things that are seen, know him that is; who by considering the works, did not consider the work master, but deemed either fire, or wind, or the swift air, or the circle of the stars, or the violent water, or the lights of heaven, to be the gods that govern the world." Solomon rebuked those who looked upon each manifestation as the act of a special deity. A single God, "the first author of beauty," as he so happily called him, "ordered all things in measure and number and weight."

It was the purpose of the *Principia* of Newton to discover and set forth the exact manner in which "all things had been ordered in measure and number and weight."

To understand the *Principia*, one must first recall something of the times in which it was written and the temper of the group of young English scientists which gave character to the age in which Newton was born. The *Principia* must

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* Presented by invitation at the meeting of the Mathematical Association of America at State College, Pa., September 7, 1937 in commemoration of the two hundred and fiftieth anniversary of the publication of Newton's Principia.

be looked upon as the masterpiece that came out of the new learning or the Experimental Philosophy that took its hold on the world about three centuries ago. The old learning was resting in quiet content in the Universities of Oxford and Cambridge. Aristotle and the seven philosophies seemed safe and fixed and unchangeable. Then from outside the universities an intellectual revolution got under way. Bacon had announced that the methods employed in the sciences and the results reached were alike erroneous, "yielding," he said, "no true fruit of learning." He announced that a new method of philosophy must be devised whose aim should be the service and welfare of men and not the pleasure and delight of scholars. The world was ripe for Bacon's ideas, and they probably would have burst forth even without his powerful aid. Within twenty years of his death a group of vigorous young men in London had taken up the new philosophy. For the first home of the new learning there stood the ever open hospitality of the London tavern. This may seem a strange place of beginning for the new learning, but the place, nevertheless, was natural enough. Private homes, for the most part, were poorly built, uncomfortable and cold. The taverns were warm and cheery and the center of life and hospitality. Hence it was quite natural that in 1645, three years after the birth of Newton, we should find a group of diners regularly getting together at the Bullhead Inn in Cheapside, for the purpose of discussing and experimenting in the new philosophy.

It was this group, called by Boyle the "Invisible College," that in 1660 organized the "Visible College," chartered as the "Royal Society of London for the Improvement of Natural Knowledge." Prominent in this group were Lord Brouncker, Bishop Wilkins, Robert Boyle, John Evelyn, Robert Hooke, Sir William Petty, Christopher Wren. These men, together with a number of less known but equally enthusiastic associates, nearly all under thirty-five years of age, confirmed one of the greatest events in British history, when on July 15, 1662, they received the charter of the Royal Society from the King.

In that charter the King saith: "We do hereby make and constitute the said Society, . . . to be a Body Corporate . . . whose studies are to be employed for the promotion of the knowledge of natural things and useful arts by *Experiments*." On this foundation has been built the great structure of British science. It was the age of scientific revolution. Experimentation became a craze and natural science in England developed outside the universities and in spite of them. The newly founded Gresham College in London was an exception. Its greatest renown in the world is that in the first fifty years of its life it became the home of the new philosophy. It was not only the cradle of the Royal Society, but its fine halls were the meeting place of the society for nearly all of the first half century of its history. Its galleries housed its libraries and the curiosities, and supplied the permanent offices for the secretaries and officials of the new society.*

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* The scientific revolution made no exception to the rule that everyone desires to belong to some club or "college"; the Royal Society of that day and Phi Beta Kappa, Elks, Eagles, Rotarians, and this Mathematical Association of our own day, are all examples which, according to Wallace

The men who constituted these first groups of organized science were not long-faced specialists or academicians in the continental sense; they were convivial Englishmen and men taken from all walks of life. This type of membership was maintained for many generations. The membership lists for the first century of its history include many prominent members of the peerage, numerous members of parliament, amiable and versatile politicians, a notable band of medical men, artists, critics, civil servants, and pamphleteers. There were bishops and explorers and travelers and antiquarians and many all-round good fellows whose principal job was to contribute enthusiasm and financial support.

To illustrate the temper of the times, here is an extract from the minutes of one of the early meetings, that of September 10, 1662. Newton was then at Cambridge, twenty years of age:

> "It was order'd, at the next meeting Experiments should bee made with wires of severall matters of ye same size, silver, copper, iron, etc. to see what weight will breake them; the curatour is Mr. Croone.
>
> "Dr. Goddard made an experiment concerning the force that presseth the aire into lesse dimensions; and it was found, that twelve ounces did contract 1/24 part of Aire. The quantity of Aire is wanting.
>
> "My Lord Brouncker was desired to send his Glasse to Dr. Goddard, to make further experiments about the force of pressing the aire into less dimensions.
>
> "Dr. Wren was put in mind to prosecute Mr. Rook's observations concerning the motions of the Satellites of Jupiter.
>
> "Dr. Charleton read an Essay of his, concerning the velocity of sounds, direct and reflexe, and was desired to prosecute this matter; and to bring his discourse again next day to bee enter'd.
>
> "Dr. Goddard made the Experiment to show how much aire a man's lungs may hold, by sucking up water into a separating glasse after the lungs have been well emptied of Aire. Severall persons of the Society trying it, some sucked up in one suction about three pintes of water, one-six, another eight pintes and three quarters, etc. Here was observed the variety of whistles or tones, which ye water made at the several hights, in falling out of the glasse again.
>
> "Mr. Evelyn's Experiment was brought in of animal engrafting, and in particular of making cock spurs grow on a cock's head."

You may smile at these early enthusiasms, but there was more good for the future of society bound up in these primitive efforts than in volumes of disputations at the universities on theses of the schoolmen. This was the world of scientific enthusiasms into which Newton was born and these were the men whom he drew upon as colleagues and worshippers.

The interesting partnership between English men-of-the-world and English scholars in promoting the New Philosophy is certainly emphasized when we

Ferguson, "provide the comforting sense of belonging to an exclusive society, a sense so dear to the hearts of gregarious men."

join together such names as Samuel Pepys and Isaac Newton, each of whom served as president of the Royal Society within an interval of twenty years. Newton became a fellow of the Royal Society in 1672 and president of the society thirty years later. Samuel Pepys became a fellow in 1664 and president in 1684. Newton's name, moreover, is immortalized with Pepys' in another and very interesting manner. The Royal Society not only issued printed transactions from time to time, but it also undertook to print, at its own cost, meritorious scientific books and treatises. It thus came about in 1687 that *Philosophiae Naturalis Principia Mathematica*, was brought out by the Royal Society. On the title page we read "I. Newton, he wrote it," and below this, "Samuel Pepys, President Royal Society, he printed it." This was a remarkable linking of names. Newton, we all know, led a simple life, quite devoid even of the convivial pleasures enjoyed by nearly all of his associates. Newton was a man of temperate and abstemious habits, and Samuel Pepys an example of the animalism of the seventeenth century. But the title page does not tell the whole truth.The Royal Society, it is true, had promised to publish the book, but it was actually Halley the astronomer who furnished the funds from his own meagre purse. It is difficult to exaggerate the indebtedness of the world to Halley in the production of the *Principia*. He not only furnished the means for the prompt issue, but he was the intermediary between Newton and the Society and others and the wise counsellor to all of the many temperamental persons that were involved in its issue. In fact, if it had not been for Halley there might never have been a *Principia*, and Newton might now be known primarily as an important experimenter in the field of optics and not as one of the two or three great scientists of all time.

Trinity College, Cambridge, was the last place on earth to which you would send a boy in the seventeenth century for training in experimental philosophy. Newton was sent there because his mother's brother, the Reverend Ayscough, was a Trinity man. It was an excellent place for one seeking holy orders. But Newton was a person who needed no teachers in science—he could profit most from the absence of tutors and freedom from task masters. Trinity College fortunately gave Newton what he needed most—hospitality, freedom, and a friendly spirit. Most colleges of Oxford and Cambridge would have considered Newton at that time as an intolerable nuisance. The vile smells from his alchemy and the noise and confusion from his work with hammers and files and chisels, and the mutilation of college property by the drilling of holes and driving of nails would have made him an unwelcome guest.

Cambridge not only tolerated Newton, the weird individual, but in Professor Barrow he found a friend and advisor and promoter of priceless value. Cambridge became, therefore, "the birthplace of Newton's genius, her teachers fostered his earliest studies, her institutions sustained his mightiest efforts," her spirit of freedom and the lack of all mechanical control promoted still further the independence and resourcefulness his early life had cultivated, and within her precincts were all his discoveries made that were made. He created of Trinity

College a temple wherein his spirit still broods and he quickened those college halls and those walls of stone with immortal life.

Newton considered himself primarily an experimental philosopher. His work on the composition of white light and the design of a reflecting telescope, he placed first in his interest. His notebooks were filled with epoch-making discoveries in mathematics and new and immortal discoveries in analysis, but he seems to have regarded these as of secondary or even of trivial interest. When his discoveries on the spectrum brought him into violent controversy with Linus and Lucas of Holland, he became so disgusted with science that he nearly quit philosophy. He longed to enter the study of law and was only saved to science by Professor Barrow, who decided in favor of his competitor when Newton applied for a fellowship in Law. Thus the world owes another debt to wise Professor Barrow for his understanding of the real Newton. Isaac Barrow was an eminent mathematician and a worthy tutor of the youthful Newton. He had wisdom enough to tolerate if not to encourage his brilliant disciple in his immortal excursions into experimental philosophy.

While Cambridge University was dismissed in 1666 on account of the "black death," Newton took refuge at the family farm at Woolsthorpe. It was then that the apple is said to have fallen. In any case he began at that time his speculations concerning gravity. But in testing out his theory of universal gravity he laid aside his computation of the moon's motion, because, as usually stated, the then accepted radius of the earth was five hundred miles too short and hence his calculations would not check properly against the known motion of the moon. Newton had plenty of other things to do. He was more than busy in his workshop or "elaboratory." He was as much enamoured with the "universal elixar" as with "universal gravity." He was grinding lenses, observing comets, playing with alchemy, and making his discoveries on the composition of white light and the properties of the spectrum.

Newton did not turn away from alchemy and return to his gravitational studies until 1684. In the meantime Picard had shown that the earth's radius was about 3,960 miles and not 3,450, the value taken by Newton. After using this new value to correct his former computation, Newton wrote: "The moon appears to be kept in her orbit purely by the power of gravity." According to Cajori the delay in the verification of the law of universal gravitation may have been due to the difficulty Newton experienced in solving the problem of the mutual attraction of two uniform spheres, which he did not clear up until the summer of 1685. Then Newton's real job began. The years 1685, 1686 and the early part of 1687 will ever be memorable in the history of science. It was in these years that the *Principia* was composed and given to the world.

Each age seems to have an outstanding problem to serve as challenge to its scholars. This is the age of the Michelson-Morley experiment. The challenge of this unsolved problem has produced more good electrodynamics and more good mathematics than thousands of successful investigations. The Michelson-Morley dilemma of Newton's day was the problem of the moon's motion, and

for more than a century after his time it was the challenge that called to the contest the great men of those days—Clairaut, MacClaurin, D'Alembert, Euler, Lagrange, LaPlace. These giants were developed in their strength by the problem of the moon. The problem had had in Newton's day a most practical appeal. If the complex course and irregular motions of the moon could be brought under a system of causes, then accurate tables could be computed and the position of a ship at sea could be determined. It seemed hopeless in those days to perfect a timepiece that would keep good enough time through long sea voyages so that longitude could be known. By the tables of the moon of Newton's day the longitude could not be determined within thirty minutes. About five of the irregularities in the moon's motion had up to that time been discovered, but no cause was known. Galileo explained the libration in latitude, and Newton promptly explained the libration in longitude. But why should the moon's speed vary in its quadrants? What effect should the changing distance from the sun have on these quadrantal speeds? Why should the perigee continuously advance? Does the moon's orbit change shape as the earth passes from the near position to the far position from the sun? Does the plane of the moon's orbit rock back and forth as the earth carries it about its own annual course? What effect has the oblateness of the earth upon the motion of the moon? These and many other questions coursed through Newton's brain as he contemplated the problem of the heavens. If the principle of universal gravitation could explain and tie together all these movements and all of the irregularities of movement, then there was hope that accurate tables of the moon could be prepared, that the seas could be securely navigated, and that truly a great service would have been rendered his race. The plea of the "practical" was strong with Newton as it was with many other great scientists. "Practical" is a very poor and much abused word; to Newton, the determination of longitude at sea meant safe journeys, and the stimulation of intercourse and the spread of civilization among the races of men. The "practical" often implies more altruism than the thoughtless are willing to admit.

Newton was able to print in the lunar theory of the *Principia* a fairly complete explanation of the variation, of the parallactic inequality, of the annual equation, of the retrogradation of the nodes, of the progression of the line of apsides, of the evection or variation in eccentricity, and of the variation in the inclination. Newton left an unpublished manuscript in which he had accounted for the motion of the perigee. No progress was made in explaining the secular acceleration for a century, until the time of Laplace, and later.

The *Principia* must be viewed, therefore, not as a mere book of science, but as an epic whose story is the sufficiency of the laws of mechanics and the principle of universal gravitation to explain the motions and all of the irregularities of the motions in the phenomena of the heavens. It is the most profound story ever put forth by human genius. The first two books of the *Principia*, if translated from their geometrical form into the language and symbolism of twentieth-century analysis, could hardly be distinguished from a modern text on dynamics

of a particle, so little have two centuries added to the work of the master. As one enters the last part of the treatise, he is overwhelmed by the uncanny intuition shown in the treatment of almost every topic. The fundamental problem of the mutual attraction of two uniform spheres, the motions of the planets about the central sun, the motions of the four satellites of Jupiter and of the moons of Saturn, are brought into the orderly march of the epic. Then the climax of the story is reached in the analysis of the complex motions of the moon. Next, by a reworking of some of the material of the lunar theory, the phenomena of the tides are explained, and, by the very ingenious device of considering the belt of excess material in the earth's equatorial zone as a continuous set of moons attached to a spherical earth, the precession of the equinoxes is accounted for. Throughout all these many chapters of the story, gravity with its law of inverse squares serves as the overmastering fate that sweeps in its control all the events to their appointed destiny. The human intellect had never before reached so noble a conception of the nature of things.

Newton worked at the composition of the *Principia* with an enthusiasm and a speed that was little short of fanatical. From 1683 to 1689 he employed a secretary, Humphrey Newton, to copy papers and prepare matter for the printer, etc. From this intimate source we know much about his personal habits. Humphrey says: "I cannot say I ever saw him laugh but once . . . his carriage was meek and humble . . . I never knew him to take any recreation or pastime either in riding out to take the air, walking, bowling or any other exercise whatever, thinking all hours lost not spent on his studies—so intent, so serious upon his studies that he ate very sparingly, nay oftentimes not at all, so that going to his chamber I have found his meal untouched, of which when I reminded him, he would reply 'Have I?' and then making to the table would eat a bite or two standing, for I cannot say I ever saw him sit at a table by himself . . . he very rarely went to bed until two or three of the clock, sometimes not until five or six, lying about four or five hours—he very seldom went to Chapel, that being the time he chiefly took his repose, as for the afternoon, his earnest and indefatigable studies retained him, so that he scarcely knew the house of prayer . . . in his chamber he walked so much you might have thought him to be educated in Athens among the peripatetic sect . . . when he has sometimes taken a turn or two he might make a sudden stand, turn himself about, run like another Archimedes with an Eureka, fall to write on his desk standing . . . his behavior was mild and meek, without anger, peevishness or passion, so free from that, you might take him for a stoic."

The *Principia* was issued by the Royal Society about mid-summer of 1687. As already stated the funds for its printing were really furnished by Halley though the Society had originally promised to bear the expense. This great work, as might have been expected, excited a warm interest in all parts of Europe. A copy could scarcely be procured by 1691 and at that time a new edition was already contemplated. While it is probable that not more than twenty people in England could understand it, nevertheless it became the

subject of parlor conversation and made great fame for its author. The universities, however, in the beginning, were little influenced by the new system. The Scottish universities, St. Andrews, and Edinburgh, were the first to teach the philosophy of the *Principia*. Oxford, with characteristic conservatism, continued to teach the vortex theory of Descartes for many years. Even Cambridge can hardly claim to have been an early supporter of the *Principia*. While Newton was at Cambridge, he practically had no hearers at all; often not a single person would show up at his lectures. When he left Cambridge in 1696 the Physics of Rohault was still in use as a text, but fortunately they used Clark's Latin translation of the French text, which contained copious notes in which the ingenious translator explained, without bias or controversy, the views of Newton on the principal objects of discussion, so that they virtually constituted a refutation of the text, but the student, naturally inclined to radicalism, would look into the notes, and of course contend for that view. Hence, we may say that the Newtonian philosophy first entered Cambridge surreptitiously and under the protection of the Cartesian philosophy.

It has never been claimed that the *Principia* is easy reading. There is a sense of unnaturalness about the book that a modern student finds difficult to cast off. In reading Archimedes' works for example, the student is not conscious of any artificiality; it is like the geometrical reasoning that he is already familiar with in more elementary form. But the processes and symbolism of theoretical mechanics are now so familiar to the reader that he finds it hard to follow reasoning without the use of the formulas and the methods of modern elementary analysis. Then there is a serious incompleteness due to the fact that Newton possessed no concept of "energy" or of "mechanical work" which are now so fundamental even in the most elementary treatment of mechanics. Then again those geometrical ratios so constantly occurring in the *Principia*, the reader knows are but trigonometric functions in disguise and he has to force himself not to pass over from the letter press to the methods and processes of the calculus. Of course Newton had used his method of fluxions in discovering the truth of theorems, but almost no vestige of the analytical process of discovery can be discerned in the text. As a matter of fact, the word "fluxion" does not appear anywhere in the book except in one lemma* where he seems to have forgotten to change over from the analytical process of discovery into the usual synthetic form of presentation. The eleven lemmas of Section I, Book I, on prime and ultimate ratios represent material intimately associated with Newton's theory of fluxions but it is hardly correct to regard these lemmas as introductory to the general methods of the calculus. They could quite as well appear independently in the *Principia*, even if Newton had never discovered his general theory of fluxions and fluents. In the famous lemma to Theorem V, Book II, he actually presents the rules for the fluxions of products and powers without using the term or its usual notation. In a scholium he refers to a letter to Collins of 1672 and to the then unpublished account of his method prepared in 1671.

* Lemma II, Book II.

There are two *Principias*. One is the formal *Principia* made up of the definitions and theorems in conventional and impersonal form; the other is the informal *Principia*, made up of the numerous and rather extended scholia liberally scattered throughout the work, to which should be added most of the matter of the third book. It is in these informal remarks and discussions that we are able to get a glance at the scientific personality of Newton. Here we begin to know him as the Natural Philospher and as an experimental and inventive genius. The *Principia* is much like a composite treatise on theoretical and experimental physics built up in layers of numerous strata from the contrasting departments —hard impersonal layers of mathematics interspersed with more pliant and unconventional layers of informal discussion and common sense tabulations and observations. It was this second *Principia* that seemed to be nearest to Newton's heart. His constant appeal was to nature itself. "*Hypotheses non fingo!*" ("I do not form hypotheses") he said, and he added in explanation: "We are certainly not to relinquish the evidence of experiments for the sake of dreams and vain fictions of our own devising; nor are we to recede from the analogy of Nature, which is wont to be simple and always consistent with itself."

"I have laid down," he says, "the principles . . . not philosophical but mathematical. These principles . . . lest they should have appeared of themselves dry and barren, I have illustrated here and there with some philosophical scholia, giving an account of such things as are of more general nature on which philosophy seems chiefly to be founded."

What is the most profound concept in the *Principia?* Perhaps it is the observation that the quantity of matter is always proportional to weight—or, as we now say, gravitational mass and inertia mass are identical—two centuries were required to bring out the full meaning of this fact in the theory of relativity. Newton was engrossed in its philosophical significance and at the apparent simplicity of structure of the universe it implies. He approached the proposition from every angle: the outer shell of a body does not insulate its interior from external gravity; changing the shape of a body does not change its total gravity at remote distances. He even contrasts these facts with magnetic action, for he says: "The power of gravity is of a different nature from the power of magnetism; for the magnetic attraction does not vary alone with the quantity of matter" but depends as well upon the kind of substance acted upon.

Genius cannot be defined, much less explained. That glover's son of Stratford-on-Avon was not selected by heredity nor by the training of schoolmen to become the poet of the ages. There was nothing in the blood of the simple farmer folk of Woolsthorpe, Lincolnshire, and little in the discipline of Trinity College, Cambridge, that could fashion a creator of modern science. The world and heredity may have rough-hewn these characters, but we must conclude that it was a divinity, above and outside of them, that shaped their ends. We can do no better, therefore, than to say with the Greeks that genius is the gift of Zeus. His servants Melpomene and Thalia ministered unto Shakespeare and

his handmaid Urania ministered abundantly unto Newton. This is as good an explanation as the biologists or others have been able to give us. The world merely accepts or rejects genius and uses or abuses the great gifts. In reckoning the total riches of mankind, we hardly know what part to credit to the labor of the sweating multitude and what part to credit to the dispensation of Zeus. Gratitude is not one of man's conspicuous qualities. He is apt to look upon the bounty of nature, and the riches arising from the work of the many, and the great gifts brought to him by genius, as his by natural right and only coldly and formally to be acknowledged in an occasional perfunctory hymn of praise.

This year is the two hundred and fiftieth anniversary of the publication of the *Principia*. On the day of its issue in July, 1687, there began a new stepping up of the power of intellect over nature and a remaking of the ancient world into a modern world. Emerson says a great man is one who administers a shock to the world, and he names Newton as one who did. If recent industrialism is a blessing, give initial credit to the *Principia*. If industrialism means that an end-point has been fixed for the happy life on earth, place the blame on the *Principia*; for the book is but the primer of man's effort to understand and predict and control natural phenomena to his own good or to his own undoing, just as his moral powers happen to determine. Three centuries ago man still lived in awe of nature as he had been living from the day of his creation. He had been limited and ruled by nature as a slave is limited and ruled—everywhere he stood in dread of nature and without mastery over her. Fate represented the tragic element in life, just as it did in the ancient drama. The new mathematical philosophy brought about a reversal in the relation of man and nature. For the first time in human affairs, the question was no longer how nature could be prevented from overcoming man; it now was to what length may man go in subduing nature. He believes he has won the mastery—not complete, it is true, but with the balance of control in his favor. This consciousness of power has changed human outlook. Man now doubts the necessity of many of the hardships of life formerly regarded as inevitable. It is not enough that the fear of the "black death," or the dread of lightning, or the perils of the sea, or the superstitions of the heavens, should vanish. It is now believed, as a corollary to the doctrine of the *Principia*, that the common hardships of daily life, the cruel pressure of economic forces, are unnecessary and intolerable. The understanding and control over natural processes given by science, he is now convinced must be matched by a control over destiny itself. Man has not yet attained this mastery, but the *Principia* inaugurated a long schooling that has led him to reach a belief in its possibility. Man is no longer willing to bow down to fate or to resign himself to all the tragic elements in life as did the ancients; rather, he is demanding deliverance through the setting up of a new *Principia* and a bringing forth of an understanding of the forces at work in society.

Newton drew up the Magna Charta of man's power over his environment. The *Principia* is but a failure if man's power and the unity of existence once being revealed, it finally comes about that the intellect is insufficient to discern

and control the vectors and the orbits and the tides of action at work in the daily affairs of men. After the *Principia*, man can no longer assign his failures or his successes to the work of demons, or to the acts of fairies, or to superstitions, or to chance, or to the perils of the age, or to the intervention of the gods. The *Principia* has placed man at the center of his own universe. The frame of reference for all social phenomena is fixed with man as origin, and nothing external offers enlargement nor escape.

We need not debate today which of Art or Science has the greater meaning for the advancement of humanity. But there is a difference, we should note, in the way in which these two universals enter the *scheme of civilization*. Of the two, Art is the more permanent. The permanence of Art is only limited by the endurance of marble and clay and metals and of paint and paper and canvas. If a marble of Phidias should now be unearthed it would still charm with the beauty once given to it by the artist, just as the marbles and bronze and papyri and tablets of Egypt and Mesopotamia still reveal the inspirations of an ancient age. But with the works of science the story is different. The discoveries of Archimedes and Newton have now joined the ever-growing stream of scientific truth and for the most part are dissolved in that river, unrecognized and unlabelled. They have become a part of a greater whole. Art builds up the spiritual continents, majestic and eternal. Science forms the rivers flowing on into the final ocean of reality, where each contribution blends and becomes lost amid all the others. Aeschylus and Shakespeare built mountain ranges; Archimedes and Newton started rivers at their source. Therefore, in giving heed to this anniversary, we must remember that much of the *Principia* is now to be found amid the contributions of many men of sagacity into whose treatises the materials of the *Principia* has threaded its way. The greatness of the *Principia* has not vanished, merely its details are becoming a shadowy but cherished memory.

The *Principia* is a good book for occasional reference for the teacher of theoretical mechanics, especially for the teacher who has a good deal of confidence in his own ability. He can quickly recover his modesty if he will but turn to the very incomplete first theory of the tides, or even to the motion of a body in a resisting medium, which the teacher has probably taught his class for several years. The uncanny genius of Newton is apt to appear in almost any topic. Take the simple problem of the rising and falling body in a medium resisting as the square of the velocity. The college junior solves this problem as a matter of routine, the details for the rising body coming out in terms of circular functions and for the falling body in terms of hyperbolic functions. Turn now to Proposition VIII of the second book. There he will find a circle and a rectangular hyperbola whose semi-axes he can take as the terminal velocity of the falling body. Two associated points t and T on the circle and hyperbola are shown. The times of ascent and descent are proportional to the sectors of the circle and hyperbola respectively. The velocities at corresponding positions are laid off on the common tangent. There the whole solution is constructed geometrically, such that any velocity of projection leads immediately by plane geometry

to the final velocity at the level of the ground. No circular and no hyperbolic functions are directly in evidence, but the problem is fully solved in geometrical terms that actually anticipated both of them.

The *Principia* possesses the contradictory power of either augmenting the self-confidence of the elementary student, or of reducing to zero the haughtiness of the doctor of philosophy. If the modern scientist wishes to liquidate his mathematical arrogance, let him turn frequently to the pages of the *Principia*. For example, in the case of the famous scholium on stream-lined bodies, let the reader supply the proof, not given in the *Principia*, in terms of the mathematics of Newton's day. Modern emphasis on problems of streamlining has again directed interest to Newton's mathematical and experimental investigation of the resistance of a moving body in a medium. In his scholium on the stream-lined body, Newton said: "I conceive that this proposition may be of use to builders of ships." After 250 years, modern automotive engineering has taken the lead and made the public streamlined conscious. Even salesmanship has helped itself to the problem and given us streamlined household devices from cook stoves to refrigerators and even streamlined children's toys. Mathematicians also have been attracted to this part of Newton's work as one of the first serious problems set up and solved in what would now be called the Calculus of Variations. A thoughtful scientist has remarked to me that the problem is a fine example of Newton's scientific courage, for, after showing that a sphere is subject to but half of the resistance of a cylinder of like dimensions, Newton had the boldness to attack the problem of the body of *least* resistance. There existed no analogies and no evidence that this type of problem even lay in a realm susceptible of study by mathematics. No better proof of his genius can be submitted than the fearlessness with which he entered such umbral and unexplored regions.

Newton was born on Christmas day, 1642, a premature and weak infant that was hardly expected to survive a few hours. But that frail infant, so small that his mother said that he could have been placed in a quart measure, lived to the ripe age of eighty-five. For thirty years he had practically ceased to contribute to philosophy. He was past eighty before he began to suffer seriously from the torments of age. He had moved from Chelsea to Kensington to ease his infirmities and there on March 20, 1727, he passed away. On Tuesday, March 28, his body was borne to Westminister Abbey and buried near the entrance to the choir on the left side. At that place there has been erected a monument to one who, of all those of the Abbey, least needs a monument, and thereon in well-poised Latin words, is the eulogy to one who least needs praise: "Who by a vigor of mind almost divine, the motions and figures of planets, the paths of comets, and the tides of the seas, first demonstrated."

Hypotheses non fingo!

REQUIRED MATHEMATICS IN A LIBERAL ARTS COLLEGE

By W. L. SCHAAF, Professor of Education, Brooklyn College

1. *Introduction.* There is a growing movement throughout the country to eliminate mathematics as a required subject for students in liberal arts colleges. This unfortunate wave of sentiment is by no means altogether new. Almost ten years ago the Mathematical Association of America recognized a similar feeling of dissatisfaction when it appointed a committee to study the problem of collateral readings in mathematics for first and second year college students. In its report* this committee said: "It is generally recognized that there has been a strong tendency to decry the traditional course of instruction in mathematics for freshmen and sophomores in American Colleges. The objections urged are familiar: . . . that the drill designed for the specialists is wasted on those who do not continue mathematical study; that no glimpse of the philosophy or history of the subject is given to the students who can really assimilate only these features; that what is needed is a survey of what mathematics aims to accomplish, and not manipulative speed or problem-solving ingenuity; that the same disciplinary training can be acquired in any other study, while the added gain of lively interest and ready application makes other subjects more suitable than mathematics; that the classical languages are being laid aside save for the professional scholar, and mathematics, which partakes of their character, should share their fate; and so forth." To be sure, these objections can and must be answered; so, too, with even more searching criticisms. It must be conceded, however, that the traditional course usually offered and frequently required, consisting as it does of some algebra, some trigonometry, and some analytic geometry, can scarcely be regarded as adequate for those students for whom such a course is to be their last systematic contact with the subject. It would seem that the extreme formalism into which the teaching of the subject has fallen furnishes ample argument for the critics who would abolish it as a required subject.

Before proceeding to an analysis of the complaints lodged against mathematics, and the formulation of an effective reply to the alleged indictment, it should be added that not all authorities have yielded to these misdirected attacks or abandoned the cause. Thus, in the preface to his recent and charming book, Dresden [1]† avers his belief that "this movement, in as far as it is guided by sound educational principles, results from the lack of understanding, among educated persons in general and among educational authorities in particular, of the essential character of mathematics. There is little doubt that for this lack of understanding the teachers of the subject are in a large measure responsible. The fact that such a movement can gain adherence after mathematics has been for many years a required subject of study in schools and colleges points to a serious flaw in the manner in which the subject has been presented. There has

* This MONTHLY, vol. 35, 1928, pp. 221–228.

† Numbers in brackets refer to references at the end of the paper.

been too much emphasis on its formal and narrowly technical aspects, even where technique is not the end to be achieved, and neglect of the wider bearings, of the broad human implications of the subject, and of its more interesting and stimulating problems. . . . Because mathematics has a contribution of fundamental significance to make to the education of our people, we cannot allow false conceptions of the subject to weaken its influence. Mankind may be in a better position to deal with the baffling problems which confront it in the modern world if an understanding of mathematics were the rule rather than the exceptions."

2. *Facing facts.* Most of the criticisms of mathematics are offered in all sincerity, but many of them are superficial and can readily be answered. Such as are more profound must be recognized as partly or wholly valid. Upon closer analysis it seems to me that most of the objections fall into one or another of the following categories: (1) dissatisfaction with the content of the courses offered, or the manner in which that content is organized; (2) dissatisfaction with the technique of teaching, or the concomitant attitudes on the part of the teachers and students, or both; and (3) the outright denial of any potential educational values resulting from the study of mathematics.

Frankly, the present writer is quite ready to agree with many of the criticisms which fall under the first two headings. In the first place, an honest and courageous appraisal of the freshman mathematics required at present in colleges the country over would compel one to confess that much of it is hopelessly worthless and futile from any point of view. It is deplorable, for example, to contemplate the prevalence even today of traditional, hidebound "trigonometry" and "college algebra," organized in watertight compartments, each a collection of completely unrelated "topics," with excessive emphasis on mechanical manipulative processes and petty details, and with almost no indication whatever of their possible relation to vital problems of physical science, technology, economics, industry, business, and finance. Similarly, we cannot escape the conviction that many of the more recent "survey" courses, "integrated" courses and "appreciation" courses are far from adequate. Many survey courses suffer from attenuation, superficiality, or general mediocrity. The so-called integrated courses frequently attempt to cover entirely too much material; or the basis of correlation is artificial and forced; or else the student fails to see the unification because of the hodge-podge set before him. And the appreciation courses that appear from time to time are often much too general or too perfunctory to be genuinely convincing; students are expected to "appreciate" things with which they are not sufficiently familiar, and so fail to achieve the desired insights and attitudes which make appreciation possible.

In the second place, personal experience and observation again compel the writer to admit that much of the teaching to which freshmen are subjected in their mathematics is sadly in need of reform. Some of it is unbelievably petty and pedantic; much of it is mechanical and uninteresting, as if it were a necessary evil; and not a little of it is unimaginative and uninspiring. There is no

use in condoning the situation. When mathematics is taught by individuals without vision, without a love for the subject, and without conviction, then it deserves the fate for which it is apparently headed. Under such circumstances, are students to be blamed if they voice distaste, disgust, or abhorrence for mathematics? I hardly think so. Let us then, in the light of honest self-criticism, make a determined and forthright effort to improve the teaching of freshman mathematics through better selection and organization of content as well as better craftsmanship in teaching.

With respect to the third category, however, I must remain adamant. I cannot yield to those critics who deny the educational possibilities of mathematical study. I cannot possibly agree with those who maintain that, inherently, mathematics has nothing to offer in the way of enduring values, when properly taught, even to individuals of but average intelligence. This conviction is based on the following considerations. Unless I am entirely mistaken, the accepted notion of what constitutes a *liberal education* implies that profound attention be given to these claims: I. Cultural and disciplinary demands; II. The demands of a technic civilization; III. The demands of social-economic intelligence. Herein lie beyond doubt the real claims of mathematics as a contribution to a liberal education. This threefold defense I believe to be intrinsically sound and unanswerable. Let us examine these demands more closely.

3. *Culture and discipline.* Precisely what goes into the making of a cultured person tends to elude us when we seek to impound it within the limitations of language. It would seem that the essence of culture might properly be regarded as *style.* What differentiates the man of culture from his untutored fellow-beings is a difference in the *manner* in which he thinks and feels. It is exactly here that mathematics has important contributions to make. The distinctive feature of mathematics, that which sets it apart from all other domains of human achievement, is that it exemplifies, or, more strictly, it is, a *unique* style of thinking; one which, it should be added, is more intimately associated with man's aesthetic and emotional sensibilities than is generally supposed. The uniqueness of mathematics as a mode of thought results from the following features: (1) the formulation of generalizations; (2) the method of postulational thinking; and (3) the ceaseless quest for greater rigor.

Consider first the abstract formulation of universalities from specific instances. In a sense, this may be said to be the *leit-motif* of all mathematics. In achieving this goal, where particular instances are regarded as special cases of larger generalities, symbolism plays a vital part; so do abstractions and imagery, as well as language and semantics. These are rigorous disciplines. Yet they are not without their implications for humanity, as Dresden [**2**, p. 201] cogently suggests when he says that, "If there is to be true progress, any fundamental betterment in human existence, the advance must be in the direction of enlarged application of general principles, of the discovery of new principles of wider scope, and of a world-wide envisagement of the important problems. Thus in no small measure will the development of man's existence on this earth depend

upon his ability to grasp and appreciate more inclusive generalities, to deal intelligently with vaster ranges of human and natural phenomena. In the cultivation of such ability the abstract formulation of problems should play an increasingly important rôle."

In the second place, mathematics also manifests its unique nature by the method of postulational thinking. Stated a little differently, the essence of the method is deduction. It is in this spirit that the mathematician says, "If P, then Q"; he may be uncertain about P, but there is no doubt that Q is inevitable whenever P obtains. The latter consideration leads to the critical distinction between autonomous thinking and empirical thinking; the former refers to the realm of possibilities, the latter to the world of actualities. In the first case, postulates are deliberately selected in advance, while in the second case they are usually tacitly hidden, and we are faced with the additional problem of uncovering postulates that might have unconsciously been taken for granted. The nature of logically rigorous thinking has been happily expressed by Hutchins [**3**] when he says: "Logic is a statement in technical form of the conditions under which reasoning is rigorously demonstrative. If the object of general education is to train the mind for intelligent action, logic cannot be missing from it. Logic is a critical branch of the study of reasoning. It remains only to add a study which exemplifies reasoning in its clearest and most precise form. That study is, of course, mathematics, and of the mathematical studies chiefly those that use the type of exposition that Euclid employed. In such studies the pure operation of reason is made manifest. The subject matter depends on the universal and necessary processes of human thought. It is not affected by differences in taste, disposition, or prejudice. It refutes the common answer of students who, comformable to the temper of the times, wish to accept the principles and deny the conclusions. Correctness in thinking may be more directly and impressively taught through mathematics than in any other way."

Closely related to the arbitrary choice of postulates and the detection of tacit assumptions we find the third characteristic feature of mathematical thought, viz., the desire for rigor. Lack of space precludes an extended examination of this feature, but its humanistic bearings have been so aptly expressed by Dresden [**2**, p. 206] that we cannot refrain from a final excerpt: "What is of consequence for our immediate purpose is to recognize that an existence theorem, particularly if it asserts unique existence, gives significance to procedures which otherwise would be purely formal. Before we set out on an elaborate quest for something or other we may well inquire as to the conditions under which the thing we are looking for is known to exist. The application of such a point of view to the study of social questions would unquestionably have a profound effect. We have only to suggest that those who are directing the affairs of nations should consider it as an essential part of the efforts to establish world peace to inquire whether this ideal state can exist no matter what may be the economic structure of society; or that those who are responsible for the educational policies of a district should inquire as to the conditions necessary for the existence

of the educational ideal which they pursue. Insistence upon at least a consideration, if not a solution of the existence problem in connection with social and economic policies would exercise a very significant, and I think wholesome influence upon the development of human life. Mathematical education can implant an understanding of the significance of this problem; we should not be satisfied with teaching that does not make this contribution to the education of its pupils."

4. *Technics and civilization.* The imperative need for a minimum understanding of the rôle played by mathematics in a civilization based upon science and technology would appear to be so obvious as to require but little elaboration. Indeed, the matter has been ably discussed by many writers, including E. T. Bell, C. J. Keyser, A. N. Whitehead, and J. W. N. Sullivan, to mention but a few. We shall therefore merely indicate a few of the reasons why even a moderate acquaintance with mathematics affords a deeper and more intelligent appreciation of the bearings of science upon contemporary civilization.

To begin with, all pure science, and especially technology, rests fundamentally on the art of measurement, both direct and indirect. Not a single achievement of modern technics would be possible without precise measurements, coupled with the art of computation. Without mathematics, man could not rear his lofty buildings, or throw his majestic bridges across a bay, or burrow his tunnels beneath a river. Gigantic motor-driven vessels, streamlined trains, and transoceanic Clipper Ships would be impossible, even as it would be beyond the realm of reality to speak to one's friend across thousands of miles of land or sea, or hear a symphony half way around the world. Less dramatic, but none the less vital, are the multitude of instances in applied technology where precision measurement is indispensable—those unsung robots in plants and factories, the huge printing presses, rolling mills, power looms, lathes, drills, trip-hammers, punch presses and countless intricate machines of modern fabrication and processing. The same is true of indirect measurements; all the refinements and remarkable achievements of surveying, navigation, and astronomy are based on mathematical methods.

Perhaps the most vivid example of the intimate relationship between science and mathematics appears when we consider the phenomenon of change. It is here that mathematics has repeatedly been of inestimable service to science. The phenomena of physical science are forever exhibiting changes, and the development of differential equations has indeed proved to be a powerful tool of thought. As a matter of fact, it is probably not too much to say that modern science, at least physics and mechanics, began with the invention of the calculus by Newton and Leibniz, or possibly a little earlier with the work of Kepler. Since that time the progress of physics, chemistry, engineering, and more recently even biology, has paralleled the progress of mathematical analysis. This is not to say that the only branch of mathematics which has contributed to the development of science is the calculus, for many other fields have played their parts as well; consider the significance of the theory of complex

numbers and vector analysis as a foundation for the theory of alternating currents, or the importance of non-euclidean and projective geometries for the relativity principle in physics. More recently the theory of matrices is being employed in the study of psychological, biological and even social phenomena in connection with the analysis of composite traits. By and large, however, differential equations may be said to comprise the key which enabled mankind to unlock the secrets of nature.

Finally, there remains the mathematical theory of probability. When we consider the quantum theory of Planck and the wave mechanics of Heisenberg, we have an example of the tremendous influence of pure mathematics upon physical and experimental science. These more recent developments have forced a reconsideration of the principles of causality from the standpoint of indeterminateness; thus the question of the probability of a phenomenon becomes of paramount importance. The philosophy of modern science is in the very act of shifting; statistical and probability theory are the scene-shifters. Biometric and statistical methods in biology were accepted with the classic work of Karl Pearson. Since that time the theory of probability has been applied effectively to many problems of atomic structure and molecular forces as well. In short, the physics, chemistry, and biology of tomorrow will find the theory of mathematical probability indispensable to further advances.

5. *Socio-economic bearings*. Many events of the past few years, both national and world-wide, have dramatically emphasized the imperative need for a citizenry that is critically intelligent concerning matters of social and economic import. Without doubt the reader, through personal experience and observation, is acutely aware of the implications of this contention, so that further elaboration is scarcely necessary. It should be equally clear that the intimate relationship of mathematical training to a proper understanding of the social sciences is germane to the demands of a liberal education. I have discussed the question of quantitative thinking elsewhere [**4**], pointing out that the average intelligent man or woman does not ordinarily possess any great degree of sensitivity to the *quantitative* aspects of relationships, since their *qualitative* side makes by far the greater and more natural appeal. Sensitive discrimination with regard to numerical and quantitative matters is by no means a general trait, and hence requires cultivation. This can be met only by suitable mathematical training.

The urgent need for quantitative thinking has been shown by Rosander [**5**], of the General Education Board, as follows:

"We cannot overlook the widespread existence of social, economic, and political illiteracy among our people. Numerous observations and investigations could be cited to illustrate this point. As long as social ignorance is rampant, it ill behooves us to maintain in the curriculum subject matter which does not directly contribute to the elimination of the deficiency. . . . Most of the reasons given for the study of mathematics appear to be little more than rationalizations, or justifications of why we should continue in the path over which we have always traveled. . . . Mathematics is the only field which deals with quan-

titative thinking; that is the unique principle underlying its organization. In a quantitative age such as ours, quantitative thinking is just as important as ordinary linguistic thinking; in fact one may argue that it is more important. . . . Experience with this type of thinking should, therefore, be required of all students. Moreover, mathematics must deal with those principles and techniques which will make the individual more accurate and more rational in his thinking and behavior. We make the assumption here, and we have some support for it from observation and investigation, that one reason why citizens are not as socially intelligent as they might be is that they are quantitative illiterates."

6. *Attempts at improvement.* It would seem reasonable to urge that considerations of cultural, technic, and social demands constitute values that mathematics can and should contribute to a liberal education. If this be so, we have at least established the possibility of a highly desirable goal. There still remains the problem of how that goal shall be effectively reached. How mathematics should be taught is just as important as why it should be taught at all. *Unless both of these claims can be satisfactorily established, the case for required college mathematics must fall.*

What has already been done in the way of attempted improvements? Perhaps the least successful solution has been in the direction of so-called "appreciation" courses. It appears that most of these efforts have been singularly unsuccessful. Some years ago the writer attempted [6] this sort of thing with more or less mature college students, but the results were none too gratifying. More recently there have been further attempts to offer materials in the appreciation of mathematics. I have in mind a very suggestive and readable book [7] by Logsdon, which has much to commend it. But to secure genuine appreciation, there would seem to be required a vigorous and thorough course of study.* Any course aiming at appreciation and based solely on such a book, however meritorious, would have to be substantially augmented by supplementary readings or definitely reinforced by inspirational lectures at the hands of a scholarly person. It should also include among its demands upon students the performance of challenging tasks. Under any other circumstances, such a course would be much too tenuous to deserve consideration. Perhaps a somewhat more hopeful method of approach may be through the so-called survey courses, which have latterly enjoyed considerable vogue, if not always popularity, in other fields as well. Such courses, however, usually entail several significant shortcomings. For one thing, they frequently attempt to include excessive amounts of material; for another, the separate topics considered are frequently entirely unrelated and detached; and finally, the individual units are in themselves often inadequate. A variant of the survey course is the course in general analysis, or so-called "integrated" course, in which the matter of intercorrelation has been more or less successfully handled by various writers. One great difficulty still remains with

* Not being familiar with the courses of study offered at the University of Chicago, the author intends no criticism in what follows.

survey courses: they require not only particularly skillful teaching, but sympathetic understanding of the purpose to be achieved. The present writer is convinced that in not a few instances, what might otherwise have been successfully conducted survey courses have proved to be miserable failures primarily because the accompanying attitudes were fatal to its success. A still more recent and rather novel solution is implied in the point of view of Dresden, as exemplified in his book, *An Invitation to Mathematics*, to which we have already alluded. This proposal offers many important contributions to the solution of the problem. It is as yet too early to evaluate this attempt, and the writer is certainly in no position to do so. It is felt, however, that the book is too idealistic and far too difficult for a *required* course; in short, the mental maturity implied is such as might be expected only of selected students who might take the course of their own choice, and then only when somewhat more mature, let us say as juniors or seniors, after having learned a little mathematics. It must be emphasized at this point that none of the above proposals makes any notable attempt to include historical material; and I do not here allude to the conventional chronological narrative, but rather to the style of development so effectively used by E. T. Bell in his *Men of Mathematics* and by Tobias Dantzig in his *Number: The Language of Science*.

7. *Conclusion*. Having ventured thus far into the rather dangerous business of proffering suggestions which might clarify the issues involved in the hope of bringing about even slight improvement, we shall be incautious enough to proceed a little further. We shall assume that, from the standpoint of a liberal education, the entire array of human achievements which comprise culture and civilization, both past and present, can be identified with one or more of a very few areas, to wit: *letters*, *fine arts*, *political science*, *pure science*, and *mathematics*. We shall assume further that no scheme of liberal education can be considered adequate unless it provides for a minimal experience, not with some, but with *each* of these five areas or domains.

In thinking further about the problem, we soon encounter another question: where shall we place the chief emphasis,—on the cultivation of intellectual power, on the appreciation of human values and social significance, or on an understanding of historical perspective in relation to other fields? This question admits of conflicting opinion. While I confess I cannot answer it, my own inclination would definitely lean to the first, calling attention to the other two matters from time to time in passing. A consideration of the possibilities of cultivating intellectual power suggests the problem of transfer of training. The possibilities of transfer lie not only in the selection of the *content* of the courses, but far more importantly in the *teaching technique* employed. Instruction should *consciously* strive to develop in the student a *desire* for the power to employ his mathematical training functionally. The techniques available for this purpose are reasonably well understood and need not be described here. We merely wish to point out that there is ample evidence for a hopeful outlook. As Judd [8] has recently suggested, "at the higher levels transfer is typical, not excep-

tional. Indeed the function of the higher mental processes is to release the mind from particulars and to create a world of general ideas. Thus, when the intellectual efforts of the race evolved a number system, it became possible to deal readily with every situation in which quantity is involved; when languages were developed, men found themselves in possession of means of communicating on every conceivable topic. The psychology which concluded that transfer is uncommon or of slight degree is the psychology of animal consciousness, the psychology of particular experiences. The psychology of the higher mental processes teaches that the end and goal of all education is the development of systems of ideas which can be carried over from the situations in which they were acquired to other situations. Systems of general ideas illuminate and clarify human experiences by raising them to the level of abstract, generalized, conceptual understanding."

To the present writer, then, the ruthless rejection of all required mathematics is unthinkable. If such a position is taken, why stop there? Why not abolish all requirements in one or more major fields? Why not, indeed, invite disaster by permitting the unguided neophyte complete freedom and discretion with regard to his choice of studies? It would seem that if one more step were needed to accelerate the trend toward increasing mediocrity and superficiality in American colleges, this is it.

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A PROJECTIVE GENERALIZATION OF CERTAIN FOCAL RELATIONS

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A problem that has long interested geometers is that of focal curves, viz., "What is the locus of the foci for various systems of conics?" Chasles, in his theory of characteristics, which he developed about 1864 [1], gives the degree of the curve and something about its singular points for many such systems. Later, in the same volume of the *Mathematische Annalen*, Durège [2] and Schröter [3] treated synthetically the focal curve of a range of conics, and the latter gave a generalization of the curve, which he generated by means of projective involutions of rays. Bobek in 1892 [4] obtained, also by the synthetic method, the focal curve of a pencil of conics; and Bauer [5] and others have treated the same curve analytically.

It has long been known [6] that if f, g, h, are the trilinear coördinates of one focus of a conic which is tangent to the three sides of the fundamental triangle of the coördinate system, the coördinates of the other focus are certain multiples of the reciprocals, $1/f$, $1/g$, $1/h$. A projective generalization of this relation between the foci leads to the transformations that it is proposed to consider. One definition of the foci of a conic is that they are the real intersections of tangents drawn to the curve from the circular points. These tangents also intersect in two imaginary points, sometimes called the imaginary foci. This conception is to be generalized by replacing the circular points by any two points. Throughout the discussion which follows, however, they are to be thought of as real external points. The four tangents then intersect in four real points.

1. *The geometrical correspondence.* Let A_1, A_2, A_3 be the vertices, and a_1, a_2, a_3 the sides, of a fixed real triangle, let Z and Z' be fixed real points not lying on a side of the triangle nor collinear with a vertex, and let X be a variable point (Figure 1). For a general position of X the three lines a_1, a_2, and a_3 and the two lines ZX and $Z'X$ may be considered as tangents of a conic, and as such they are just sufficient to determine it. From each of the points Z and Z' a second tangent to the curve can be drawn and these intersect in a point X'. These tangents can be found from the five given tangents by an application of Brianchon's Theorem. They intersect ZX and $Z'X$ in two further points which we denote by Y and Y'. We seek the relations existing between the four points of intersection.

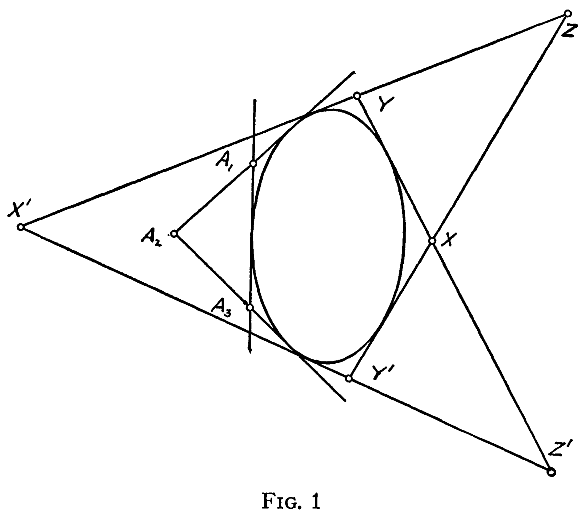

FIG. 1

2. *Analytic form of the relations.* If a system of projective coördinates be introduced with $A_1A_2A_3$ as fundamental triangle, with $Z=(z_1, z_2, z_3)$ and $Z'=(z_1', z_2', z_3')$, the Brianchon constructions can be followed through analytically step by step and yield the coördinates of $X'(x_1', x_2', x_3')$ in terms of those

of $X(x_1, x_2, x_3)$. After discarding a common quadratic factor these equations take the form

$$\rho x_1' = z_1 z_1' x_2 x_3, \qquad \rho x_2' = z_2 z_2' x_3 x_1, \qquad \rho x_3' = z_3 z_3' x_1 x_2. \tag{1}$$

This is a well-known quadratic transformation of exactly the same form as that discussed by Daus in a recent article in this MONTHLY [7].

The points $Y(y_1, y_2, y_3)$ and $Y'(y_1', y_2', y_3')$ may now be found as the intersections of ZX' with $Z'X$ and ZX with $Z'X'$, respectively:

$$\begin{aligned} \rho y_1 &= z_1(zx)_1(z'x)_2(z'x)_3, \\ \rho y_2 &= z_2(zx)_2(z'x)_3(z'x)_1, \\ \rho y_3 &= z_3(zx)_3(z'x)_1(z'x)_2; \end{aligned} \tag{2}$$

$$\begin{aligned} \rho y_1' &= z_1'(z'x)_1(zx)_2(zx)_3, \\ \rho y_2' &= z_2'(z'x)_2(zx)_3(zx)_1, \\ \rho y_3' &= z_3'(z'x)_3(zx)_1(zx)_2. \end{aligned} \tag{3}$$

These cubic relations are to be investigated further.

3. *The transformation* (2). Equations (2) represent a birational transformation; they can be solved for the x's and yield equations of the same form:

$$\begin{aligned} \rho x_1 &= z_1(zy)_1(z'y)_2(z'y)_3, \\ \rho x_2 &= z_2(zy)_2(z'y)_3(z'y)_1, \\ \rho x_3 &= z_3(zy)_3(z'y)_1(z'y)_2. \end{aligned} \tag{4}$$

In this transformation, to the straight line whose equation is $v_1x_1+v_2x_2+v_3x_3=0$ there corresponds a cubic curve

$$v_1z_1(zy)_1(z'y)_2(z'y)_3 + v_2z_2(zy)_2(z'y)_3(z'y)_1 + v_3z_3(zy)_3(z'y)_1(z'y)_2 = 0 \tag{5}$$

which passes through the fixed points A_1, A_2, A_3, Z and Z', and has a double point at Z'. Two such curves have, therefore, one free point of intersection, the image of the point of intersection of the lines to which they correspond. The tangents of (5) at the double point may be found as follows. Let $K(k_1, k_2, k_3)$ be an arbitrary point. Any point on the line $Z'K$ has the coördinates $z_i' - \lambda k_i$, and lies on the curve if these satisfy (5). Two of the values of λ obtained by substituting $z_i' - \lambda k_i$ for y_i in (5) are immediately seen to be zero, and the third will also be zero if

$$v_1z_1(zz')_1(z'k)_2(z'k)_3 + v_2z_2(zz')_2(z'k)_3(z'k)_1 + v_3z_3(zz')_3(z'k)_1(z'k)_2 = 0. \tag{6}$$

This is therefore the equation of the tangents at Z' in the running coördinates k_i. To write this in convenient factored form we put $w_i = v_i z_i z_i'(zz')_i$, and further let $W = w_1^2 + w_2^2 + w_3^2 - 2w_2w_3 - 2w_3w_1 - 2w_1w_2$. The equations of the tangents are then

$$(7)\qquad 2w_1\frac{k_1}{z_1'}-\frac{k_2}{z_2'}(w_1+w_2-w_3)-\frac{k_3}{z_3'}(w_1-w_2+w_3)=\pm\left\{\frac{k_2}{z_2'}-\frac{k_3}{z_3'}\right\}\sqrt{W}.$$

If $W>0$, the curve has an ordinary node, if $W=0$, a cusp, and if $W<0$, an isolated point at Z'. To interpret this condition geometrically, note that the equation $W=0$, considered as an equation in line coördinates v_1, v_2, v_3, if transformed into point coördinates can be recognized as the conic determined by the five points A_1, A_2, A_3, Z, and Z'. If the line v is a tangent of this conic the corresponding cubic has a cusp; if v is an external line, Z' is an isolated point; if v is a secant, there is an ordinary node at Z'.

The cubic relation (2) belongs to a well known class of transformations, that bear the name Central Arguesian Transformations [**8**], and that may be described as follows: let C be a point on a cubic curve; the transform P' of any point P of the plane lies on the line PC and is the mate of P in the involution whose double points are the other two intersections of PC with the cubic. The cubic itself thus appears as the locus of self-corresponding points in the transformation. The equation of the locus of double points for the transformation (2) is easily seen to be the cubic

$$(8)\qquad x_2x_3z_1z_1'(zx)_1+x_3x_1z_2z_2'(zx)_2+x_1x_2z_3z_3'(zx)_3=0,$$

a curve passing through Z and Z', through the vertices A_1, A_2, and A_3 of the fundamental triangle, and through the projection of Z upon each side of the

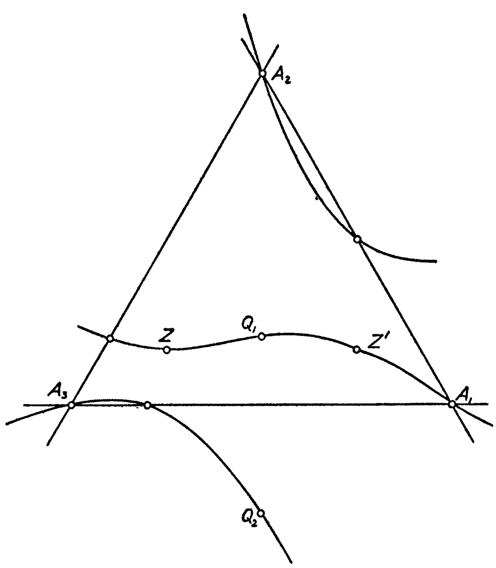

FIG. 2

triangle from the opposite vertex. The tangents of the curve at A_1, A_2, A_3, and Z pass through Z', and on these lines the involutions are degenerate. Further

points on the curve are the four points $Q_i=(\sqrt{z_1z_1'}, \pm\sqrt{z_2z_2'}, \pm\sqrt{z_3z_3'})$, the self-corresponding points of the quadratic transformation (1) (Figure 2). This situation resembles that recently discussed in this MONTHLY by Weaver, Musselman, and Goormaghtigh [**9**, **10**, **11**].

4. *The transformation* (3). This differs from (2) only in the interchange of the two points Z and Z'. This does not affect W; hence cubics corresponding to the line v under (2) and (3) have the same sort of a double point, one at Z', the other at Z. The cubic curve of double points for this transformation has in common with (8) the nine points Z, Z', A_1, A_2, A_3, Q_1, Q_2, Q_3, Q_4.

5. *A multiple correspondence.* The transformations (1), (2), and (3) may be combined into one by the following considerations. For a fixed position of X the conic determined by the five tangents a_1, a_2, a_3, ZX, and $Z'X$ has as its equation in line coördinates

$$\begin{vmatrix} u_2u_3 & u_3u_1 & u_1u_2 \\ (zx)_2(zx)_3 & (zx)_3(zx)_1 & (zx)_1(zx)_2 \\ (z'x)_2(z'x)_3 & (z'x)_3(z'x)_1 & (z'x)_1(z'x)_2 \end{vmatrix} = 0.$$

Expanding and removing the factor (xzz'), we obtain

$$(zx)_1(z'x)_1x_1u_2u_3 + (zx)_2(z'x)_2x_2u_3u_1 + (zx)_3(z'x)_3x_3u_1u_2 = 0. \tag{9}$$

The conic determined by a_1, a_2, a_3, ZX', and $Z'X'$ would have an equation of exactly the same form, x' replacing x throughout. If X and X' are to be corresponding points in the transformation with which we started, these two conics must be the same, and that requires that the coefficients of u_2u_3, u_3u_1, and u_1u_2 be proportional. Hence we get two equations connecting the coördinates of X and X',

$$\begin{aligned} (zx)_3(z'x)_3x_3(zx')_2(z'x')_2x_2' - (zx)_2(z'x)_2x_2(zx')_3(z'x')_3x_3' &= 0, \\ (zx)_1(z'x)_1x_1(zx')_2(z'x')_2x_2' - (zx)_2(z'x)_2x_2(zx')_1(z'x')_1x_1' &= 0. \end{aligned} \tag{10}$$

If X be regarded as fixed so that the x's are constant, these are cubic equations in the x''s; each of them taken by itself represents a cubic curve and the point X' must be a common point of these curves. Of their nine points of intersection five are the fixed points A_1, A_2, A_3, Z and Z'. This leaves four points X' corresponding to a general point X, a multiple correspondence instead of the 1–1 correspondence effected by the original construction. The reason is not far to seek. The condition that equations (10) impose upon X' is that lines drawn from it to Z and Z' shall determine with a_1, a_2, and a_3 the same conic that is determined by ZX, $Z'X$, and a_1, a_2, and a_3. In addition to the point X' constructed there are three other points that meet this requirement, viz., the point X itself and the points we have called Y and Y'. Equations (10) may therefore be considered as combining the transformations (1), (2), and (3), together with the identical transformation. From their form it is obvious that an interchange of X and

X' does not alter this relationship; it may therefore be called a 4–4 involutory transformation.

6. *The range of conics.* We have viewed (9) as the equation of a conic determined by a fixed point X. If, on the contrary, we consider u as a fixed line and X as variable, this same equation is satisfied by the coördinates of those points X for which the conic touches the four lines a_1, a_2, a_3, and u; that is, it represents the locus of the point X for the range of conics determined by these four tangents. It is a cubic curve which clearly passes through the points A_1, A_2, A_3, Z, and Z', and it is not difficult to see that it also contains the points in which u cuts the sides of the triangle $a_1a_2a_3$. This is the curve discussed by Schröter [**3**].

But further, if in this equation the values of x_1, x_2, and x_3 in terms of x_1', x_2', and x_3' are substituted from the quadratic transformation (1), the result is

$$x_1' x_2' x_3' \left[(zx')_1(z'x')_1x_1' u_2u_3 + (zx')_2(z'x')_2x_2' u_3u_1 + (zx')_3(z'x')_3x_3' u_1u_2\right] = 0,$$

which shows that the locus of X' is made up of the sides of the coördinate triangle and the same curve that X traces. The curve is then said to be invariant under the quadratic transformation (1) [**12**].

Similarly the locus of Y is made up of this same cubic, the conic $W=0$, and the lines $Z'A_1$, $Z'A_2$, $Z'A_3$, and $Z'Z$, and the locus of Y' consists of the cubic, the conic $W=0$, and the lines ZA_1, ZA_2, ZA_3, and ZZ'.

7. *Summary.* The results obtained may be summarized as follows: Consider the web of conics tangent to the sides of the fundamental triangle $A_1A_2A_3$. Let a complete quadrilateral be circumscribed to one of these conics, and designate its pairs of opposite vertices by X, X'; Y, Y'; Z, Z'. If one pair of these points, Z, Z', is kept fixed while the other four points are allowed to vary, subject only to the condition that the quadrilateral circumscribe a conic of the web, the relation between opposite vertices is quadratic, while that between points of different pairs is a Central Arguesian Involution. The two cubic curves on which X coincides with Y and X' with Y'; X with Y' and X' with Y belong to a pencil with the base points A_1, A_2, A_3, Z, Z', and the four common self-corresponding points of the quadratic transformation between X and X' and that between Y and Y'. When one of the four variable points coincides with one of these self-corresponding points, the other three coincide with it.

If from the web of conics a range is selected by the introduction of a fourth tangent u, the loci of the four points include a certain common cubic curve through the vertices of the quadrilateral $a_1a_2a_3u$ and the points Z and Z'.

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THE TRISECTION OF AN ANGLE

By W. B. GIVENS, Fresno State College

Of the three famous problems in geometry which were unsolved by the ancient Greeks, the trisection of an angle has received the most attention. The fact that it cannot be solved under the classical restrictions, which permit the use of compasses and straight edge only, does not deter the poorly informed,

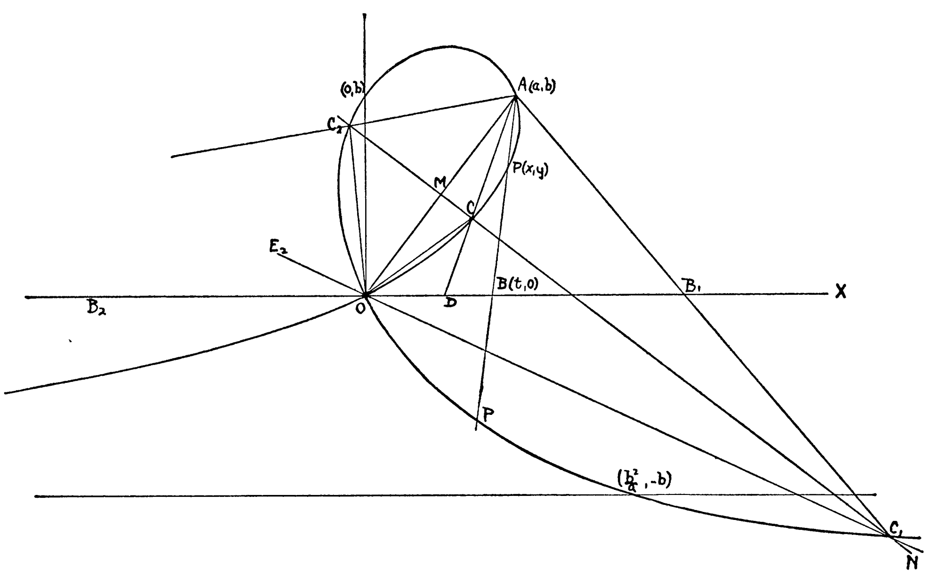

FIG. 1

but rather stirs him to make the effort to solve it [1]. So many persons submit what purport to be solutions that editors of mathematical and scientific journals no longer give consideration to most papers having the title of the present note,

but refer the writer to some proof like Klein's [2] or Dickson's [3] without comment. An interesting article, written for the general reader, appeared in a recent number of a scientific magazine [4].

The problem of trisecting an angle can, however, be solved, as mathematicians know, by the use of other implements or curves than those permitted by the classical restrictions. The ancient Greeks knew how to trisect an angle by use of the conchoid of Nicomedes or the quadratrix of Hippias. In an article published in this MONTHLY last year [5], Bussey discussed a number of devices for solving the problem, using marked straight edges, carpenter's squares, T-squares, etc.

It is my purpose to present what may be a new method for solving the problem by the use of an auxiliary curve. This curve may be described as follows: Let O be a point on a given line and A a point not on the line. Let B be a variable point on the line and let P be a point on the line AB such that $BO=BP$. The locus of the point is the curve. (See Figure 1.)

We shall derive an equation for the curve, taking O as the origin of coördinates, and taking the given line as the X-axis. Letting (a, b) be the coördinates of A and $(t, 0)$ the coördinates of B, and (x, y) the coördinates of P, we have

$$(x-t)^2+y^2=t^2, \qquad \text{and} \qquad \frac{y-b}{x-a}=\frac{y}{x-t}\cdot$$

On eliminating the parameter t, we obtain the desired equation

$$y^3+x^2y+bx^2-by^2-2axy=0.$$

An analysis of this curve shows that it passes through the point A, intersects the Y-axis at $(0, b)$ has a node at the origin, has a horizontal asymptote $y=-b$, is below the asymptote when $x>b^2/a$, and is above the asymptote when $x<b^2/a$. It is clear that the curve depends upon the position of the point A with respect to the point O and the line.

To trisect an angle AOB, we first construct the curve we have described, taking A at a convenient distance from O on one arm of the angle. Draw MN the perpendicular bisector of OA. Let C be the point of intersection of MN with the curve which lies inside the angle, AOB. Draw line AC, intersecting line OB at D. Then $DO=DC$ and $CO=CA$. It readily follows that angle $AOC=\frac{1}{2}$ angle $COB=\frac{1}{3}$ angle AOB.

It is not difficult to show that OC_2 is a trisector of the obtuse angle AOB_2, and that OC_1 is a trisector of the reflex angle AOB_2. By producing line C_1O, it can also be readily shown that line E_2O is the trisector of reflex angle AOB.

An exceptional case which we have failed to discuss is that in which $a=0$. In this case, the point C_1 does not exist.

Strictly speaking, we need to prove that the line MN intersects the curve in order to know that we have a right to speak of the points C, C_1 and C_2. To do this we note that the equation of line MN may be written: $2ax+2by=a^2+b^2$. By substituting the value of x or y from this equation in the equation for the curve, it can readily be shown that the three roots are real.

In a review of a first draft of this article, Professor W. H. Bussey notes that the characteristic property of the curve used here is given in a work by Gino Loria [6]. He calls it "Die Fokale von Quetelet oder schiefe Strophoide." The definition of the "Fokale" which Loria first gives is in terms of the foci of conic sections of a right circular cone, cut by planes through a tangent perpendicular to an element of the cone. The name "Fokale" comes from the fact that the curve is the locus of foci. Apparently, however, Loria has nothing to say about the use of Fokale for trisecting angles. After the wide investigation made by Professor Bussey together with some investigations made by Professor F. R. Morris and myself, I have been unable to find any place in which this curve has been used for the purpose of trisecting angles.

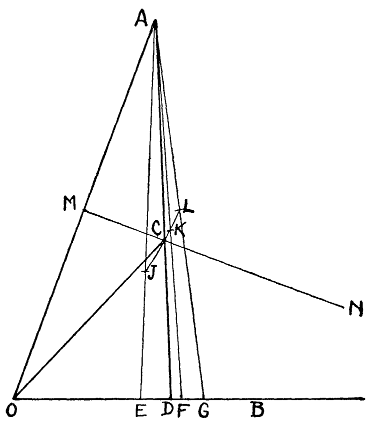

FIG. 2

As a construction problem, this method here developed offers an easy and accurate way for trisecting angles. (See Figure 2.) We wish to trisect angle AOB. Draw the perpendicular bisector MN to line AO. On OB lay off convenient distances such as EO, FO and GO. Draw lines EA, FA and GA. Lay off $EJ=EO$, $FK=FO$ and $GL=GO$. Through points J, K and L, draw a smooth curve intersecting MN at C. Draw line AC and produce to its intersection with OB at D. $DC=DO$ and $CO=CA$. Then angle $AOC=\frac{1}{2}$ angle $COB=\frac{1}{3}$ angle AOB.

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QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions, and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

ON THE CONGRUENCE $(p-1)!\equiv -1 \pmod{p^2}$

By EMMA LEHMER, Bethlehem, Pa.

In connection with my note on *Wilson's Quotient*, this MONTHLY, vol. 44, 1937, pp. 237–38, Dr. N. G. W. H. Beeger has kindly called my attention to an article by him in the *Messenger of Mathematics*, vol. 49, 1920, pp. 177–8, where a table of w_p (W_p in his notation) appears for $p<300$. This table was calculated directly without the use of Bernoulli numbers. Dr. Beeger has asked me to point out that the entries in his table for $p=127$, 167 and 173 are incorrect and should read as in my paper cited above. Also the entry for $p=241$ should read 196 instead of 34.

A METHOD FOR SOLVING QUADRATIC EQUATIONS*

By J. W. CIRUL, Chicago, Illinois

While on a farm, at the age of 13 or 14, and without access to a book of algebra, I decided that I would invent a way to solve equations of the second degree, and finally solved them by my own method, the like of which probably exists, but I have never seen it described in any algebra book as yet.

Consider $x^2+x=a$. The left side can be factored into $x(x+1)$, or two factors, one of which is greater than the other by *one*.

Whenever we find how to factor the given number a into two such factors we will in fact have solved equations of type $x^2+x=a$, as the smaller factor will be equal to x.

That was my starting point and method of reasoning. Then, I figured, since obviously $\sqrt{a}\sqrt{a+1}\neq a$, why not try $(\sqrt{a}-\frac{1}{2})(\sqrt{a}+\frac{1}{2})$, as the two factors are differing by unity only? Multiplying this out I found that the result was always equal to $a^2-\frac{1}{4}$, independently of the value of a.

By choosing factors $(\sqrt{a+\frac{1}{4}}-\frac{1}{2})(\sqrt{a+\frac{1}{4}}+\frac{1}{2})$ the product became equal to a, and one of the factors was greater than the other by unity. Therefore, the smaller factor was the root of the equation $x^2+x=a$.

From there it was but a simple step to solve equations of the form $x^2+bx=a$ by a similar chain of equations.

$$x(x+b) = a,$$

$$(\sqrt{a})(\sqrt{a}+b) \neq a,$$

* This note was submitted on the recommendation of Professor H. S. Everett, of the University of Chicago, with whom the author took correspondence study in mathematics. *Editor.*

$$\left(\sqrt{a} - \frac{b}{2}\right)\left(\sqrt{a} + \frac{b}{2}\right) = a - \frac{b^2}{4},$$

$$\left(\sqrt{a + \frac{b^2}{4}} - \frac{b}{2}\right)\left(\sqrt{a + \frac{b^2}{4}} + \frac{b}{2}\right) = a,$$

and therefore

$$\sqrt{a + \frac{b^2}{4}} - \frac{b}{2} = x.$$

Calling Signals

By B. C. Zimmerman, Corozal, British Honduras

Referring to the note on page 238 of the current Monthly, *Some Unfamiliar Ordinals*, it is a real misfortune from a pedagogic point of view that in very many languages there exists a varying irregularity in calling numbers in the second decade, and I regret that Mr. Campbell had no suggestion for renaming the cardinals of that decade.

For some years now in the several schools under my care the teachers have been introducing the second decade to the children by calling *ten*, *ten-y-one*, *ten-y-two*, $\cdots$, *ten-y-nine*, with a consequent orderly number concept that is truly gratifying.

I chose the "y" as the connector because nearly all the children speak Spanish [y=and], and because of the similarity to twenty-one, thirty-one, etc. The ordinals follow the same nomenclature, and the probability of Santiago's saying "The ten-y-second of May, ten-y-nine hundred thirty-seven" for 12 May, 1937 is high.

Not conspicuously more desirous for world reform than Mr. Campbell, I have instructed the teachers to use the scheme only as a bridge, teaching the children the ordinary names as they advance.

An Etymological Excursion

By A. S. Householder, Washburn College

In Professor W. B. Campbell's interesting remarks about the ordinals, made recently in this department, his point is not affected by his erroneous assumption that the words *eleven* and *twelve* are relics of an older duodecimal system. But the true etymologies of these words disclose some points of even greater interest, at least to those mathematicians who happen to be also dictionary fanatics. Referring to the *Oxford English Dictionary* (I shall speak of it as the OED), and to Fowler's abridgment, *The Concise Oxford Dictionary*, we find these words classified as common Teutonic forms, along with the German *elf* and *zwölf*, derived from the Old Teutonic **ainlif* and **twalif*. The first syllables of these words mean *one* and *two*. Among the Aryan non-Teutonic languages the only corresponding forms are said to be found in Lithuanian, where from

venolika = *eleven* to the word for *nineteen*, the number words are formed with the suffix *-lika* = *left over*. The OED compilers are cautious about giving the etymology for the Teutonic suffix *-lif*, but Fowler is more positive. It seems, then, that this is related to the Old Teutonic **liban* = *to leave*, and comes from the Old Aryan **lip* = *to adhere*, and hence *to remain*. This same Old Aryan root gives also *leave, live, believe*; Greek *lipos* = *grease* and *leipo* = *leave* (whence *ellipse* and *eclipse*); German *bleiben*, to list only a few. Thus the words *eleven* and *twelve* are clearly based upon a decimal system. It may be remarked in passing that the French *onze* is derived from the Latin *undecim*.†

But if we wish to find in language the vestiges of a duodecimal system, we may find it, surprisingly, in *hundred*, which often stands (the OED is still my authority) in older texts for the duodecimal "hundred." Other points about this word are of interest. The first syllable alone means "100," the suffix *-red* being related to Gothic *rathjan* = *to reckon*, and cognate with English *read* and German *Rede* and *Rat*. *Hund*, as well as Greek *ekaton* and Latin *centum*, comes from the Indo-European **kmtom*, and the various forms taken by this word furnish the philologists with one of their means of classifying the Indo-European languages.

Glancing at the ordinals, it is interesting to note that *first*, Old English *fyrest*, is a superlative formed on the Old Teutonic *fur-*, a stem which gave the English *fore* and the German *vor*. Hence *first* = *foremost*, and is the same as the German *Fürst*. Another superlative on the same stem, and a former synonym, is the now obsolete *forme*, earlier *frume* = *beginning*, identical with Latin *primus*. *Forme* acquired another superlative ending, becoming *formest* and finally *foremost*. From the double superlative *formest*, a superlative-comparative *former* was made by analogy. This same Old Teutonic *fur-* represents the Old Aryan *pr-*, and in most of the Aryan languages the words for *first* are formed on this stem, chiefly with superlative suffixes as in English. However, the German *erst* is a superlative of *ere*, and the identical form existed in early English. It is now practically obsolete except in the compound *erstwhile*.

It is interesting that along with the native *first*, *third*, etc., we have the French *second*. The reason is that Old English had no word for this but used the word for *other*. The French form was thus readily adopted to avoid ambiguity.

And should any discussion of number words neglect to recall that *zero* and *cipher* come from the same Arabic word *cifr* = *empty*?

On the Convergence of Newton's Method of Approximation

By G. T. Coate, Tulane University

Consider any function $f(x)$ satisfying the following conditions:

1. The equation $f(x)=0$ has a single real root within an interval $0<a\leqq x\leqq b$, and this root is irrational.

† However, Professor G. K. Anderson considers that O.E. *endleofantig* (110) and *twelftig* (120) may properly be regarded as remnants of a duodecimal system, and also possibly the use of *hund* in all O.E. cardinals beginning with *seventy* and running to *one hundred twenty*. Editor.

2. The function $f(x)$ is single-valued and continuous in this interval.

3. The derivatives $f'(x)$ and $f''(x)$ are continuous and have no real zero in this interval.

It is shown in Dickson's *First Course in Theory of Equations* that the graph of such a function in such an interval must have one of the four forms shown in the following figures:

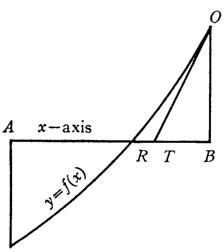


FIG. 1

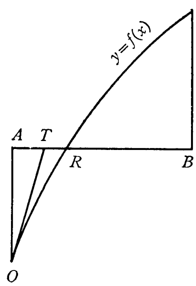


FIG. 2

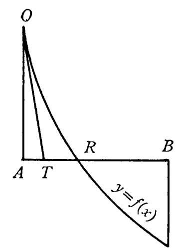


FIG. 3

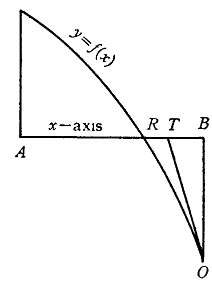


FIG. 4

Here the points A, B, T, and R have the abscissas a, b, t, and r, respectively, a and b being defined above, and r being the root of $f(x)=0$.

It is further shown that if Newton's Method is applied to this function, using as a first approximation to R the end of the interval at which $f(x)$ and $f''(x)$ have the same sign, an improved approximation is always obtained. If O be a point on the curve at this end of the interval, and OT the tangent to the curve through O, then t, the abscissa of the intersection of OT and the x-axis, is this improved approximation.

We shall prove the following theorem:*

If Newton's Method is applied a sufficient number of times to this function, the value of the root of $f(x)=0$ *may be found to any desired degree of accuracy.*

Let the abscissas of the points O and R be p and q, not respectively but so chosen that $p<q$. Then in each of the four figures $p<t<q$. For example, in Figure 1, $f''(x)$ is positive throughout the interval, and $f'(r)<f'(b)$. Hence the slope of chord RO is less than $f'(b)$. As $f'(b)$ is the same as the slope of TO, we have $TB<RB$. But in this figure $f'(x)>0$, hence $t<b$, and hence $r<t<b$. This is equivalent to $p<t<q$.

In the second application of the method, we draw a tangent to the curve through O_1, a point on the curve having the same abscissa as T. By the same reasoning, if T_1 is the intersection of this tangent with the x-axis, and if t_1 is its abscissa, then $p<t_1<q$. Similarly, after any number n of applications of the method, the foot of the tangent is T_n, its abscissa is t_n, and $p<t_n<q$. In Figure 5

* Cours de Mathématiques Générales, tome I, René Garnier, Gauthier-Villars et Cie, Paris, 1930, page 223 et seq. (Here a different and much longer proof is given.)

these points are indicated, but because of difficulties in drawing the tangent lines O_iT_i distinctly they are represented only schematically.

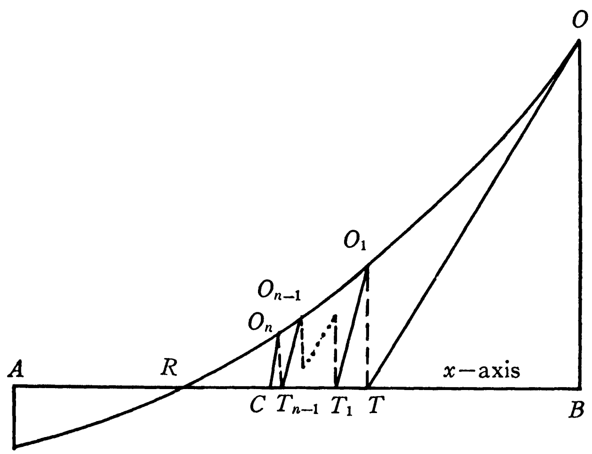


FIG. 5

Since t_n remains in this interval, and since we already know that t_n is a better approximation to r than t_{n-1}, we know that one of two things must occur as $n \to \infty$:

(1) T_n approaches R as a limiting position; or

(2) T_n approaches a point C, whose abscissa is c, as a limiting position and $p < c < q$.

Assume that (2) occurs. Then the series $TT_1 + T_1T_2 + T_2T_3 + \cdots$ converges to TC, and hence

$$\lim_{n \to \infty} T_{n-1}T_n = 0.$$

Also, the abscissa of O_n is t_{n-1} and $|t_{n-1} - r| > |c - r|$. Hence, since $f'(x)$ has no zero in the interval,

$$|O_nT_{n-1}| > |f(c)| > 0.$$

Hence

$$\lim_{n \to \infty} |\text{slope of } O_nT_n| = \lim_{n \to \infty} \left| \frac{O_nT_{n-1}}{T_{n-1}T_n} \right| = \text{infinity},$$

which contradicts the assumption that $f'(x)$ is continuous. Hence (2) cannot be true, and (1) is therefore true. Statement (1) implies that $|r - t_n|$ may be made less than any previously assigned positive number, and this completes the proof of the theorem.

AN EXAMPLE OF A CONTINUOUS FUNCTION WITH FINITE DISCONTINUITIES IN ITS SECOND DERIVATIVE

By W. R. LONGLEY, Yale University

The following problem is so hoary with age that it is difficult to trace its source:

Into a full conical wine glass of depth a and generating angle α there is carefully dropped a sphere of such size as to cause the greatest overflow. Show that the radius of the sphere is

$$\frac{a \sin \alpha}{\sin \alpha + \cos 2\alpha}.$$

This problem has caused thousands of students to give up in despair so that teachers are wise to avoid it in general assignments. Yet it has an interesting aspect and a peculiar value which, so far as the writer knows, has never been pointed out.

Students of elementary calculus can be expected to appreciate both finite and infinite discontinuities in functions but their experience will usually extend only to *infinite* discontinuities in the *derivatives* of functions. Illustrations of finite discontinuities in the derivatives of continuous functions are usually arbitrary and, to the practical minded, highly artificial. The problem above has the virtue of providing a concrete example, easily visualized, of a real function which is continuous and has a first derivative which is continuous for all positive values of the argument, but whose second derivative has finite discontinuities at two points.

The function involved is the volume V of the sphere which lies below the rim of the glass and this volume depends only on the radius r of the sphere. As r increases from zero, there are three stages to be considered.

In the first stage r is so small that the sphere is entirely submerged and, as r increases, this stage continues until the top of the sphere reaches the top of the glass. This occurs when

$$r = b = \frac{a \sin \alpha}{1 + \sin \alpha}.$$

In the second stage the sphere is partly submerged and rests on the side of the glass tangent to the cone along a circle below the rim. As r increases, the circle of tangency rises and the second stage continues until the circle reaches the rim of the cone. This occurs when

$$r = c = \frac{a \sin \alpha}{\cos^2 \alpha}.$$

In the third stage the sphere rests on the rim of the glass and this stage continues as r increases indefinitely from c.

Expressing the volume in terms of the radius by means of the proper geometric formulas, we have a function V of r defined for all positive values of r as follows. When $0<r\leqq b$,

$$V = f_1(r) = \frac{4}{3}\pi r^3;$$

when $b<r\leqq c$,

$$V = f_2(r) = \frac{\pi}{3\sin^3\alpha}\left[r(\sin\alpha - 1) + a\sin\alpha\right]^2\left\{r(2\sin\alpha + 1) - a\sin\alpha\right\};$$

when $r>c$,

$$V = f_3(r) = \frac{\pi}{3}\left[2r^3 - (2r^2 + a^2\tan^2\alpha)\sqrt{r^2 - a^2\tan^2\alpha}\right].$$

By direct substitution we find

$$f_1(b) = f_2(b) = \frac{4\pi a^3\sin^3\alpha}{3(1+\sin\alpha)^3};$$

$$f_2(c) = f_3(c) = \frac{\pi a^3\sin^3\alpha(2+\sin\alpha)}{3(1-\sin\alpha)(1+\sin\alpha)^3}.$$

Hence V is continuous at $r=b$ and at $r=c$.

For the first derivative of V we have, when $0<r\leqq b$,

$$\frac{dV}{dr} = f_1'(r) = 4\pi r^2;$$

when $b<r\leqq c$,

$$\frac{dV}{dr} = f_2'(r) = \frac{\pi}{\sin^3\alpha}\left[r(\sin\alpha - 1) + a\sin\alpha\right]\left\{r(2\sin^2\alpha - \sin\alpha - 1) + a\sin\alpha\right\};$$

and when $r>c$,

$$\frac{dV}{dr} = f_3'(r) = \pi\left[2r^2 - \frac{r(2r^2 - a^2\tan^2\alpha)}{\sqrt{r^2 - a^2\tan^2\alpha}}\right].$$

By direct substitution we find

$$f_1'(b) = f_2'(b) = \frac{4\pi a^2\sin^2\alpha}{(1+\sin\alpha)^2};$$

$$f_2'(c) = f_3'(c) = -\frac{\pi a^2\sin\alpha}{(1+\sin\alpha)^2}.$$

Hence dV/dr is continuous at $r=b$ and at $r=c$.

Carrying out the necessary calculations we find

$$f_1''(b) = \frac{8\pi a \sin\alpha}{1+\sin\alpha};$$

$$f_2''(b) = -\frac{2\pi a(1-\sin\alpha)(1+3\sin\alpha)}{\sin\alpha(1+\sin\alpha)};$$

$$f_2''(c) = -\frac{2\pi a(1-\sin\alpha)}{1+\sin\alpha};$$

$$f_3''(c) = \frac{\pi a(1-\sin\alpha)(1+2\sin\alpha-\sin^2\alpha)}{\sin^2\alpha(1+\sin\alpha)}.$$

Since α is necessarily an acute angle, it follows that

$$\lim_{r\to b}\frac{d^2V}{dr^2} = \text{a finite } positive \text{ number when } r < b.$$

$$\lim_{r\to b}\frac{d^2V}{dr^2} = \text{a finite } negative \text{ number when } r > b.$$

Hence the second derivative has a finite discontinuity at $r=b$.

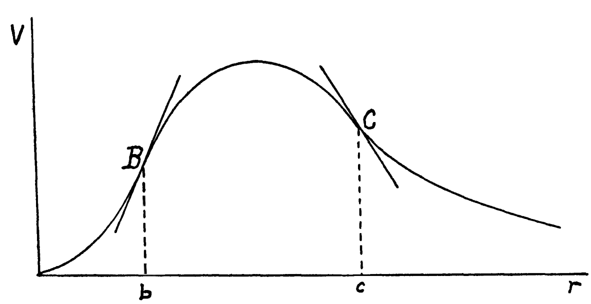
Also

$$\lim_{r\to c}\frac{d^2V}{dr^2} = \text{a finite } negative \text{ number when } r < c.$$

$$\lim_{r\to c}\frac{d^2V}{dr^2} = \text{a finite } positive \text{ number when } r > c.$$

Hence the second derivative has a finite discontinuity at $r=c$.

Investigation shows that dV/dr can vanish only in the second stage. The factor in square brackets in $f_2'(r)$ cannot vanish in the interval of definition.*



* The vanishing of this factor would make $V=0$ and would correspond geometrically to the situation in which the bottom of the sphere is on a level with the rim of the glass and the sphere is tangent to the side of the glass *extended.*

Setting the last factor of $f_2'(r)$ equal to zero, we obtain the result announced in the problem.

The graph of the function V has the shape shown in the figure. It is a continuous curve with a continuously turning tangent. The nature of the point B may be open to question; is it a point of inflection of the curve? The second derivative does not approach zero as r approaches b from either side. However, at the left of B the curve is concave up and at the right of B it is concave down. Hence, according to one definition, at least, B is a point of inflection. But the second derivative is not zero at B. Whatever judgment is passed upon B must apply also to C.

Elementary Development of Certain Infinite Series

By J. P. Ballantine, University of Washington

We find it is possible to obtain the usual series for certain fundamental functions rigorously without presupposing any knowledge of infinite series or of convergence. The method will be illustrated by developing $\log(1+x)$; other functions which could be so treated include $(1+x)^n$ when n is fractional or negative, arc tan x, sin x, cos x, and e^x. We assume a knowledge of integration of x^n, of differentiation of x^n and the particular function being developed, and of one further theorem, namely:

Theorem 1. If $f(x)>0$ for values of x in the interval $a>x>b$, then

$$\int_a^b f(x)dx > 0.$$

From Theorem 1, we may prove immediately three other theorems cited below. The present development calls only for Theorem 2, but the proofs here given of Theorems 3 and 4 may be of interest.

Theorem 2. If $f(x)<g(x)$ for the interval $a<x<b$, then

$$\int_a^b f(x)dx < \int_a^b g(x)dx.$$

Theorem 3. If $f(x)<M$ for the interval $a<x<b$, then

$$\int_a^b f(x)dx < M(b-a).$$

Theorem 4. If $f(x)<M$ and $g(x)>0$ for the interval $a<x<b$, then

$$\int_a^b f(x)g(x)dx < M\int_a^b g(x)dx.$$

Each theorem follows directly from Theorem 1 by replacing the function $f(x)$ of Theorem 1 by a particular positive function, namely:

$g(x) - f(x)$	in the case of Theorem 2,
$M - f(x)$	in the case of Theorem 3,
$\{M - f(x)\}\{g(x)\}$	in the case of Theorem 4.

Development of log $(1+x)$. The derivative of $\log(1+x)$ is $1/(1+x)$. It is easy to find simple polynomials larger or smaller than $1/(1+x)$. Thus, for $0<x<b$,

$$\frac{1}{1+x} < 1, \qquad \frac{1}{1+x} > 1 - x, \qquad \frac{1}{1+x} < 1 - x + x^2,$$

and so on. These inequalities are readily proved by clearing of fractions. Therefore, by Theorem 2, integrating from 0 to x, where $0<x$, we have

$$\log(1+x) < x, \quad \log(1+x) > x - \frac{x^2}{2},$$

$$\log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}, \qquad \text{etc.}$$

These inequalities hold, even if x is given a large value, but for practical purposes, x should be small. Then the expressions that are smaller than $\log(1+x)$ are very close to those that are larger, and a close determination can be made.

Since these relations have not been established for negative values of x, we shall develop $\log(1-x)$. Its derivative is $-1/(1-x)$. As before, we have, for $0<x<b<1$,

$$\frac{1}{1-x} > 1, \qquad \frac{1}{1-x} > 1 + x, \qquad \frac{1}{1-x} > 1 + x + x^2.$$

and so on; from which we readily obtain (by integration and change of sign)

$$\log(1-x) < -x, \quad \log(1-x) < -x - \frac{x^2}{2},$$

$$\log(1-x) < -x - \frac{x^2}{2} - \frac{x^3}{3}, \qquad \text{etc.}$$

Thus we have any number of expressions which are greater than $\log(1-x)$. We now seek expressions which are smaller. By long division with remainder, we find that

$$\frac{1}{1-x} = 1 + \frac{x}{1-x}, \qquad \frac{1}{1-x} = 1 + x + \frac{x^2}{1-x}, \qquad \text{etc.}$$

Consider x limited to the interval $0<x<b<1$. Then using the relation

$$\frac{1}{1-x}<\frac{1}{1-b}=c \qquad 0<x<b<1$$

in the equations just considered, we obtain the inequalities

$$\frac{1}{1-x}<1+\frac{x}{1-b}=1+cx, \quad \frac{1}{1-x}<1+x+cx^2,$$

$$\frac{1}{1-x}<1+x+x^2+cx^3, \qquad \text{etc.}$$

If we integrate from 0 to b, change signs, replace c by $1/(1-b)$, and b by x, we have

$$\log(1-x)>-\frac{x}{1-x}, \quad \log(1-x)>-x-\frac{x^2}{2(1-x)},$$

$$\log(1-x)>-x-\frac{x^2}{2}-\frac{x^3}{3(1-x)}, \qquad \text{etc.}$$

Consider one of the expressions which is larger than $\log(1-x)$ and the corresponding one which is smaller; we have, for example,

$$-x-\frac{x^2}{2}-\frac{x^3}{3(1-x)}<\log(1-x)<-x-\frac{x^2}{2}-\frac{x^3}{3}.$$

They are precisely the same, except for the extra factor $1-x$ in the denominator of the last term of one of them. For these expressions to be of practical use, x should be small, in which case $1-x$ is near 1, and the two final terms are nearly equal. If enough terms have been taken so that the final terms are negligible, then the factor $1-x$ makes very little difference, and either member of the above inequality may be taken as the desired value of $\log(1-x)$.

While the above series for $\log(1-x)$ is not new, the remainder term is better than the usual one. For example, if $\log(1-.9)$ is computed by taking ten terms of the series, the error is found to be less than

$$\frac{.9^{10}}{10(1-.9)}=.9^{10}=.35.$$

The classical remainder formula, $R_n(x)=f^n(\xi)x^n/n!$, $0<\xi<x$, in this case shows only that

$$R_{10}(.9)=\frac{9!}{(1-\xi)^{10}}\frac{.9^{10}}{10!}<\frac{.9^{10}}{10(.1)^{10}}=350{,}000{,}000.$$

RECENT PUBLICATIONS

EDITED BY W. R. LONGLEY, Yale University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Men of Mathematics. By E. T. Bell. New York, Simon and Schuster, 1937. 21+593 pages. $5.00.

The Handmaiden of the Sciences. By E. T. Bell. Baltimore, Williams and Wilkins, 1937. 8+216 pages. $2.00.

Construction, Classification and Census of Magic Squares of an Even Order. By A. L. Candy. Ann Arbor, Edwards Brothers, 1937. 6+182 pages. $1.00.

Biologie Mathêmatique. By V. A. Kostitzin. Paris, Librairie Armand Colin, 1937. 220 pages. 13 fr.

Mathematics for the Million. By L. Hogben. Illustrations by J. F. Horrabin. New York, W. W. Norton and Company, 1937. 647 pages. $3.75.

Calculus. By J. V. McKelvey. New York, The Macmillan Company, 1937. 9+420 pages. $3.00.

Plane Trigonometry. By K. B. Patterson and A. O. Hickson. New York, F. S. Crofts and Company, 1936. 9+219 pages. $1.75.

General Mathematics for Students of Business. By W. S. Schlauch. New York, F. S. Crofts and Company, 1936. 9+393 pages. $3.75.

Plane Trigonometry. By A. M. Harding and G. W. Mullins. Revised edition. New York, The Macmillan Company, 1937. 7+172 pages. $1.60.

First Year College Mathematics. By V. H. Wells. Part I. Trigonometry. 7+133 pages. $1.25. Part II. Mathematical Analysis. 9+276 pages. $2.75. New York, D. Van Nostrand Company, 1937.

Analytic Geometry and Calculus. By M. Morris and O. E. Brown. New York, McGraw-Hill Company, 1937. 10+507 pages. $3.75.

Trigonometry. With tables. By J. W. Branson and J. O. Hassler. New York, Henry Holt and Company, 1937. 8+198+73 pages. $1.75.

Plane Trigonometry. With tables. By H. A. Simmons and G. D. Gore. New York, John Wiley and Sons, 1937. 8+201+81 pages. $2.00.

REVIEWS

A First Course in the Differential and Integral Calculus. (Revised Edition.) By W. B. Ford. New York, Henry Holt and Company, 1937. 7+369 pages. $3.00.

In this revised edition of Professor Ford's text the most noteworthy change is a rearrangement of the material. A more desirable unity in the content of the chapters has been accomplished by the breaking up of the original seventeen chapters into twenty-four. The work on formal integration is now divided into

two chapters with a welcome intervening chapter on some preliminary applications of integration to relieve the monotony. Many sections of the text have been re-written and numerous new excercises have been added. There are also several new illustrative figures. One regrets that in this revision, as in the first edition, there is no proof of the Law of the Mean, or of the theorem relative to the convergence of an alternating series. The general appearance of the book is neat and attractive.

M. C. Foster

Das Grenzgebiet der elementaren und höheren Mathematik. By K. Kommerell. Leipzig, Koehler, 1936. 8+249 pages.

This book was written for use in intermediate schools in Germany and is an attempt to present certain topics in mathematics in a more rigorous manner than is customary at that stage. In particular, the author's concern is for processes involving limits. He says, "Since today in all schools which prepare for the secondary schools (Höhere Schule), infinitesimal methods are dealt with more or less in instruction, it is out of the question for elementary mathematics to shun, as formerly, the question when limits are met with."

The book covers a wide range of material. Significant topics treated in the first chapter are Heron's method for computing roots; use of a binary number system in computing logarithms and roots; the arithmetic-geometric mean (legacy of Gauss) used to compute almost anything including the periods of elliptic integrals of the first kind; continued fractions used to find the solutions of the diophantine equation $ax-by=1$; an interesting geometric interpretation of this equation; continued fractions used to find rational approximations of irrational numbers and of the transcendental numbers of Liouville.

Chapter II deals with certain geometric transformations, e.g., inversion, the circle construction of Mascheroni, the Dupin cyclides, central projection, collineations, the Lorentz transformation and the special relativity theory, the spiral transformation, conic spirals, and others.

Chapter III deals with vector calculation and algebra. He develops many of the formulas of spherical trigonometry, line coördinates, the null system, vector systems, equations of degree three and four and closes with a geometric construction of the regular 17-gon.

The exposition is lucid but the development of the topics is highly computational. A number of the very interesting subjects treated were new to the reviewer and in her opinion are complicated *excursi* from the direct development of mathematics needed for preparation for secondary school training even if this training has as its sole purpose that of producing mathematical scholars.

The work is carefully done, well edited, and well printed. Only one typographical error was noted; on page 238 the sentence $BE=BC$ should read $BE=EC$.

Mayme I. Logsdon

Analytic Geometry, Alternate Edition. By W. A. Wilson and J. I. Tracey. New York, D. C. Heath and Company, 1937. 288 pages. $2.12.

This is an elementary text treating both plane and solid analytic geometry, the latter being confined to the last chapter containing 34 pages. This book contains sufficient material for a year's course and forms a part of the preparation for the calculus. The authors have not attempted to solve the problem of shortening the road to the calculus; they leave the preparation in trigonometry and in college algebra to other texts.

The material of the text is clearly and adequately treated and the problems are given in sufficient number. There are no innovations or variations from the traditional methods of presentation of the topics of analytic geometry. However, the content of the chapter, Graphs of Functions and Empirical Equations, is not ordinarily found in such texts.

The book is practically free from errors, having passed through several editions. The authors published the first edition of this book in 1915 and revised it in 1925. The present revision is called the alternate edition and is not materially different from the former editions. The sets of problems have been revised and extended. A section on the method of moments has been added to the subject of curve fitting and sections on cylindrical and spherical coördinates have been added to the chapter on solid geometry. The inclusion of historical notes should be welcomed by those who find this text suited to their needs.

The authors are to be commended for the simple treatment of conics in the first chapters, reserving the study of the more general forms of the equations for the chapter on Transformation of Coördinates.

While this text is excellent in many ways and while the previous editions have been widely used, the reviewer feels that an honest appraisal has not been given unless he points out that the text is conventional and that it can be recommended only where formal methods of presentation prevail. It is left to the instructor, for the most part, to supply interesting applications and relations with other fields of study. Had the authors introduced the subject of tangents earlier, some of these applications and relations might have been presented with the study of each conic.

It is no longer possible to hold a student's interest and attention by training him in the technique of mathematics. He must be encouraged to have an inquiring mind and be taught in a manner which will enable him to acquire ability to be independent and to answer his own questions. He must understand why each subject is a necessary part of his education. For such a procedure, we must justify the study of a large part of the subject of analytic geometry.

V. H. WELLS

MATHEMATICS CLUBS

EDITED BY F. W. OWENS AND HELEN B. OWENS, State College, Pa.

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to F. W. Owens, 462 East Foster Ave., State College, Pa.

A NEW PUBLICATION

Log in Rhythm is the name of the magazine sponsored by the Bronx Mathematics Club of Hunter College. Volume 1, Number 1 is a brightly bound, clever compilation of mathematics, club news, and nonsense. "The Sad Tale of Miss Poly Gon" is an amusing addition to the rapidly growing collection of romances about that fascinating young person.

Columbia Mathematics Society

The Columbia Mathematics Society, Washington, D. C., an informal organization with bi-monthly meetings held at George Washington University is the direct outgrowth of a college mathematics club. When the Mathematics Club of the George Washington University was reorganized in 1933, (see this MONTHLY, vol. 43, pp. 423) as an undergraduate club, the graduate members, together with faculty members of various local schools and colleges, and others interested in mathematics formed the Columbia Mathematics Club.

The fields of work of the members may be indicative of possibilities for future mathematicians. Besides teachers of mathematics there are men from the Signal Corps, the Bureau of Ordnance, the Naval Model Basin, the Coast and Geodetic Survey, the Hydrographic Office, the Naval Observatory, the Weather Bureau, and the Fixed Nitrogen Laboratory. The meetings are open to all interested. Most of the papers are on research in pure mathematics carried on by members in fields quite distinct from their professions. This club is noteworthy because it carries forward into mature years the outstanding ideal of the undergraduate club, to foster interest in mathematics outside the classrooms and assigned tasks.

National Convention of Kappa Mu Epsilon

The third national convention of Kappa Mu Epsilon met at State College, Mississippi, April 30 and May 1, 1937, with representatives present from each of the sixteen active chapters. Over one hundred members came from outside State College. Sessions devoted to mathematical papers and the business of the society were supplemented by a reception and a banquet. The following national officers were elected to serve two years:

President Pythagoras, Professor J. A. G. Shirk, Kansas State Teachers College, Pittsburg, Kansas.
Vice-President Euclid, Professor C. V. Newsom, Albuquerque, New Mexico.
Secretary Diophantus, Miss E. Marie Hove, Wayne, Nebraska.
Treasurer Newton, Professor L. E. Pummell, Springfield, Missouri.
Historian Hypatia, Miss Orpha Ann Culmer, Florence, Alabama.

DIRECTORY OF MATHEMATICS CLUBS IN COLLEGES AND UNIVERSITIES OF THE UNITED STATES AND CANADA

The following is a supplement to a list published in this MONTHLY, vol. 43, 1936, pp. 420–431. Dates, meetings, conditions of membership, etc., conform to the previous listings.

Alabama College, Montevallo, Alabama.

Kappa Mu Epsilon, Alabama Gamma. April 24, 1937.

Adelphi College, Garden City, New York.

The Mathematical Klub of Adelphi College. 1898. Anyone interested in mathematics. Issues at irregular intervals, mimeographed paper, *Math Log.*

Bowling Green State University, Bowling Green, Ohio.

Kappa Mu Epsilon, Ohio Alpha. April 24, 1937.

Brooklyn College, Brooklyn, New York.

Mathematics Club. 1931. Sponsors Integration Contests. Awards medals for scholarship. Publishes, annually, *The Math Mirror.*
Mathematics Club of the Evening Session of Brooklyn College. 1935. Cooperates in publication of *The Math Mirror.*
Pi Mu Epsilon. 1933.

College of Saint Teresa, Winona, Minnesota.

Mathematics-Physics Club. 1932. Majors in mathematics and physics. Membership 33.

Creighton University, Omaha, Nebraska.

Mathematics Club of Creighton University. 1929. Membership 30.

Kansas State Teachers College, Emporia, Kansas.

Mathematics Club. 1915. All interested in mathematics.

Louisiana State University, Baton Rouge, Louisiana.

Kappa Mu Epsilon, Louisiana Alpha. 1936.

Marquette University, Milwaukee, Wisconsin.

Mathematics Club of Marquette University. 1936. All interested in mathematics.

Massachusetts State College, Amherst, Massachusetts.

Mathematics Club. February, 1937. All interested in mathematics.

Michigan State College, Lansing, Michigan.

Michigan State College Mathematics Club. 1936.

Mount Saint Scholastica College, Atchison, Kansas.

Mathematics Club. 1936.

Regis College, Weston, Massachusetts.

Regis College Mathematics Club. 1936. Students enrolled in mathematics. Annual mathematical exhibit. Annual prize, $10, for best solution of 300 assigned problems.

Transylvania College, Lexington, Kentucky.

Mathematics Club of Transylvania College.

Trinity College, Washington, D. C.

Mathematics Society. 1936.

University of Kansas City, Kansas City, Missouri.

Delta X. 1936. Semimonthly meetings. Membership 25.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 288. *Proposed by W. B. Campbell, Ithaca, New York.*

If $\cos(a+bi)=p+iq$, where a, b, p, and q are real, find explicit expressions for all permissible values of $\cos a$, $\sin a$, $\sinh b$, in terms of p and q, classifying where necessary, and showing that they all lead to real values for a and b.

E 289. *Proposed by A. A. Bennett, Brown University.*

Show that the Pappus configuration in the plane (consisting of three triads of points, the triads by pairs being triply perspective from the points of the remaining triad), comprises nine points, P_{xy}, and nine lines, p_{uv}, $(x, y, u, v=0, 1, 2)$, where the notation may be so assigned that P_{xy} is incident to p_{uv} if and only if $ux\equiv v+y \pmod 3$.

E 290. *Proposed by Meyer Karlin, Student, Yeshiva College.*

If $S_k=1+x+x^2+\cdots+x^k$, decompose the fraction, $1/(S_{2n}S_{2n+1}S_{2n+2})$, into the algebraic sum of three fractions (also proper) whose denominators are respectively S_{2n}, S_{2n+1}, and S_{2n+2}.

E 291. *Proposed by V. Thébault, Le Mans, France.*

With the digits, 1, 2, 3, 4, 5, 6, 7, 8, and 9, used once each, form a perfect square which is divisible by 99, and show that the solution is unique.

E 292. *Proposed by K. W. Miller, Utilities Research Commission, Inc., Chicago.*

Verify the following trigonometric identities:

$$\text{(a)}\quad \sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}\frac{\cos k\theta}{\cos^k\theta}=0 \text{ if } n \text{ is odd, but } =(-1)^{n/2+1}\tan^n\theta \text{ if } n \text{ is even.}$$

$$\text{(b)}\quad \sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}\cos^k\theta\cos k\theta=(-1)^{(n+1)/2}\sin^n\theta\sin n\theta \quad \text{if } n \text{ is odd,}$$

$$=(-1)^{n/2+1}\sin^n\theta\cos n\theta \quad \text{if } n \text{ is even.}$$

Here $\binom{n}{k}$ represents a binomial coefficient, as usual.

E 293. *Proposed by J. H. Butchart, Phillips University, Enid, Oklahoma.*

Construct three circles through a point P so that the sum of the directed segments cut off by the circles on any line through P is zero.

SOLUTIONS

E 216 [1936, 305]. *Proposed by J. Rosenbaum, Hartford Federal College.*

Find

$$\lim_{x\to\infty}\left[x\sin\frac{1}{x}+\frac{1}{x}\right]^x.$$

Comment by B. P. Gill, College of the City of New York

The solution of this problem as published in the January 1937 number of this MONTHLY, page 54, is not adequate, though it is easily modified so as to become valid.

First "infinitesimals of the same order" should be replaced by "asymptotical equivalent" or by "infinitesimals whose ratio approaches unity." But the essential point is that none of these statements is sufficient to justify the substitution of one of the infinitesimals for the other. What is needed is the stronger statement, $\sin 1/x-1/x=\mathrm{O}(1/x^2)$, where $\mathrm{O}(\quad)$ means "infinitesimal of higher order than." Then $x\sin 1/x=1+\mathrm{O}(1/x)$, and the required result follows from the standard theorem that if $f(x)$ is an infinitesimal, and $f(x)\cdot g(x)$ has the limit c, then $(1+f)^g$ will have the limit e^c.

To see the necessity for this more clearly, consider the limit

$$\lim_{x\to\infty}\left[(e^{x^{-1}}-1)x+1/x\right]^x.$$

Although $e^{x^{-1}}-1$ is asymptotically equivalent to $1/x$, to replace it by $1/x$ and conclude that the given limit is e, would be entirely wrong. The correct solution is $e^{1/x}-1=1/x+1/2x^2+\mathrm{O}(1/x^2)$. Hence the expression to be treated is $\lim_{x\to\infty}[1+3/2x+\mathrm{O}(1/x)]^x=e^{3/2}$, by the theorem already mentioned.

E 252 [1937, 49]. *Proposed by N. A. Court, University of Oklahoma.*

If the planes drawn through a given point M parallel to the faces BCD, CDA, DAB, and ABC of the tetrahedron $ABCD$, whose centroid is at G, meet the respective medians of $ABCD$ in the points P, Q, R, and S, prove that,

$$\frac{GP}{GA}+\frac{GQ}{GB}+\frac{GR}{GC}+\frac{GS}{GD}=0.$$

Solution by C. E. Springer, University of Oklahoma

The proposition can be proved for a simplex in a linear space of n dimensions. Let the vertices $A_t(t=1,2,\cdots,n+1)$ of the simplex have homogeneous coördinates x_t^i $(i=1,2,\cdots,n+1)$, and let the hyperplane H_t through $M(x_0^i)$ parallel to the hyperplane opposite A_t have the equation $X_i^t(x^i-x_0)=0$, where

the left member is summed on the index i in accordance with the usual convention, and X_i^t is the cofactor of x_t^i in the determinant $|x_t^i|$. If the hyperplane H_t is intersected by the line GA_t in P_t, and if G is the origin of coördinates, we have

$$GP_t/GA_t = (X_{n+1}^t - X_i^t x_0^i)/(X_i^t x_t^i - X_{n+1}^t), \qquad \text{[summed on } t\text{]},$$
$$= (x_t^j X_{n+1}^t - x_t^j X_i^t x_0^i)/(x_t^j X_i^t x_t^i - x_t^j X_{n+1}^t), \quad \text{[summed on } i \text{ and } t\text{]}.$$

If now $j \neq i$, and $j \neq n+1$, the numerator of the last expression is zero, while the denominator is not zero. Hence $GP_t/GA_t = 0$, [summed on t].

Also solved by L. M. Kelly and the proposer.

E 253 [1937, 49]. *Proposed by V. Thébault, Le Mans, France.*

Find two positive integers differing by five, with the sum of their squares a perfect cube, and show that the solution is unique.

Solution by E. P. Starke, Rutgers University

Let $x^2+(x+5)^2=z^3$; or if we put $2x+5=s$, $s^2+25=2z^3$. Since z^3 is the sum of two squares, it must contain no factor of the form $4n+3$. Evidently z is odd, hence contains only factors of the form $4n+1$. The last digit of z cannot be 1 or 9, since s^2 cannot end in 7 or 3. Testing values of $z<100$ which fit these requirements, we obtain two satisfactory solutions, namely, $5^2+10^2=5^3$ and $47^2+52^2=17^3$.

Editorial Note. The proposed problem obviously does not have a unique solution, although several of our readers "proved" that it did. W. B. Carver states that the second solution above is the only one where the integers are not multiples of 5, but that it seems very difficult to determine whether or not there exist other solutions like the first above, in which the integers are all multiples of 5.

Also solved by Mary L. Constable, Daniel Finkel, M. A. Heaslet, H. R. Mutch, K. B. Patterson, W. R. Talbot, C. W. Trigg, Simon Vatriquant, and the proposer.

E 254 [1937, 49]. *Proposed by D. L. MacKay, Evander Childs High School, New York.*

Given the vertices B and C, and the altitude from A, construct the triangle ABC so that $a^4=b^4+c^4$, where a, b, and c are the sides of the triangle.

Solution by K. B. Patterson, Durham, North Carolina

The solution is possible only for $y^2 \leqq a^2(\sqrt{8}-1)/4$, where y is the altitude, but within these limits the construction is determined as follows:

If we take the midpoint of the base a as the origin, and let the base lie along the x-axis, then the locus of the vertex (x, y) is

$$2x^4 + 3a^2x^2 + 2y^4 + 4x^2y^2 + a^2y^2 - 7a^4/8 = 0,$$

from which we find $x^2 = a\sqrt{a^2+y^2} - (y^2+3a^2/4)$. The problem now is to construct the value of x by ruler and compass. To do this, we first construct the

mean proportional between a and $\sqrt{{}^2a+y^2}$ by the usual means, and call it w. Next we construct a right triangle whose legs are y and $a\sqrt{3}/2$, and call its hypothenuse z. Finally, we construct a right triangle having z as a leg and w as hypothenuse. Its other leg, x, must be the abscissa of the vertex.

Also solved by Elmer Schuyler, C. W. Trigg and the proposer.

B. H. Brown gives the following references for the treatment of these "pseudo-Pythagorean" triangles. *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht* (de Hoffmann) XXXI-524, XXXIV-335, XXXV-123, 488, XXXVI-84, XXXXI-26, and *Mathesis* 1905, p. 259. A summary is in Neuberg's *Bibliographie des triangles spêciaux* in *Memoires de la Sociêtê royale des Sciences de Liège*, 3e série, tome XII, 1924.

E 255 [1937, 49]. *Proposed by Cezar Coşniţă, Roumanian Mathematical Institute.*

Determine the locus of the centers of the spheres which pass through the two fixed points, A and B, and are tangent to the given plane P. What is the locus of the points of tangency on the plane?

Solution by Simon Vatriquant, Brussels

If M denotes the point of intersection of the line AB with the plane P, and T the point of contact of the sphere with P, then we have $MA \cdot MB = MT^2 = \text{const}$. The locus of T is a circle in the plane P, with center M. The perpendicular line to P through T passes through the center C of the sphere, and the same center lies in the plane Q, perpendicular to AB at the mid-point of the segment. The locus of the center is consequently the section by Q of the cylindric surface whose directrix is the locus of T.

If AB is parallel to P, point M is at infinity, the locus of T is a straight line, and the locus of C is a parabola.

In his solution, N. A. Court quotes as reference his *Modern Pure Solid Geometry*, p. 194, art. 607.

Also solved by Fred Discepoli, C. E. Springer, E. P. Starke, and C. W. Trigg.

E 256 [1937, 49]. *Proposed by L. J. Adams, Santa Monica Junior College, Calif.*

Prove that

$$\frac{\sin\theta + \sin 2\theta + \cdots + \sin n\theta}{\cos\theta + \cos 2\theta + \cdots + \cos n\theta} = \tan\frac{n+1}{2}\theta.$$

Solution by H. H. Downing, University of Kentucky

In Rothrock, *Plane and Spherical Trigonometry*, 1914, page 109, we find

$$\sin\theta + \sin 2\theta + \cdots + \sin n\theta = [\sin\tfrac{1}{2}(n+1)\theta \sin\tfrac{1}{2}n\theta]/\sin\tfrac{1}{2}\theta,$$

$$\cos\theta + \cos 2\theta + \cdots + \cos n\theta = [\cos\tfrac{1}{2}(n+1)\theta \sin\tfrac{1}{2}n\theta]/\sin\tfrac{1}{2}\theta.$$

Division of the first of these equations by the second gives the desired result.

Editorial Note. The above trigonometric identities apparently appear in

many different modern texts, as several of our readers gave different references where they might be found.

Also solved by F. A. Alfieri, Norman Anning, George Bingley, Romeo Falciani, Daniel Finkel, G. E. Forsythe, Bernard Greenspan, W. R. Hardman, G. G. Harvey, M. A. Heaslet, L. H. Kanter, Elmer Latshaw, D. L. MacKay, H. R. Mutch, Harris Rice, C. E. Springer, E. P. Starke, C. W. Trigg, and Simon Vatriquant.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

SOLUTIONS

3748 [1935, 454]. *Proposed by Harry Langman, Brooklyn, N. Y.*

If we set

$$\sum_{i=0}^{n-1}(-1)^i\frac{{}_{n-1}C_i}{(i+1)^{t+1}}=a_t,$$

where the C's are binomial coefficients, show that

$$\begin{vmatrix} a_1 & -a_2 & a_3 & \cdots & (-1)^{n-1}a_{n-2} & (-1)^n\ a_{n-1} & (-1)^{n+1}a_n \\ -a_0 & a_1 & -a_2 & \cdots & (-1)^{n-2}a_{n-3} & (-1)^{n-1}a_{n-2} & (-1)^n\ a_{n-1} \\ 0 & -a_0 & a_1 & \cdots & (-1)^{n-3}a_{n-4} & (-1)^{n-2}a_{n-3} & (-1)^{n-1}a_{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & -a_0 & +a_1 & -a_2 \\ 0 & 0 & 0 & \cdots & 0 & -a_0 & a_1 \end{vmatrix} = \frac{1}{n!n^n}.$$

Solution by Frank Ayres, Jr., Dickinson College.

If we set $na_t=A_{tn}$, it may be shown that

$$A_{tn}-A_{t\,n-1}=\frac{1}{n}A_{t-1\,n},\quad A_{0n}=1,\quad A_{t1}=1,\quad A_{1n}=\sum_{j=1}^{n}\frac{1}{j}. \tag{1}$$

The determinant of the problem becomes D_{nn}/n^n, where D_{nn} has in its jth row

$$0\ 0\cdots 0-A_{0n}\,A_{1n}\cdots(-1)^{n-j+2}A_{n-j+1\,n}. \tag{2}$$

The first row begins with A_{1n} and ends with $(-1)^{n+1}A_{nn}$. Replace the jth row R_j by R_j+R_{j+1}/n, $j=1, 2, \cdots, n-1$. From (1) it will be seen that, except for the last row, the new determinant is obtained from the original by replacing

A_{tn} by $A_{t\,n-1}$. The last row may be written by (1)

$$0\ 0 \cdots 0 - A_{0\,n-1}\ A_{1\,n-1} + \frac{1}{n}, \tag{3}$$

and we have

$$D_{nn} = D_{n-1\,n} + \frac{1}{n} D_{n-1\,n-1} = \frac{1}{n} D_{n-1\,n-1}, \tag{4}$$

where the second subscript is the order of the determinant. Since we may use the same reduction repeatedly, we have

$$D_{n-1\,n} = D_{n-2\,n} + \frac{1}{n-1} D_{n-2\,n-1} = \sum_{j=1}^{n-1} b_j D_{1\,j+1} = 0.$$

Since $D_{22}=1/2$, we have

$$D_{nn} = 1/n!$$

and this completes the proof.

Editorial Note. The results in (1) were developed in more detail in the solution; but since these results were also developed by E. G. Olds in his solution of 3701 [1936, 197] some of the details of the proof were omitted.

3750 [1935, 515]. *Proposed by V. F. Ivanoff, San Francisco, Calif.*

A circle with a fixed tangent rolls along a straight line AB. Show that the envelope of the tangent is the locus of the point of intersection of the tangent with the perpendicular to it drawn through the point of contact of the circle with AB.

Solution by W. B. Campbell, Ithaca, N. Y.

Let the unit circle be initially in contact with the x-axis at (0, 0), with the line T tangent to it at (0, 2). When the circle has rolled along the axis through angle θ, T will be tangent at $(\theta+\sin\theta,\ 1+\cos\theta)$, with slope $=-\tan\theta$. If we write the equation of T, differentiate with respect to parameter θ, and solve the resulting equation with the original equation, we find for the coördinates of the envelope

$$x = \theta + \sin\theta(1+\cos\theta), \qquad y = \cos\theta(1+\cos\theta). \tag{1}$$

For a given θ, the point of contact with the axis is $(\theta, 0)$, and the line through this point perpendicular to T, that is with slope $=\cot\theta$, intersects T at the point (x, y) given by (1).

The locus is symmetrical with respect to $x=\pi$, for $0\leqq x\leqq 2\pi$. In general, $dy/dx=-\tan\theta$, $d^2y/dx^2\leqq 1$. It starts horizontally at (0, 2), proceeds with negative and decreasing slope, passes vertically through $(\frac{1}{2}\pi+1,\ 0)$, for $\theta=\frac{1}{2}\pi$; then back to $(2\pi/3+3^{1/2}/4,\ -1/4)$, for $\theta=2\pi/3$, where it has a cusp of slope $3^{1/2}$, then with positive but decreasing slope to horizontal contact at $(\pi, 0)$.

Also solved by C. V. L. Smith, E. P. Starke, and F. Underwood.

Editorial Note. If the circle (C) with center C at a given moment is tangent to AB at N, its instantaneous center of rotation is at N. Hence the tangent T attached to (C) has also its instantaneous center of rotation at N. From this it follows that the foot Q of the perpendicular from N to T is the point of contact of T with its envelope. A proof of the first theorem may be obtained by considering the angular velocity of rotation α of (C) at the given moment. Any point P on the circumference of (C) moves with a linear velocity which is the resultant of two linear components, $r\alpha$ parallel to AB, $r\alpha$ perpendicular to CP. Hence the resultant linear velocity of P is perpendicular to NP, and every point of (C) rotates instantaneously about N. If P is the point of contact of T, NP is the normal at P to the locus (P) of P. The second part is easily proved. An elementary treatment of the center of rotation may be found in *Traité de Géométrie*, Rouché et Comberousse, vol. 1, 1912, pp. 118, 119.

3751 [1935, 515]. *Proposed by V. Thébault, Le Mans, France.*

Let Q_b, Q_c, Q_b', Q_c' be the intersections of the bisectors of the angles B and C with the opposite sides of the triangle ABC; D' and D_a', the points on the inscribed and escribed circles in the angle A diametrically opposite to their points of contact with BC; P and P', the intersections of Q_bQ_c and $Q_b'Q_c'$ with the interior bisector of angle A. Show that the lines $D'P$ and $D_a'P'$ meet in the foot of the altitude AA' and are symmetric with respect to BC.

Solution by L. M. Kelly, Lawrence, Mass.

Let the interior bisector at A cut the opposite side BC in N, and let D be the point of contact of the inscribed circle (I) with the same side. From similar triangles we see that DA' divides the segment IA at N in the ratio r/h_a. But $PI/PA = IN/NA = r/h_a$, and P divides the segment IA in the ratio r/h_a. Hence the points D', P, A' are collinear. Similarly, D_a', P', A' are collinear. The range $APNP'$ is harmonic and so is the pencil $A'(APNP')$. Since $A'A$ and $A'N$ are perpendicular, the angle $PA'P'$ is bisected by $A'N$ which is in the same line with BC. This completes the proof.

Solved also by the proposer.

Editorial Note. It is of interest to observe that AI, BC, Q_bQ_c', $Q_b'Q_c$ meet in the point N, and we adjoin the proof. Since AQ_bCQ_b' and AQ_cBQ_c' are harmonic ranges, Q_bQ_c, $Q_b'Q_c'$, BC meet in a point K. Hence $APNP'$ is harmonic. Also since Q_bB and Q_cC meet in I, $KCNB$ is harmonic and AK is the external bisector of the angle at A. Thus Q_bQ_c' and $Q_b'Q_c$ intersect on both BC and AI, and this proves the theorem.

The theorem of the problem is an easy consequence of the harmonic properties of this figure. For the figure shows that $ANIP$ is harmonic and so is also the pencil $A'(ANIP)$. Obviously $A'(ADID')$ is an harmonic pencil. Since the first three pairs of corresponding rays coincide, the fourth pair must also coincide, and A', P, D' lie in a straight line. The rest of the proof is just as easy.

3753 [1935, 515]. *Proposed by Frank Ayres, Jr., Dickinson College.*

Let P_4 be the orthocenter of the triangle $P_1P_2P_3$; and O_i the circumcenter of the triangle $P_jP_kP_4$, where i, j, k is a permutation of 1, 2, 3. Prove that the sum of the squares of the radii of the circles having O_i as centers and orthogonal to the nine-point circle of the given triangle is one-fourth of the sum of the squares of the edges of the given triangle.

Solution by L. M. Kelly, Lawrence, Mass.

Let N be the center of the nine-point circle (N) and P_i', the foot of the altitude from P_i in triangle $P_1P_2P_3$. An examination of the figure shows that $O_1O_2O_3$ is similar to $P_1P_2P_3$ with N as center of similitude, and with the ratio -1. The two triangles have the same nine-point circle, and it suffices to prove the theorem for either triangle. The sum of the powers of P_2 and P_3 with respect to (N) is $P_2P_1' \cdot P_2P_3/2+P_1'P_3 \cdot P_2P_3/2=\overline{P_2P_3}^2/2$. Hence the sum of the squares of the radii of circles with centers at P_1, P_2, P_3 (or at O_1, O_2, O_3) orthogonal to (N) is

$$\tfrac{1}{4}(\overline{P_2P_3}^2 + \overline{P_3P_1}^2 + \overline{P_1P_2}^2).$$

Solved also by H. A. DoBell and Leon Recht.

Editorial Note. The solutions by the other solvers were different from the above and involved more complicated computations. We shall give a brief outline of one form of proof of the first theorem in the solution. It is obvious that O_iC is normal to both O_jO_k and P_kP_j, and that C is the orthocenter of $O_1O_2O_3$. The two triangles are similar. Since N is the mid-point of P_4C, and $M_1C=P_4P_1/2$, where M_1 is the mid-point of P_2P_3, we easily find from the rectangle with two sides $P_1'M_1$ and M_1O_1', that $P_1'P_4=CO_1'$, etc. Hence N is the center of similitude and the ratio is -1.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items of interest to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

Professor John von Neumann of the Institute for Advanced Study has been elected to membership in the National Academy of Sciences, and Professor Oswald Veblen, also of the Institute for Advanced Study, has been elected a member of the Council of the Academy.

Assistant Professor A. A. Albert of the University of Chicago has been promoted to an associate professorship.

Dr. D. B. Ames of the Rensselaer Polytechnic Institute has been promoted to an assistant professorship.

Dr. Max Astrachan of Antioch College has been promoted to an assistant professorship.

Professor L. C. Bagby of the Linsley Institute of Technology, Wheeling, West Virginia, has been appointed dean of the College of Engineering.

Professor E. T. Bell of the California Institute of Technology addressed the New Brunswick Scientific Society and the Mathematics Clubs of Rutgers College and the New Jersey College for Women on May eleventh. His topic was "Is there an economic history of mathematics?"

Dr. Samuel Borofsky of Brooklyn College has been rpomoted to an assistant professorship.

At the University of Cincinnati the mathematics departments of the College of Liberal Arts and the College of Engineering have been combined. Professor Louis Brand, formerly head of the mathematics department in the College of Engineering, has been made head of the combined department.

Professor H. E. Buchanan of Tulane University has been appointed "The A. B. Dinwiddie Graduate Professor of Mathematics" in recognition of "conspicuous achievements in higher mathematics."

J. W. Calhoun, professor of applied mathematics, has been named acting president of the University of Texas.

Dr. R. H. Cameron of the Massachusetts Institute of Technology has been promoted to an assistant professorship.

Associate Professor Lennie P. Copeland of Wellesley College has been promoted to a professorship.

Professor H. T. Davis of Indiana University has resigned to accept a professorship of mathematical statistics at Northwestern University.

Assistant Professor J. E. Davis of Central Y.M.C.A. College, Chicago, has been promoted to an associate professorship.

Professor E. L. Dodd of the University of Texas has been given leave of absence from September 15 to November 1 to enable him to attend the Congress in the Theory of Probability at Geneva.

Professor B. F. Finkel has become professor emeritus of mathematics and physics after a service of forty years at Drury College.

Associate Professor J. H. Fithian of Newark College of Engineering has been promoted to a professorship.

Dr. B. C. Getchell of Butler University has been promoted to an assistant professorship.

Professor Harris Hancock of the University of Cincinnati has retired with the title professor emeritus.

Professor G. A. Harter of the University of Delaware has retired with the title professor emeritus.

Dr. M. A. Heaslet of San Jose State College has been promoted to an assistant professorship.

Dr. M. R. Hestenes of the University of California at Los Angeles has been appointed an assistant professor at the University of Chicago.

Assistant Professor A. S. Householder of Washburn College has been awarded a Rockefeller Fellowship and will spend the coming year at the University of Chicago working with Professor Nicolas Rashevsky in mathematical biophysics.

Professor Alfred Hume of the University of Mississippi has retired with the title professor emeritus.

Dr. Ralph Huston of Rensselaer Polytechnic Institute has been promoted to an assistant professorship.

Dr. H. D. Larsen of the University of New Mexico has been promoted to an assistant professorship.

Dr. D. H. Lehmer of Lehigh University has been promoted to an assistant professorship.

Professors D. N. Lehmer and C. A. Noble of the University of California have retired.

Associate Professor F. A. Lewis of the University of Alabama has been promoted to a professorship.

Professor W. R. Longley of Yale University has been appointed acting dean of freshmen.

Professor C. N. Moore of the University of Cincinnati has been made director of graduate studies in mathematics.

Assistant Professor C. B. Morrey of the University of California has been granted leave of absence for the coming year during which time he will be at the Institute for Advanced Study in Princeton.

Assistant Professor C. O. Oakley of Haverford College has been promoted to an associate professorship.

Dr. G. B. Price of Brown University has been made assistant professor of mathematics at the University of Kansas.

Dr. W. T. Reid of the University of Chicago has been promoted to an assistant professorship.

Professor M. H. Stone of Harvard University acted as delegate of the Mathematical Association and of the American Mathematical Society to the Third Inter-American Conference on Education at Mexico City in August.

Dr. Otto Szasz has been appointed research lecturer in mathematics at the University of Cincinnati.

Assistant Professor H. S. Thurston of the University of Alabama has been promoted to an associate professorship.

Associate Professor H. S. Uhler of Yale University has been promoted to a professorship.

Assistant Professor Henry Van Engen of Kansas State College has been appointed head of the department of mathematics at Iowa State Teachers College.

Associate Professor H. S. Wall of Northwestern University is on leave of absence for the academic year 1937–1938, and has accepted a fellowship for study at the Institute for Advanced Study.

Professor W. M. Whyburn of the University of California at Los Angeles has been appointed acting chairman of the department of mathematics to succeed Professor E. R. Hedrick who is now Provost of the University.

Assistant Professor H. N. Wright of the College of the City of New York has been promoted to an associate professorship.

The following appointments to instructorships are announced:
Cornell University: Dr. J. W. Givens
University of Chicago: Dr. Saunders MacLane
Harvard University: (Benjamin Peirce Instructor) Dr. Israel Halperin
New York University: Dr. J. T. C. Wright
Rutgers University: Dr. M. S. Robertson
University of Saskatchewan; Miss V. A. Ames
Spring Hill College: Spring Hill, Alabama, Rev. E. H. Larguier
Texas Technological College: Mrs. Opal L. Miller, H. E. Woodward
Rensselaer Polytechnic Institute: Dr. R. W. Rempfer

Professor W. I. Foster of Northern Montana College, Havre, Montana, died on January 1, 1937.

Dr. A. P. Wills, professor of mathematical physics at Columbia University since 1909, died April 18, 1937 in St. Augustine, Florida. He was sixty-four years old.

It is planned to make the 1938 volume of this MONTHLY a memorial volume to the late Professor H. E. Slaught. The January number will contain biographical sketches of his life. A shorter sketch has recently appeared in the September number of the *Bulletin of the American Mathematical Society*.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, E. J. MOULTON, Northwestern University, Evanston, Ill.

BOOKS FOR REVIEW should be addressed to REVIEW EDITOR, American Mathematical Monthly, 531 West 116th Street, New York, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of addresss to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-first Summer Meeting, Pennsylvania State College, Sept. 6–7, 1937.
Twenty-second Annual Meeting, Indianapolis, Ind., December 30–31, 1937.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1937 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Waynesburg, Pa., May 1; Pittsburgh, October 23.
ILLINOIS, DeKalb, May 14–15.
INDIANA, Greencastle, April 30–May 1.
IOWA, Dubuque, April 16–17.
KANSAS, Wichita, April 3.
KENTUCKY, Louisville, May 1.
LOUISIANA-MISSISSIPPI, Hammond, La., March 5–6.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Lynchburg, Va., May 8.
MICHIGAN, Ann Arbor, March 20.
MINNESOTA, St. Paul, May 15.
MISSOURI.
NEBRASKA, Lincoln, May 7.
OHIO, Columbus, April 1.
OKLAHOMA, Tulsa, February 5.
PHILADELPHIA, Haverford, Nov. 27.
ROCKY MOUNTAIN, Greeley, Colo., April 16–17.
SOUTHEASTERN, Nashville, Tenn., April 16–17.
SOUTHERN CALIFORNIA, Los Angeles, March 6.
SOUTHWESTERN, State College, N.M., April 2–3.
TEXAS, Houston, April 23–24.
WISCONSIN, Milwaukee, May 8.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

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RESOLUTION ADOPTED BY THE COUNCIL OF THE AMERICAN MATHEMATICAL SOCIETY SEPTEMBER 9, 1937

The Council notes with deep regret the death of Herbert Ellsworth Slaught on May 21, 1937. An active member of the Society almost from its foundation, secretary of the Chicago Section of the Society from 1906 to 1916, and always an enthusiastic and intelligent promoter of mathematics, of mathematical education, and of the interests of mathematicians, he contributed uniquely to the development of mathematics in America. As a teacher and an editor, he rendered unusual service, but probably his greatest contribution lay in his success in founding the Mathematical Association of America, devoted to the interests of collegiate mathematics. After the American Mathematical Society in 1915 voted to restrict its activities to the promotion of mathematical research, it was under Professor Slaught's inspiration and guidance that the Association was formed; and it was through his wise counsels that its policies were developed in such a manner that it has so effectively and so harmoniously supplemented the work of the Society. The Council of the Society takes this occasion to express its appreciation of the fine achievements of the Association, and of the uniformly friendly and effective cooperation which has obtained between the Society and the Association since the latter was founded.

THE MAY MEETING OF THE ILLINOIS SECTION

The eighteenth annual meeting of the Illinois Section was held at the Northern Illinois State Teachers College, De Kalb, Illinois on May 14–15, 1937. The chairman of the Section, Professor C. N. Mills, Illinois State Normal University, presided.

There were seventy in attendance including the following thirty-three members of the Association: Edith I. Atkin, S. F. Bibb, C. E. Comstock, H. B. Curtis, D. R. Curtiss, J. E. Davis, W. M. Davis, Elinor B. Flagg, R. E. Gadske, A. E. Gault, G. D. Gore, E. W. Hellmich, Martha Hildebrandt, E. C. Kiefer, J. M. Kinney, W. C. Krathwohl, Luise Lange, Mayme I. Logsdon, Ruth G. Mason, C. N. Mills, G. E. Moore, E. J. Moulton, Mary W. Newson, Rufus Oldenburger, J. W. Peters, E. W. Ploenges, W. C. Randels, H. A. Simmons, L. W. Sims, F. C. Smith, Norma K. Stelford, W. B. Storm, F. E. Wood.

The following officers were elected for next year: Chairman, Professor W.B. Storm, Northern Illinois State Teachers College; Vice-Chairman, Professor J. R. Mayor, Southern Illinois State Normal University; Secretary, Edith I. Atkin, Illinois State Normal University. It was decided to hold the next meeting at the Southern Illinois State Normal University at Carbondale.

An unusually pleasant social feature was the excellent dinner in Williston Hall served under the direction of the Hall dietitian, Miss Donalda Morrison. Professor Slaught, an inspirational leader of the Section until illness detained him at home, was remembered by a greeting signed by those in attendance.

The following papers were presented at the three sessions:

1. "Remarks concerning special Miquel points" by Dr. G. E. Moore, University of Illinois.

2. "The removal of certain restrictions from Simpson's Rule" by Professor W. C. Krathwohl, Armour Institute of Technology.

3. "A simplified calculation of a statistical problem" by Professor C. N. Mills, Illinois State Normal University.

4. "Affine differential geometry of curves and ruled surfaces" by Professor H. A. Simmons, Northwestern University.

5. "Explorations in new dimensions" by Dr. Luise Lange, Woodrow Wilson Junior College, Chicago.

6. Illustrated lecture, "At the International Congress last summer" by Professor Rufus Oldenburger, Armour Institute of Technology.

7. "The differential equations of certain transversal surfaces and their transformations" by Professor G. D. Gore, Central Y.M.C.A. College, Chicago.

8. "Symposium on mathematics in the junior college" conducted by Dr. J. W. Peters, University of Illinois.

Abstracts of these papers follow:

1. Dr. Moore called attention to a theorem which was stated and proved by Miquel in 1838; namely, "If points $P_i (i=1, 2, 3)$ are marked, one each, on the sides of a triangle and through each vertex A_i of the triangle and the marked points on the adjacent sides a circle is drawn, these three circles meet in a point P. Furthermore, the lines PP_i make equal angles with the respective sides." Special consideration was given to the case in which the lines PP_i were perpendicular to the sides, and relationships between the point P and a point Q (the intersection of the lines A_iP_i) were noted. The fact that there are two cubics associated with the locus of the points P and Q was pointed out and some of the corresponding points on the two curves were identified.

2. Professor Krathwohl stated that Simpson's Rule can be extended to the case where the range of the definite integral is divided into an odd number of divisions by finding the area defined by one division. There are two ways of doing this. If Y_1, Y_2, and Y_3 are three equally spaced ordinates, the area between Y_1 and Y_2 is given by the formula $h/12(5Y_1+8Y_2-Y_3)$. The area between Y_2 and Y_3 is given by $h/12(-Y_1+8Y_2+5Y_3)$. Simpson's Rule frequently can be applied to the case of an infinite range by a proper change of variables. In some instances it is necessary to divide the range into two parts.

3. In the determination of the standard error of the difference of the means of paired items, the formula usually used is

$$\sigma(M_1 - M_2) = (\sigma^2 M_1 + \sigma^2 M_2 - 2r\sigma M_1 \sigma M_2)^{1/2}. \tag{I}$$

In experimental research where one is interested only in the difference of the means, much time and energy is necessary in applying the above formula. Professor Mills presented a simplified method of obtaining the same result. Given two series of paired items, a_i, b_i $(i=1, \ldots, n)$, the standard error of the

difference of the means is given by the formula

$$\sigma(M_1 - M_2) = \frac{1}{N}\left(\sum (b-a)^2 - \frac{\sum (b-a)^2}{N}\right)^{1/2}. \tag{II}$$

The paper was concerned with the mathematical validation of formula (II). Examples were given to illustrate the method.

4. After a few comparisons between the rigid motion, affine, and projective groups of transformations of a plane, Professor Simmons gave a method for constructing an affine differential geometry of curves in a linear n-space, then certain results for the special case of plane curves. A method of procedure for studying ruled surfaces under affine transformations was also given. The procedures used for plane curves and ruled surfaces are analogous to corresponding projective theories of Wilczynski.

5. In constructing graphs of equations between two variables, x and y, two rectangular space coördinates were used by Doctor Lange to represent the whole field of complex values for one of the two variables (e.g. for y), while the third rectangular space coördinate was used to represent (1) the real values of x, (2) the imaginary values of x, (3) one-dimensional fields of complex values of x, in particular those which, in a complex x-plane, lie on the straight lines $\theta = \text{const.}$ (θ being the amplitude of the complex number x). The third scheme is thus the generalization of (1) and (2), ($\theta = 0°$, $\theta = 90°$), and allows us to show the gradual transition from (1) to (2). A number of elementary algebraic and transcendental functions were represented in this way. Conic sections in the real plane appear associated with complementary conic sections in the imaginary plane (ellipses with hyperbolas and vice versa, parabolas with parabolas); the three-dimensional graphs of $y = x^{1/n}$ are n-valued throughout; isolated points in the real plane are seen to belong to imaginary branches of the locus, etc. Many graphs of these loci were demonstrated.

6. Professor Rufus Oldenburger reported on the quadrennial International Mathematical Congress held at Oslo, Norway, July 13–18, 1936. About 500 delegates attended, representing 40 countries of the world, all of the larger countries with the exception of Russia and Italy. The Congress was devoted to general sessions with lectures of 45–60 minutes duration, and special sessions each of which consisted of short papers in a particular field of mathematics. Professors Oswald Veblen, G. D. Birkhoff, Norbert Wiener, and Oystein Ore were the American lecturers on the general program. The Congress members were guests of their royal highnesses, King Haakon and Queen Maude, at a palace tea, dinner guests of the city of Oslo at the Hotel Bristol, and guests of the government aboard the S.S. Stavangerfjord on a cruise in company with the crown prince and princess. The next Congress will be held in the United States in 1940.

7. A set of transformations for a surface bearing a family of curves in autoconjugacy of type $\nu(\nu > 1)$ was submitted by B. Segre (*Annales Scientifiques de l'Ecole Normale Supérieure*, (3) vol. 44, 1927, pp. 152–212). In the transforma-

tions of Segre, corresponding points of a given surface and its transform are connected by osculating linear spaces of the curves of a family on the given surface. Professor Gore (*Transactions of the American Mathematical Society*, vol. 36, 1934, p. 538) has shown that transformations similar to those of Segre are applicable to any surface which cuts across the above osculating spaces in such a way that the points of the cutting surface are in one-to-one united relation with the osculants of the set. Such a surface is said to be *transversal* to the osculants of the set. It is desirable to determine the point differential equations of these transversal surfaces, and to express the transformations of them in terms of the elements of their point differential equations. While these desiderata have not been attained for the general case, results were submitted by Professor Gore for the case $\nu=2$.

8. A problem of great importance at the present time in junior college mathematics is to present material suitable for those students who wish to take a single course in the subject. Dr. Peters stated his belief that by a discussion of topics closely related to the experiences and problems of the average individual, the student's interest in the subject can be awakened most successfully, and that upon realizing the fundamental importance of mathematics in many fields he may become interested in the subject for its own sake. Some of the topics that could be discussed are the number system; the mathematics of finance, probability and insurance; functions and their graphs; simple problems illustrating the uses of the differential and integral calculus; and elementary statistics.

EDITH I. ATKIN, *Secretary*

THE 1937 MEETING OF THE TEXAS SECTION

The annual meeting of the Texas Section of the Mathematical Association of America was held at the Rice Institute, Houston, Texas, on Friday afternoon, April 23, and Saturday morning, April 24, 1937. All sessions took place in the Chemistry Lecture Room of the Rice Institute and were presided over by Professor L. R. Ford, Section chairman for 1937.

Among the thirty-seven persons attending were the following twenty-two members of the Association: E. F. Beckenbach, J. H. Binney, L. W. Blau, H. E. Bray, Myrtle C. Brown, H. E. Buchanan, J. E. Burnam, Alice C. Dean, Nat Edmonson, H. J. Ettlinger, L. R. Ford, E. H. Hanson, J. A. Hurry, E. C. Kennedy, J. N. Michie, R. L. Moore, M. E. Mullings, J. W. Querry, W. A. Rees, C. R. Sherer, F. E. Ulrich, H. S. Vandiver.

Those attending the meeting were guests of the Rice Institute at a dinner served in the Faculty Clubhouse on Friday evening, and at a luncheon on Saturday, also at the Faculty Clubhouse. Professor J. H. Binney of the Agricultural and Mechanical College of Texas, and Professor H. J. Ettlinger of the University of Texas, were elected chairman and vice-chairman respectively for the coming year. The decision as to the time and place for the 1938 meeting was left to the Section officers.

The following papers were read:

1. "Roots of polynomials" by Professor H. E. Bray, Rice Institute.

2. "Polynomial expansions in a Borel region" by Dr. J. T. Hurt, A. and M. College of Texas, introduced by Professor Ford.

3. "Simple types of semi-rings" by Professor H. S. Vandiver, University of Texas.

4. "The relations between solutions and integrals of systems of differential equations with applications to the three body problem" by Professor H. E. Buchanan, Tulane University.

5. "The structure of continua" by Professor R. L. Moore, University of Texas.

6. "On surfaces of negative curvature" by Dr. E. F. Beckenbach, Rice Institute.

7. "A new approach to Euler summability" by W. C. Mitchell, A. and M. College of Texas, introduced by Professor Edmonson.

8. "Fuchsian groups of genus 2" by E. C. Kennedy, Rice Institute.

9. "Mathematics as a prescribed subject for graduation from college" by Professor J. N. Michie, Texas Technological College.

Abstracts for some of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Professor Bray discussed the following inequalities:

$$\text{(a)}\qquad \sum_{i=1}^{n} x_i^m/n - \sum_{i=1}^{n-1} \xi_i^m/(n-1) \lesseqgtr 0, \qquad m \lesseqgtr 1,$$

$$\text{(b)}\qquad \sum_{i=1}^{n} (x_i/n)^m - \sum_{i=1}^{n-1} [\xi_i/(n-1)]^m \lesseqgtr 0, \qquad m \lesseqgtr 1,$$

where the quantities x_i are real, positive, and not all equal, $\{x_i\}$ are the roots of a polynomial $P(x)$, $\{\xi_i\}$ are the roots of the derivative $P'(x)$. A method of proving these inequalities was demonstrated, the method consisting in determining the differential changes in the left members regarded as functions of the variables x_i. A similar method of proving certain theorems, in the case where $P(z)$ has complex roots, was demonstrated. This consisted in moving the roots $\{z_i\}$, of $P(z)$, on circles with centers at the origin in order to study changes in the absolute values of the roots $\{\xi_i\}$ of the derivative $P'(z)$ or of other related polynomials.

2. Dr. Hurt discussed the expansion of analytic functions into a series of polynomials. An explicit form for the polynomials was obtained in terms of the coefficients of a Taylor's expansion and of a sequence of positive constants. Conditions were imposed upon the constants such that a region can be found within which the series will converge to the function. This region is in general larger than the circle of convergence of the Taylor's series.

3. A semi-group is defined as a system closed under an associative operation and subject also to the usual equivalence and substitution postulates governing

equality. A semi-ring is defined as a system which forms a semi-group under an operation which we may call addition, a semi-group under an operation called multiplication and both right- and left-hand distributive laws hold:

$$A(B+C) = AB + AC; \qquad (B+C)A = BA + CA.$$

Employing these definitions the well known system called a ring may be defined as a semi-ring in which the elements form a group under addition. Professor Vandiver (*Bulletin of the American Mathematical Society*, 1934, vol. 40, pp. 915–16) set up a commutative semi-ring consisting of the distinct elements $C_1, C_2, C_3, \cdots, C_j$ with $C_i = C_{j+1}$ where $j+1>i$ and noted that from $C_a + C_b = C_a + C_k$ we cannot infer $C_b = C_k$ in general, provided $i>1$. In view of this it was shown in the present paper that this particular semi-ring for $i>1$ cannot be embedded in any ring whatsoever, hence the ring is not the fundamental system for associative algebra of double composition. The possible types of additive semi-groups contained in the semi-ring were also studied.

6. Dr. Beckenbach showed that a necessary and sufficient condition that the Gaussian curvature of a surface S be $\leqq 0$ is that for all conformal maps of S on (u, v)-domains D, and for all circles C which together with their boundaries B lie in D, and for all positive integers n and all sets of n equally spaced radii of C, the sum of the lengths of the images on S of the n radii of C is $\leqq \frac{1}{2} \csc \pi/n$ times the length of the image on S of B. The sign of equality holds when S is a plane or a developable surface and C is mapped on a regular polygon P of n sides, and the n equally spaced radii of C are mapped on radii of the circle circumscribing P.

7. The E_1-transformation of the sequence (S_n) is defined by Knopp to be $S_n' = 1/2^{n+1}\sum_{i=0}^{n} \binom{n+1}{i+1} S_i$, and the pth order, obtained by iteration in a manner entirely analogous to the method of Hölder, is given by $S_n^{(p)} = 1/(2^p)^{n+1}\sum_{i=0}^{n}(i+1)(2^p-1)^{n-i}S_i$. But our development, on the contrary, is analogous to the Cesàro method, and whereas the Hölder and Cesàro methods are equivalent for all orders, Knopp's integral orders are not only equivalent to, but his means are identical to, certain of ours. This point of view, Mr. Mitchell stated, is fruitful in giving rise to several theorems which contain some of Knopp's results as special cases. It also leads to a new definition of the product of two infinite series which affords a product theorem for E-summability similar to the Cauchy product theorem of Cesàro. Also, we have generalized E_r-summability in the same way that Sannia generalized the Borel process, although we obtain processes of different power only when $r \leqq -\frac{1}{2}$ (or, in the Knopp notation, when $p \leqq -1$). Corresponding to this we have a more general product theorem.

8. In this paper Mr. Kennedy dealt with a class of Fuchsian groups of genus two whose fundamental region is bounded by eight equal arcs. He pointed out the connection between these groups and the theory of the hypergeometric differential equation and discussed the associated Riemann surfaces along with their corresponding algebraic functions and Abelian integrals of the first kind.

9. In this paper Professor Michie reviewed the position of mathematics in

relation to the requirements for the A.B. degree since the introduction of the elective system into American "higher" education, and pointed out the reason for its early inclusion as a required course, namely, that it was recognized as affording a mental discipline not provided by other studies. The opposition of present-day educators to this older attitude was briefly summarized, their disbelief in the concept of "mental discipline" and the "transfer" of learning skills, and their current tendency to meet what they call "individual differences" by altering degree requirements, often at the expense of academic standards. That none of their arguments for dropping mathematics from the list of required courses is entirely valid was asserted, and was supported on the grounds that whereas most studies are concerned with the matter of thought, mathematics is concerned with a method of thought, a method which is essential for logical thinking in any field.

NAT EDMONSON, *Secretary*

RENÉ DESCARTES*

By R. E. LANGER, University of Wisconsin

"While others walked in formulas and hearsays, contented enough to dwell there, this man could not so screen himself. The great mystery of existence glared in upon him with splendours no hearsays could hide. Innumerable men had passed across this universe with wonder, with a fruitless wonder, till this great Thinker, this original man came. He was a son of order, a calmly seeing eye, whose mission it was to make what was chaotic into a thing ruled and regular. A great intellect, his insight was not a transitory glance, it was an illumination of the whole. His thought awakened the slumbering capabilities of others. The word of such a man, men must listen to."

These words, written by Thomas Carlyle, in other contexts to be sure, epitomize, it seems to me, as none could much better do, the immortal significance of Descartes. A great and versatile genius, his kinship is claimed by the philosopher, the scientist, the man of letters, and the mathematician, and by none more justly than the last. For, founder of modern mathematics as he was, by virtue of his invention of analytic geometry, he also carried through life a conviction that true knowledge in any sphere is attainable only by the mathematical mode of thought. In this type of thought his critical mind found nothing to criticize. Its flawlessness and logical keenness satisfied him, its relentless powers for significant penetration and for the progressive reduction of difficulties impressed him, to the extent that he saw in it the standard upon which all thought should perpetually seek to pattern itself. The great Leonardo da Vinci had in his time said: "The man who undervalues mathematics nourishes himself upon confusion," and Galileo, following him, had asserted that "The book

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* Written on invitation of the editor in commemoration of the tercentenary of the publication of the "Geometry."

of nature is written in the language of mathematics." Descartes, going further than these, conceived of a mathematical basis for the whole field of possible knowledge.

The period of Descartes's life—he was born in the year 1596—marks the culmination of that supremely interesting and remarkable epoch in human history, in which one after another, in a bewildering succession, the shackles which bound the minds and conceptions of men were struck away, and the world, liberated from its thousand year stay in medievalism, passed into the modern time. This epoch had begun in 1453 when Constantinople fell to the Turks, to the extinction of the Roman Empire of the East. Byzantine scholars, dispersing over western Europe, had carried with them their stock of the learning of ancient Greece and Rome. For Europe this meant the rediscovery of the glories and achievements of antiquity in more nearly their true and original forms, without the distortions incident to countless commentaries and translations, and without the general heavy patina of perversion, which had accumulated itself during the ages when the custody of learning was lodged almost solely with the Arabian nations.

This, as it proved to be, was but the beginning of a cataclysmic sequence of events, the powerful wake of which was to sweep away generally the hemming barriers of the traditional, and to effect an intellectual release from the thralldom of many an ancient belief and superstition. In the course of little more than a century there followed, namely, the development of printing, the great voyages of Columbus, the explorations and conquests of the Americas, the Protestant Reformation, the achievement of the Copernican theory of the universe, and finally, in a very large measure through Descartes himself, the emancipation of philosophic and scientific thought from the confining bonds of the medieval scholasticism.

An awakening so general could not but be accompanied by the profoundest of social readjustments. Romantic chivalry passed from the scene. The adventurous knight gave way to the professional soldier, the fortified feudal castle, with its towers and moat, yielded to the chateau and palace with their parks and gardens. Wars, which were next to universal, extended sometimes over periods as long as human lifetimes, and served to mold the great nations of Europe from erstwhile bands of petty chiefs and feudal tyrants. In the higher social classes there was privilege, and a refinement of manners and dress; in the lower ones oppression, with increasing poverty and distress.

Descartes's birthplace was La Haye, a small town about two hundred miles southwest from Paris, in the province of Touraine, France. He was born there on March 31, 1596, into a family of some financial means, which belonged to the lesser nobility. His father, Joachim, was a councillor of the Parlement of Brittany, which is to say that he was a man of professional training and practice—in effect a provincial judge. As such, his family enjoyed some political privileges and exemptions from taxation, matters which were regarded with pride as enhancing its dignity, and which undoubtedly made for a conservative, conven-

tional, and pious outlook upon life. The family was respectable, but aside from Descartes, himself, undistinguished. For his father, Descartes always felt a warm and sympathetic filial affection. His mother he never knew, as she died soon after his birth. A brother and a sister completed the family circle, but with them his contacts seem never to have been intimate. In later years they apparently regarded his mode of life as somewhat mildly disgraceful, and seem never to have guessed of his great intellectual significance.

Descartes's formal education was that customary at the time for a young gentleman. It began when he was eight years old, extended into his sixteenth year, and was imparted at a large, and then newly founded, Jesuit school, La Flèche. Here a five year curriculum in the humanities was followed by two years of logic, physics, and metaphysics, and finally, to cap the matter off, by a year in mathematics. The preceptors were amiable, cultured men. Order and discipline were punctiliously, but not harshly, observed, and the academic instruction was supplemented by training in manners, dancing, riding, fencing, etc., that is to say, in the social graces. From his own assertions Descartes's school years were happy ones. He was accorded the privileges of a young noble, and, all in all, formed for his school and teachers an affectionate attachment, which even in the later years of his life made him desirous, to the point of anxiety, to retain the good will of the Jesuit fathers.

The studies of his earlier school years, the languages, history, eloquence, etc., Descartes pursued without either aversion or great enthusiasm. With science, philosophy, and mathematics, however, things stood differently. Even at this age his bent for thought in these fields made him not only an eager student, but also a profound one. The science which was then taught was the science of the scholasticists, a very different thing from what we today regard as science. The known facts of nature were then of a meagre stock. The versions of them, though they generally derived from Aristotle, had been to a large extent adapted, interpreted, elaborated, even peradventure distorted, by the earlier fathers of Christianity, to bring them into a suitable conformity with the theology of the Church. The foundations of this science were accordingly most uncertain, its concepts were often diluted by a generous admixture of mysticism and superstition, and the whole was buttressed, not by any reference to experimental evidence, but by reference to the writings of acknowledged, although perhaps mistaken, authorities. Men were taught to accept, not to inquire. Scientific works were works filled with error, with plausible argument, with elaborate reasoning and with hair-splitting distinction, but with little or no real fact or logical deduction. In this the boy Descartes found little to satisfy him. Where he sensed the need of fact, he found often in its stead only a labyrinth of words, which left him in doubt and confusion. This, however, as well as his equal dissatisfaction with philosophy, and his contrasting satisfaction with mathematics, can perhaps be read best in words of his own, which he wrote in later manhood. He wrote:

"I had been assured I could acquire a clear and certain knowledge of all

that is useful in life. I had an extreme desire to learn them. But as soon as I had completed the course of study, at the end of which one is usually received into the rank of the learned, I entirely changed my opinion. For, I found myself embarrassed by so many doubts and errors, that I thought I had gained nothing else from trying to instruct myself, than to have more and more discovered my ignorance. I had learnt all that others learnt;---I had run through every book treating of such matters---that I could lay hands on---I did not see that I was deemed inferior---my own times appeared to me as fertile of good wits as any that had gone before. All of which made me think that there was in the world no such learning as I had been led to hope for. I took pleasure, above all, in mathematics, because of the certainty and the absoluteness of its reasons; but I had not yet found out its true use; and, thinking that it served only for the mechanical arts, was astonished that, its foundations being so firm and solid, nothing had ever been built on them that was more exalted. Concerning philosophy I will say nothing, except that, seeing it had been cultivated by the most powerful minds that had lived for many centuries, and that nevertheless there was not yet to be found in it one single thing which is not disputed, and therefore open to doubt, I had not the presumption to hope that I should succeed better than others; and considering how many different opinions there are, touching one and the same matter, all of which are maintained by learned persons, while it was impossible that more than one of them could be true, I regarded as little better than false everything that was merely probable. Then for the sciences, since they all borrow their principles from philosophy, I judged that nothing solid could have been built on foundations so far from secure."

The issue of all this was, that Descartes, having finished school, laid away his books, and planned upon seeing something of the world. What more fitting for a young cavalier—and what a better beginning for it than a sojourn in the great city of Paris! And so to Paris he went, to learn amidst its gaieties, at seventeen, other lessons than those taught him by the religious fathers.

The fashionable center of Paris at that time was to be found in the gardens of the Tuileries. The Cours la Reine, an alleyed thoroughfare, with stately arch and shaded by elms, ending where the present Champs Élysées begins, was the daily rendezvous of the fashionable and of the court. Here the carriages, often several abreast, emblazoned with heraldic devices and upholstered and curtained with silk, bore their fair passengers. The gallants, meanwhile, on foot with hat in hand, bowed to all, acquaintance and stranger alike, for by the custom of the time the ladies were generally masked, and indiscriminate politeness was therefore hardly less than a necessity. We may assume the young Descartes a frequent participant in throngs like these. He was always fond of dress and fastidious of his appearance; of his large feathered hat; lace collar; silk shirt puffed on the chest and at the waist; of his voluminous breeches of green taffeta or velvet ending in a large frill at the knee; of his silk stockings; shoes with bows and colored heels; of his ribbons and lace all over; and the rapier to mark his noble birth.

Dancing was then in favor, as it was in no other period of French history, and gambling was a universal passion among those with means. Descartes was not proof against their fascination, nor probably against many other fashionable vices of the times. His excesses, however, were probably not great. For a year, not longer, this life of glamor, frivolity, and idleness held him. Then an urge for intellectual occupation returned, and asserting itself caused him to abandon his associates and to seclude himself in a less fashionable, a bourgeois, section of Paris, a Paris far different from that which environed the court, or from the Paris of the present day. Here the streets were narrow, and crooked, and unpaved, and were filled, like the streets of all larger cities of the earlier seventeenth century, with filth and stench. Refuse and sewage were naively thrown or dumped into them from the windows, with a warning cry for passers-by, and flowing down their gullied centers, were spattered about by rider or carriage to the dismay of those on foot. Noises were many, as varied and distinctive cries announced the sand-man, the water-carrier, the rat-poisoner, the chimney-sweep, or the peddlers of wines, perfumes, fish, milk, or pastry.

In surroundings such as these Descartes spent two years in serious study. The subject of his attentions during this period was primarily mathematics. He had met, since his arrival in Paris, a former schoolmate, the monk and mathematician Marin Mersenne, and through him had come to know the somewhat older Mydorge, who was at the time regarded as the leading mathematician of France, the successor of Vieta. Mersenne was an intellectual person, whose eagerness to know whatever was going on in the world of learning had resulted in his becoming a general intermediary for communication of scientific discoveries and opinions. He was in correspondence with practically all learned persons of his day, and acted for everyone as a transmitter of books and criticisms. He became in many respects the closest of Descartes's friends, and in later years acted often for him as agent and Parisian representative. Descartes had thus found in Paris an atmosphere full of incentive for his mathematical studies, although, to be sure, he required little in the way of outside incentive for study or contemplation at any time.

Descartes's manner of work, which he followed at this time, and in fact persisted in throughout his life, was, to say the least, peculiar and unique. He slept much, and, after allowing himself to awaken naturally in the morning, would remain in bed, spending the hours there engaged in his work, his reading, or meditation. He thought this practice beneficial to his health, which was never robust, and at the same time found in his bed the quiet and comfort and freedom from disturbance which he needed for concentration. To outward appearances he was a young man leading a life of indolence, one which was regarded his privilege by virtue of his financial independence.

As the novelty of life in Paris wore down, Descartes's desire to see more of the world grew. Travel, so easy and so widespread in our time, was much less so in the Europe of three hundred years ago. The land was covered with forests, the roads were few and perilous, and many days on horseback were consumed in

traversing relatively short distances. Towns were walled and fortified, and senseless customs barriers impeded intercourse between them. Under conditions such as these the resort of young adventurers was by custom the military life. And so Descartes took up the profession of arms, joining for that purpose the army of the Netherlands.

The life of a soldiering adventurer in those days was not too austere. The armies were then almost wholly composed of mercenaries, and so were rarely embarrassed by any ties of feeling or sentiment with the causes in which they were engaged. Adventurers of means and rank associated themselves in considerable numbers with the favorite or more renowned commanding generals, and as they ordinarily accepted no pay, they in turn held themselves aloof from any strict discipline. They often came and went as they chose, and during an encampment quite customarily spurned the rigors of the tent for the greater comfort of private quarters in nearby towns. This was the manner of Descartes's soldiering. Holland had but just won her independence from Spain, and was in an armed state prepared to retain it. During two years with her army Descartes met scientists, mathematicians, and engineers, and continued his studies, but saw no fighting.

It was the year 1620 when, like the American Pilgrim Fathers, Descartes first departed from Holland. The outbreak of the Thirty Years War at this time promised much greater activity and excitement, and so Descartes, changing his allegiance, travelled to Germany and enlisted in the army of the Catholic League. In the campaigns which followed he saw first Germany and Austria, then Bohemia and Hungary. It is, however, doubtful that he ever actually took part in any battle.

Though Descartes later professed himself to have been repelled by much of the rawness of soldiers' deportment, and though he deprecated much that he experienced in army life, he nevertheless, and paradoxically enough, found in the army an excellent environment for his intellectual work. His mental labors were invariably concentrated into short periods of time, and these he always sought to intersperse with extended periods of idleness or complete distraction. The doubts and dissatisfactions which had troubled him in his school days had never left him. They had, on the contrary, grown with him in his maturity, until they had come to absorb him completely. In his privacy he thought of nothing else, until during the winter which he was spending with the Catholic army, encamped on the Danube, he came to what he himself styles the crisis of his life. It appears that here, and of a sudden, the road to mental clarification and the dispersal of troublous uncertainties had finally revealed itself to him. One cannot do better than to take this entirely in his own words. He says:

"I had always a most earnest desire to know how to distinguish the true from the false, in order that I might be able clearly to discriminate the right path in life, and proceed in it with confidence. . . . In considering the manners of men I found scarce any ground for settled conviction and remarked hardly less contradiction among them than in the opinions of the philosophers. So . . .

I learned not to entertain too decided a belief, in regard to anything of the truth of which I had been persuaded merely by example and custom, and thus I gradually extricated myself from many errors powerful enough to darken our natural intelligence, and incapacitate us in great measure from listening to reason. . . . I at length resolved to make myself an object of study, and to employ all the powers of my mind in choosing the paths I ought to follow. . . . I was in Germany, attracted thither by the wars, and as . . . the setting in of winter arrested me in a locality where I found no society to interest me, and was besides fortunately undisturbed by any cares or passions, I remained the whole day shut up in a warm room where I had leisure to occupy my attention with my own thoughts. . . . As for the opinions which up to that time I had embraced, I thought that I could not do better than resolve at once to sweep them wholly away, that I might afterwards be in a position to admit either others more correct, or even perhaps the same when they had undergone the scrutiny of reason. I firmly believed that in this way I should much better succeed in the conduct of my life, than if I built only upon old foundations, and leant upon principles which, in my youth, I had taken upon trust.

"The long chains of simple and easy reasonings by means of which geometers are accustomed to reach the conclusions of their most difficult demonstrations led me to imagine that all things, to the knowledge of which man is competent, are mutually connected in the same way, and that there is nothing so far removed from us as to be beyond our reach, or so hidden that we cannot discover it, provided only we abstain from accepting the false for the true, and always preserve in our thoughts the order necessary for the deduction of one truth from another. . . . Considering that of all those who have sought truth in the sciences, the mathematicians alone have been able to find any demonstrations, that is, any certain and evident reasons, I did not doubt but that such must have been the rule of their investigations. I resolved to commence, therefore, with the examination of the simplest objects and believed that the four following (precepts) would prove perfectly sufficient for me, provided I took the firm and unwavering resolution never in a single instance to fail in observing them. The first was, never to accept anything for true which I did not clearly know to be such. The second, to divide each of the difficulties under examination into as many parts as possible and as might be necessary for its adequate solution. The third, to conduct my thoughts in such order that, by commencing with objects the simplest and easiest to know, I might ascend by little and little, and as it were, step by step, to the knowledge of the more complex. And the last, in every case to make enumerations so complete and reviews so general that I might be assured that nothing was omitted.

"The chief ground of my satisfaction with this method, was the assurance I had of thereby exercising my reason in all matters, if not with absolute perfection, at least with the greatest attainable by me."

It is not easy for us to recognize now the profundity of these conclusions, which then exalted Descartes almost to the point of ecstasy. They were, how-

ever, revolutionary enough, and in their development in later years was to be consummated the transition from the medieval to the modern philosophy. It was in every sense a feat of magnificent intellectual heroism, to conclude that the spell of the past could and must be broken, that the oracular authority of antiquity had to be disregarded, and that a true reliance could be placed, and could only be placed, upon one's self and upon one's power to think.

Descartes finally left the army in Hungary. An immediate return to Paris, however, he found not feasible, for the plague was raging there. And so he continued his travels privately through Poland, along the shores of the Baltic, through the Netherlands and so to France. The call of the road, however, was still upon him, and shortly thereafter he resumed his peregrinations, this time journeying to Switzerland and the Tyrol and into Italy. In Rome the great Church of St. Peter's had then but just been completed, and the year—it was 1625—had been declared a jubilee. The city was accordingly filled with pilgrims and visitors from all over Europe, and Descartes, who had always loved to witness throngs and ceremony and pageantry, thought that what he saw here would form an impressive and fitting climax for his travels. He felt that, now that he had attained the age of twenty-nine, the unrest of his youth had burned itself out, and so returning to Paris he set about in all seriousness to develop the thoughts which for years had been seething within him.

Although to outward appearance Descartes still led the life of an idler, with nothing to pursue but his pleasure, the actual fact was now far different. He was mentally mature, an accomplished mathematician, a great philosopher, and withal one who knew and had confidence in his own powers. He refreshed his intimacy with Mersenne, renewed his friendship with Mydorge, and made the acquaintance of the geometer Desargues. He was introduced to Cardinal Richelieu and became intimate with the writer Balzac. His philosophical ideas had ripened to the point where he began giving utterance to them, with the result that he became an object of much interested attention. The spirit of the times was in sympathy with his views. The work of Francis Bacon in England, of Galileo in Italy, of Kepler in Germany, and of many others, had so weakened the position of scholasticism that its obsolescence was presaged despite the continued support of it by the Church.

Though he was never himself a fluent conversationalist Descartes always did enjoy conversational intercourse. His place of residence, therefore, became in time a sort of academic meeting place, in fact an increasingly popular one, as he was all too soon to become aware of. In fact, by the time he had been in Paris for three years the distractions had grown to such proportions, that he began to foresee the complete frustration of his work, unless by some means he withdrew himself from access to others. The prospect of a hermit's existence would have held no appeal for him. He liked the life and bustle of a big city. What he wanted was to find himself alone, unnoticed, and therefore undistracted in a throng, and so, having quietly made his decision, he secretly left Paris in 1629, and, settling himself at Amsterdam, began a twenty years'

residence in Holland. He never again resumed life in his native country.

Despite the cruelty and the almost interminable duration of the war by which Holland had won its independence, it was at that time a country of abounding prosperity. For its size it was the richest country in Europe. With a great and rapidly developing colonial empire, the Dutch had outstripped all rivals to become the foremost commercial people of the world, and the canals of their larger cities were filled with the vessels of all nations. With flourishing trade they had found luxuries and wealth. And the center of all was Amsterdam, a quaintly picturesque city, charming to an artist's eye, upon whose streets were to be seen strangers of all nationalities. Nor did the Netherlands present a picture of trade alone. On the contrary their power and riches had risen to no greater heights than their intellectual significance. In literature, in science, and in education they were the peers of any, while in the world of art they were beyond comparison, what with Rembrandt and Hals, and Brouwer and Ruysdael, and a positive host of other immortals, painting at Amsterdam. Holland was no hermitage.

Descartes let his whereabouts be known to Mersenne, but otherwise took elaborate precautions to keep his residence secret. The search after knowledge without prejudice, on the basis of precepts which he had formulated in Germany a dozen years before; the construction of a philosophy upon these precepts and of a science upon that philosophy—this was a project the immenseness of which even he did not guess. He feared his life might be terminated before he could bring it to a proper completion, never dreaming that a host of thinkers working for centuries after him could not so complete it. He immersed himself in the subject and it grew and grew. His philosophical and mathematical cogitations, while they continued to occupy him, were supplemented now by serious work in the sciences. Optics, acoustics, mechanics, astronomy, chemistry, medicine, physiology, and anatomy, all claimed his time. The facts and materials upon which calculations and deductions were to be made required experiment. It was one of his primary principles to accept nothing upon hearsay. And so experiments were made; on the one hand, on the weight of air, on the refraction of light, on the falling of bodies and the inertia of matter, etc., while, on the other hand, the butcher-shop and the abattoir were visited for parts of animals which were carried home for dissection. The great versatility of his genius began to show itself. A critical analytical thinker, he revealed himself now as also a superb experimenter and keen observer, who combined with these faculties a preëminent constructive imagination. He saw, not only inwardly, but outwardly as well, and the kaleidoscope of empirical detail which he first uncovered, then tested and mentally dissected, he lastly built into theories to yield a great, consistent and understandable interpretation of the universe.

His first efforts to embody his results in essays foundered upon the continual expansion of his subject. He conceived therefore of writing an almost encyclopaedic treatise which he chose to entitle *The World*. The work progressed, and was nearing completion when disaster came upon it, for Descartes was sud-

denly faced with the annihilating spectre of conflict with the Church. Having sought to obtain a copy of Galileo's *Dialogues on the System of the World, according to Ptolemy and Copernicus* which had but recently been published and in which the motion of the Earth was maintained, he learned to his consternation of Galileo's summons to the Inquisition at Rome, of his condemnation and enforced retraction, and of the proscription and burning of his book. This meant for Descartes a complete change of plans. He was alarmed. The publication of his own book would have made him equally guilty with Galileo, and even without publication the guilt was there, and would most certainly be punished if it became known. Descartes's character was not one of a heroic stamp. His temperament rather was cautious, cold, calculating, and selfish. He was timid, and in short had nothing of the crusader about him. The Church, on the other hand, was at that time the most powerful and ruthless enemy one could have. It claimed an absolute authority and against its decrees the opinions or convictions of no individual could withstand. Its punishments could be severe, in extreme cases it still took recourse to death by fire at the stake.

To be sure, Descartes was in no danger of physical harm so long as he remained in free Holland. This, however, was far from settling the matter insofar as he was concerned. He had, namely, identified himself all his life as an orthodox Roman Catholic. The precepts of his early Jesuit teachers were firmly ingrained in him. Though it seems certain that religion was never really deeply and spiritually felt by him, it is equally certain that he wanted conscientiously to conform to its tenets, and to give his obedience to the Church. An excommunication would have meant disgrace for him, and the condemnation of those whose good will he had every desire to retain.

There was, therefore, for Descartes but one way out of his difficulty. He decided to abandon the results of his four years of work, gave up his project of publication and destroyed many of his papers. It was a pity! Not that the material was all lost, for much of it was incorporated in later works, and certainly not that the treatments given would all have been scientifically acceptable today. Far from it; for in Descartes there is almost everywhere a strange blending, sometimes even to the point of absurdity, of modern viewpoints with antiquated speculation. His theories are often pure fiction, but even when they are so they are almost invariably splendid fiction. The scale upon which he attempted his *World* can best be judged from his own later summary of the suppressed work. He writes of it:

"It was my design to comprise in it all that I thought I knew of the nature of material objects, but fearing lest I should not be able to comprise all, I resolved to expound my opinions regarding light, to take the opportunity of adding something on the sun and the fixed stars, since light proceeds from them, on the heavens, since they transmit it, on the planets, comets, and earth, since they reflect it; on the bodies that are on the earth, since they are either colored or transparent or luminous; and finally on man, since he is the spectator. I explained what the nature of light must be, and how in an instant of time it tra-

verses the immense spaces of the heavens, and how from the planets it is reflected toward the earth. To this I added much respecting the substance, the situation, the motions, and the qualities of the heavens and stars. I came to speak of the earth, to show how its parts tend to its center, how with water and air on its surface the disposition of the heavenly bodies, more especially the moon, must cause a flow and ebb, how the mountains, seas, fountains and rivers might naturally be formed, and the metals produced in the mines, and the plants grow in the fields, and I set forth all that pertains to the nature of fire, how it induces various colors upon different bodies, how it reduces some to a liquid state and hardens others, how it can convert bodies into ashes and smoke, and how from these ashes it forms glass. I passed to the description of animals and particularly to man. I gave the explication of the motion of the heart and arteries,—what must be the fabric of the nerves and muscles, what changes take place in the brain to produce waking, sleep, and dreams, how light, sounds, odors, tastes and heat impress it with different ideas, how hunger and thirst can impress it, what must be understood by the common sense, by the memory, by the imagination. I had after this described the soul, and showed that it could by no means be educed from the power of matter but that it must be expressly created.

"I was not, however, disposed to conclude that this world had been created in the manner I described; for it is much more likely that God made it at the first such as it was to be. I spoke only of what would happen in a new world, if God were now to create somewhere matter sufficient to compose one, and were to agitate confusedly the parts so that there resulted a chaos, and after that allowed nature to act in accordance with the laws which He had established. I pointed out what are the laws of nature, and thereafter I showed how the greatest part of the matter in this chaos must in accordance with these laws arrange itself to present the appearance of the heavens, how some parts must compose an earth and some planets and comets, and others a sun and fixed stars."

Shot through with faults as this great work unquestionably was, there was nevertheless much in it that was of inestimable service to the development of modern thought. Above all, the conception of the universe as a mechanism permitting of an understandable theory, and based upon reason and mathematical demonstration, was monumental—an immense advance out of the realms of mystery and obscurantism and superstition in which things such as these had up to then been lodged.

As Descartes's fright over the Galileo incident wore off with time, his fear that the destiny of his ideas would be oblivion and futility returned to him. He had been pondering and studying and meditating for years, he was in the prime of his life, his ideas were mature, but the world had as yet had nothing from him. He became convinced at last that he must publish. And so in June of the year 1637—just three hundred years ago—he published at Leiden, Holland, his first work, his *Discourse on the Method of rightly conducting the Reason in the search for Truth in the Sciences. With the Dioptric, the Meteors, and the*

Geometry as *Essays in the Method*. The work was epoch-making in almost every respect. The *Method* itself, which in common English translations covers no more than half a hundred pages, announced the new philosophy; the discard of scholasticism and tradition and the establishment of the supremacy of thought and the reliance upon rational deduction. It is a literary gem. Written, utterly contrary to custom, in French rather than in Latin, it showed at once the power of the vernacular even for the exacting requirements of philosophic discussion, and created, in its attempt to lay its subject clear, a new standard for simplicity, forcefulness of expression, and freedom from affectation.

The first two appendices were intended as illustrations of the use of the *Method* in the sciences. The *Dioptric*, which was made possible by the recent invention of the telescope, deals with the nature and refraction of light, giving a minute account of the eye, of optical delusions, of the theory of spectacles, the telescope, the manufacture and grinding of lenses, etc. The *Meteors* is concerned with general natural science; with the constitution of matter, of heat, of vapors, of salt, of winds, clouds, rain, snow, thunder, lightning, the rainbow, falling bodies, etc. It is not very likely that these two scientific appendices were ever very widely read. They were dry and difficult, and in their details soon superseded. Their lasting significance lay in their bold ostracism of all occult forces, and their insistence upon the fact that all natural phenomena are dependent upon, and mathematically deducible from, specifiable laws, which are themselves in turn expressible by means of algebraic formulas and equations. They thus inaugurated the subjects of mathematical science.

The final appendix to the *Method*, i.e., the *Geometry*, is in many respects Descartes's most famous work. By common consent the era of modern mathematics is dated from its publication. Descartes himself was especially fond of his discovery here, and seems to have realized, as but few of his contemporaries did, the immense significance it was to have. The use of algebraic symbols and equations in connection with geometric configurations may be traced back, of course, into antiquity. However, that mode of the use of coördinates may be described as a use of abbreviations for the recording of geometric facts and relations, rather than as a general and unified method for the discovery of such relations, a method depending upon a really intrinsic interplay of algebra and geometry. This last is Descartes's invention. He describes his motivation of it in the following words:

"Observing that the sciences of mathematics, however different their objects, all agree in considering only the various relations or proportions subsisting among them, I thought it best to consider these proportions in the most general form possible, without referring them to any objects in particular, and without restricting them, that afterwards I might be the better able to apply them to every class of objects to which they are legitimately applicable. Perceiving that in order to understand these relations I should sometimes have to consider them one by one, and sometimes only to bear them in mind, or embrace them in the aggregate, I thought that, in order the better to consider them individually, I should view them as subsisting between straight lines, than which

I could find no objects more simple, or capable of being more distinctly represented to my imagination and senses; and on the other hand that in order to retain them in the memory, or embrace an aggregate of many, I should express them by certain characters, the briefest possible. In this way I believed that I could borrow all that was best both in geometrical analysis and in algebra, and correct all the defects of the one by help of the other. The accurate observance of these few precepts gave me such ease in unravelling questions embraced in these two sciences, that not only did I reach solutions of questions I had formerly deemed exceedingly difficult, but even as regards questions of the solution of which I continued ignorant, I was enabled to determine the means whereby, and the extent to which, a solution was possible."

In contrast to the *Method* itself and its other two appendices, the *Geometry* was not written to be generally read, but was designed only for accomplished mathematicians. It bears not the slightest resemblance to the systematic approaches and developments of the subject to which we are accustomed by the modern text book, but rather plunges directly into the midst of things. Steps in the demonstrations are often omitted, and one soon ceases to feel any astonishment at statements such as these: "I shall not stop to explain this in more detail, because I should deprive you of the pleasure of mastering it yourself," or "I find nothing here so difficult that it cannot be worked out by any one at all familiar with ordinary geometry and with algebra." The humor sometimes involved in such remarks was, moreover, not wholly accidental, for in his letters Descartes chuckles here and there over the fact that some professed mathematicians were unable to open their mouths in criticism of him, because they were unable to follow his arguments. The clarity with which he himself saw the method is well shown by his description of it. He says:

"If we wish to solve any problem, we first suppose the solution already effected, and give names to all the lines that seem needful for its construction—to those that are unknown as well as those that are known. Then making no distinction between known and unknown lines, we must unravel the difficulty in any way that shows the relations between these lines, until we find it possible to express a single quantity in two ways. This will constitute an equation. We must find as many such equations as there are supposed to be unknown lines, but if after considering everything involved, so many cannot be found, it is evident that the question is not entirely determined. If there are several equations, we must use each in order, either considering it alone or comparing it with the others, so as to obtain a value for each of the unknown lines: we must combine them until there remains a single unknown line which is equal to some known line."

To introduce the fundamental notion of a curve as a locus, Descartes proposed to extend the set of geometric postulates of the Greeks by adding the assumption that "Two lines can be moved, one upon the other, determining by their intersection other curves." With this postulate the derivation of the equation of a curve from its geometric properties is given by him in quite the familiar manner, and is illustrated in a considerable number of cases. He says then

rightly: "When the relation between all points of a curve and all points of a straight line is known, in the way I have explained, it is easy to find the relation between the points of the curve and all other points and lines—thence to conceive various ways of describing the curve, and to find out all that can be determined about it."

The arrangement of the *Geometry* which is given in three "books" is in brief the following: In the first book, a description of the manner in which the operations of algebra are to be carried out geometrically is followed by the rather elaborate consideration and solution of a set of locus problems, which Descartes found in a work of Pappus, and which the Greeks had not been able to solve completely. The second book deals with curves and a classification of them which is based upon the degrees of their equations, and, following this, with Descartes's method for constructing the tangent to a curve at a given point. In this basic problem Descartes, of course, already anticipated the calculus, and that he appreciated something of its significance is clear, for he says in connection with it: "I dare say that this is not only the most useful and most general problem in geometry that I know, but even that I have ever desired to know." The third book of the *Geometry*, finally, is largely algebraic. It deals primarily with matters now classical in the theory of equations, and was characterized by its author as beginning where Vieta left off.

Descartes, whose remuneration for his great work had been two hundred copies of the book for distribution to his friends, awaited the reception of his theories with confidence. Of the reactions which came to him in time the most important by far was that from the great mathematician Pierre Fermat. Fermat, it is now known, had at that time also come to the conception of an analytic geometry, and might well have given one to the world had Descartes failed to do so. In the present instance he sent Descartes several pages of objections to his *Dioptrics*, and, enclosing a book of his own on *Maxima and Minima*, pointed out what he regarded as important omissions in the *Geometry*. Now Descartes was a self-centered man. He was not a man who could take criticism. He always resented it in the extreme, and never seems to have been able to even conceive of himself as being in the wrong. As a result of this he was almost always embroiled in disputes of one sort or another and as a rule his conduct of these was at the best ungenerous. When hard pressed he would pass from testiness to bitterness and acrimony. To extricate himself he would often twist and distort the point at issue and, by no means above resorting to unkind personalities, frequently attributed the worst possible motives to his critics. Lesser opponents, he scorned, as in such utterances of his as these: "I should not be less ashamed to write against a man of that sort than to stop and pursue any little cur that comes barking after me in the street," or "I think no more of him than I do of the abuse given me by a parrot hanging in a window as I pass." A criticism of his *Dioptrics* by the English philosopher Hobbes, by way of instance, was resented by him to the point where he desired no further dealings with Hobbes, and remarked that they could never meet, but as enemies. In the case of Fermat's criticisms the inevitable result was, therefore, a dispute, in fact one which con-

tinued until both men were thoroughly weary of it. Fermat is thought to have had the right of it, but it was from him that overtures for conciliation had to come.

Meanwhile Descartes continued his residence in Holland, not in one single abode—on the contrary, moving about frequently, living sometimes in the city and then again in the country. Wherever he was, and his whereabouts were invariably secret except to a very few, his quarters were always ample and genteel, extending in some cases, at least, to the occupancy of extensive and beautiful chateaux. His mode of living was always orderly and as domestic as his state of bachelorhood would permit. As throughout his life he continued to do his work abed in the mornings. His evenings he generally devoted to the consideration of his correspondence, which was mainly scientific, rarely personal, and of which he was painstakingly careful, while the intermediate part of the day he gave to relaxation. In matters of money he was neither extravagant nor parsimonious, showing himself in this respect a true philosopher. He always did some entertaining, now more, now less, professing to find considerable enjoyment in conversation, though he was himself rather taciturn.

In appearance Descartes was a small man of rather slight figure with a large head. His nose was prominent, his lower lip somewhat protruding, his beard and mustache of a semi-military type, and his hair growing down upon his forehead almost to his eyebrows. He wore a wig of natural color to which he always gave fastidious attention, as he did also to his clothes which were now invariably of black cloth. In demeanor he was generally cheerful, rarely gay. His manners were always refined, gentle, and polite, and his temper tranquil and easy. As a personality he was proud, somewhat aristocratically reserved, sensitive, a bit angular, and, though a shade domineering, was preëminently obliging.

Intimate details of Descartes's life are very obscure. The source to which one naturally would turn for such, namely his correspondence, fails in this respect, because of the fact that he formed very early in life the plan of sometime publishing it. His interests, aside from those to which he principally devoted his energies seem, however, to have been few. He had some love and enthusiasm for gardening, and was interested in music, at least insofar as its structural aspects were concerned. Natural scenic beauty, however, seems to have left him unmoved and neither literature nor art seem to have interested him. At any rate he never mentioned any of his great creative contemporaries in these fields, though he did sit to the painter Frans Hals for a portrait which hangs now in the Louvre.

In the year 1641 and 1644 Descartes published his second and third great works, his *Meditations* and his *Principles*, that is, more explicitly, his *Meditations concerning the First Philosophy; in which are demonstrated the Existence of God and the Immortality of the Soul*, and his *Principles of Philosophy*. The *Meditations*, as its title indicates, is concerned entirely with questions of metaphysics. It is the great classic of Cartesian philosophy, with which brief remark we must pass it by.

The *Principles* is Descartes's most systematic work. In the first part there is

a reorganized presentation of the material of both the *Method* and the *Meditations*. This is followed by parts two, three, and four, which are concerned with science. Part two treats of general physics, and embodies Descartes's attempt to refer the properties of every natural substance to the size, figure and state of motion, of the ultimate particles of which it is composed. Part four deals with the earth, with fire, water, minerals, magnets, etc., and was to a considerable extent an adaptation of his earlier suppressed work on the world. Part three is astronomical and cosmogonal. It deals with the familiar systems of Ptolemy, Copernicus, and Tycho Brahe, and contains Descartes's famous vortex theory of the universe, a theory which can hardly be given better than in the following, his own words. He says:

"We must think that the matter, not only of the sun and the fixed stars, but also of the whole sky, is fluid. Let us think that the earth is indeed at rest, but let us not think that this hinders it from being carried away by its sky, and, itself unmoved, from following the motions of that sky. And let us think that the entire matter of the sky in which the planets rest is continually revolving after the fashion of a certain vortex in the center of which is the sun, and that the parts of this matter nearer the sun move more quickly than the farther, and that all the planets—of which the earth is one—always stay in the same parts of that celestial matter. From this only, without any devices, all their phenomena are very easily understood."

This theory, a magnificent conception, was worthy of a better fate than was to be in store for it. For at its very date of publication there already lived, in Lincolnshire, England, Isaac Newton, a boy of two, who before many years was to demolish it with a thoroughness utterly without mercy, to clear the way for his greater theory of gravitation.

Newton, it must be said, never wasted either much liking or veneration upon Descartes, and in this there was certainly much just retribution. For, regrettable though it be, Descartes in his time showed neither love nor magnanimity to anyone in whom he saw a rival. The attainments of others he was always too ready to disparage, their works he almost invariably discussed, if he discussed them at all, with a carping and often undeserved criticism. He had travelled in Germany and in Italy but had disdained to seek out either Kepler or Galileo. Indeed, he found "no peculiar merit" in Galileo's great work. There can be no question but that Descartes, consciously or unconsciously, cultivated a somewhat exaggerated opinion concerning his own originality, and vastly underestimated his debt to others. He often disavowed all knowledge of, and even interest in, what other scientists were doing or had done, although posterity has in many cases awarded these same scientists the honors of priority in ideas which Descartes at least tacitly put forth as his own. On this point none other than Leibniz wrote: "Descartes made greater use of books than he wished it to appear. . . . He very admirably turned the thoughts of others into his own. . . . A large part of his best thoughts are taken from elsewhere: to which nobody could object, if he had acknowledged it in good faith. It is well known that Descartes read much more than he makes believe he did."

The productive and literary activity of Descartes during the last decade of his life was dominated by his interesting platonic friendships with two ladies, a princess and a queen. The Princess Elizabeth was a daughter of the Elector Frederick of Palatine, the Protestant Prince whose acceptance of an election to the Throne of Bohemia had precipitated Europe into the Thirty Years War, and had caused him to lose his own hereditary domain. The family was living in exile at The Hague, when Descartes was introduced into its circle, at about the time of the publication of his *Meditations*. The most intellectual member of the family was the Princess. Although she was then but twenty-three years of age, she was already competently learned, and had in particular become much interested in Descartes's work, both in philosophy and in mathematics. Though perhaps hardly a real beauty, she was a sensitive person of notable refinement, who was described as elegant and graceful. She welcomed Descartes as her intellectual master, and Descartes, no doubt flattered by her interest, became on his part deeply interested in her. There was a wide discrepancy in their social rank, a gap of twenty-two years in their ages, and their religious connections were irreconcilable. Yet a true friendship developed, and when the exile of the Princess carried her to other parts, contact between them was maintained by a correspondence which, although it was attended by all sorts of difficulties, was yet pursued with evident pleasure on both sides. Under such circumstances the letters had, of course, inevitably to be phrased with the utmost of discretion. There is in them, however, besides the primary matter of scholarly content, also much reference to more personal matters, which indicates that the friendship extended to a degree of intimacy and was assuredly sincere. There has been conjecture as to whether the relationship was, as it outwardly pretended to be. only that to be expected between the teacher and a favorite disciple, or whether perchance there was beneath the coating of appearance the unavowed and distant romantic courtship of a princess by a gentleman. The facts are that he dedicated to Elizabeth his *Principles of Philosophy*, and did so in terms of elaborate admiration and praise, and that generally, wherever she was concerned, he, the otherwise staid and confident philosopher, forgot his assurance and indulged himself with apparent enjoyment in many and extravagant gallantries. Thus he wrote her that her face was " like that which painters give to angels," that he wished "to treat her letters as misers do their treasures," and so on, and asserted that he could instruct her better in writing because of the confusion which overcame him when in conversation with her. He wrote for her, and sent to her in serial form, his treatise on *The Passions of the Soul*, and she replied regularly with comments upon each installment. Whatever the real significance of this relationship may have been, and it is not safe to plunge toward conclusions without at least recalling the customary extravagance of court manners and speech of the time, it seems assured that Descartes was actually more fond of the Princess than he was perhaps of any other person.

Christina, Queen of Sweden, was the daughter of the famous champion of Protestantism, Gustavus Adolphus. Even in her youth she had showed herself possessed of a fine intellect and scholarly interests, gifts which she combined on

the one hand with an ability in statesmanship sufficient to bring her an international fame, and on the other hand with a considerable modicum of eccentricity, which took principally the direction of a scorn for all things feminine. The growing repute of Descartes, as the founder of the new Cartesian school of thought, which was everywhere finding enthusiastic adherents, attracted the Queen's attention in about the year 1648, four years after the appearance of Descartes's *Principles*. The Queen, who was then but twenty-two years of age, had already played with the idea of founding at Stockholm a learned academy of the most noted contemporary figures in the world of letters and arts. She began, therefore, a correspondence with Descartes on the basis of a plea for instruction by him. With the development of this contact there came to Descartes in the course of time the inevitable invitation of the Queen for an extended visit to Sweden. Descartes, though much gratified, was filled with reluctance. Sweden, to him, in his own words was "a land of bears and rocks and ice" where "men's thoughts freeze in winter just as the water does." The invitation was repeated, however, and more importunately, a Swedish admiral, in fact, having been dispatched to present himself to Descartes and to offer him assistance and convoy. Descartes yielded, and after a month's journey arrived in Stockholm in October of the year 1649. He had no cause for complaint until the time ripened for his personal instruction of the Queen to begin. Then he was dismayed. The Queen, thinking that she would find herself most receptive to his teaching while her mind was yet unfatigued, appointed for his daily appearance at the palace the hour of five in the morning. The philosopher and man in Descartes cried out against such an unheard of violation of the habits of a lifetime, but the courtier in him felt compelled to obey the royal command. He did so. The winter came on. It was arctic and detestable to him, and fatal. He died of its rigors on the eleventh of February in 1650, in the fifty-fourth year of his life.

My essay must close. How futile, after all, would be any brief summary of the significance of such a man. He made history; for, as Carlyle says, "The history of the world is the biography of great men." He awakened mankind out of the sleep of dogma, and, again quoting from Carlyle, he continues "ruling from his grave whole nations and generations."

"Now comes the real power of [Descartes's] method. *We start with equations of any desired or suggested degree of complexity and interpret their algebraic and analytic properties geometrically*. Thus we have not only dropped geometry as our pilot; we have tied a sackful of bricks to his neck before pitching him overboard. *Henceforth algebra and analysis are to be our pilots to the uncharted seas of 'space' and its 'geometry.'* " E. T. Bell, *Men of Mathematics*, p. 54. Simon and Schuster, New York, 1937.

ON FOUR LINES AND THEIR ASSOCIATED PARABOLA

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Introduction. The representation of a point in the plane by a single complex number simplifies the analytic solution of certain types of problems. During the past few years in this MONTHLY have appeared articles by DoBell [1], Weaver [2], and the author [3] which explain and exemplify the use of this method. A recent book by the Morleys [4] employs this representation throughout. In this paper we use it to study a parabola and its tangent lines. We shall let z be a complex variable, $\bar{z}$ its conjugate, and t a complex variable whose domain is the unit circle, $|t|=1$.

1. *The centric center of n lines associated with a parabola.* There is always one parabola tangent to four non-specialized lines. We write its equation in the form

$$z = 2/(1-t)^2, \qquad |t| = 1. \tag{1.1}$$

The focus of the parabola is at $z=0$, the vertex at $z=\frac{1}{2}$. The equation of the directrix is $z+\bar{z}=2$, that of the tangent at the vertex is $z+\bar{z}=1$, while the axis is $z-\bar{z}=0$. The equation of the tangent to the parabola at the point t_1 can be written

$$z = 2/(1-t_1)(1-t), \qquad |t| = 1. \tag{1.2}$$

The tangents at the points t_1 and t_2 meet at

$$z_{12} = 2/(1-t_1)(1-t_2). \tag{1.3}$$

Since

$$z_{12} = 2(1-t_3)/\pi_3,$$

where $\pi_3=(1-t_1)(1-t_2)(1-t_3)$, we see that

$$z = 2(1-T)/\pi_3, \qquad |T| = 1, \tag{1.4}$$

is the equation of the circumcircle of $z_{12}z_{23}z_{31}$, since for $T=t_3$, t_1, t_2 respectively the circle (1.4) passes through the points z_{12}, z_{23}, z_{31}. Also for $T=1$, we have $z=0$, and the circle is on the focus of the parabola. Its center is

$$C_4 = 2/\pi_3 = 2(1-t_4)/\pi_4, \tag{1.5}$$

where $\pi_4=(1-t_1)(1-t_2)(1-t_3)(1-t_4)$; whence for any four tangents of the parabola, taking the lines three at a time, we have four such centers as (1.5) which lie on a circle, called the *centric circle*, (Fig. 1), also on the focus, whose equation is

$$z = 2(1-T)/\pi_4, \qquad |T| = 1, \tag{1.6}$$

and whose center is the point

$$c = 2/\pi_4. \tag{1.7}$$

This is Wallace's theorem [**4.** p. 230].

Evidently a similar proposition holds for n tangent lines of a parabola: *For* n *tangent lines of a parabola there is a circle which passes through the* n *centers of the* n *circles each associated with* n−1 *of the tangent lines of the parabola.*

We shall call the center of the circle just mentioned the *centric center* of n lines.

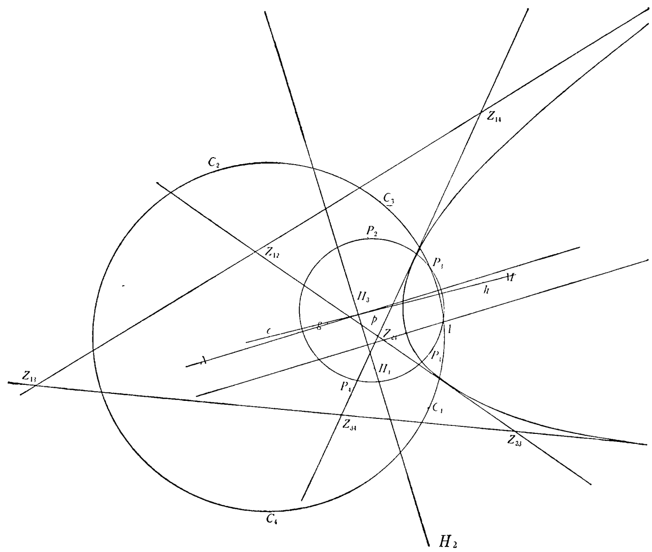

FIG. 1

2. *Certain concurrent lines*. We shall denote by S_i^n the elementary symmetric functions of n turns* $t_1, t_2, \cdots, t_n$. The equation of the line through z_{12} perpendicular to the tangent at t_3 is

$$t_3 z + \bar{z} = 2(t_3 + t_1 t_2)(1 - t_3)/\pi_3.$$

The equation of the line through z_{23} perpendicular to the tangent at t_1 is

$$t_1 z + \bar{z} = 2(t_1 + t_2 t_3)(1 - t_1)/\pi_3.$$

Hence H_4, the orthocenter of the triangle $z_{12}z_{23}z_{31}$, is the point

$$z = 2(1 - S_1^3)/\pi_3. \tag{2.1}$$

Since

$$\bar{z} = 2(S_2^3 - S_3^3)/\pi_3,$$

and $z + \bar{z} = 2$, we have the well known theorem [**5.** p. 178, problem 6]:

The orthocenter of any three tangents of a parabola lies on the directrix.

* A *turn* is a parameter whose values are complex and whose absolute value is unity.

The midpoint of (1.5) and (2.1) is F_4, the center of the nine-point circle of $z_{12}z_{23}z_{31}$. Its co rdinate is

$$F_4 = 2(1 - S_1^3)/\pi_3. \tag{2.2}$$

If we drop a perpendicular from F_4 to L_4, any fourth tangent line to the parabola, its equation will be

$$t_4 z + \bar{z} = \{t_4(2 - S_1^3) + S_2^3 - 2S_3^3\}/\pi_3. \tag{2.3}$$

For four tangent lines to a parabola we shall have four points such as F_4, and four lines such as (2.3). These four lines are concurrent at the point R, whose coördinate is

$$z = (2 - 2S_1^4 + S_2^4)/\pi_4. \tag{2.4}$$

Since

$$\bar{z} = (2S_4^4 - 2S_3^4 + S_2^4)/\pi_4,$$

we see that R lies on the directrix of the parabola. Hence we have the theorem:

For each of the four triangles, formed by taking from four tangent lines to a parabola the lines three at a time, if we draw perpendiculars from the nine-point center to the omitted fourth line, these four perpendiculars meet at a point on the directrix.

If we join F_4, the center of the nine-point circle of the lines L_1, L_2, L_3, to the focus F and lay off on that line the distance $\rho\overline{F_4F}$ where ρ is any real number, and from this point drop a perpendicular to L_4, and similarly for all four triangles formed from three out of four lines, these perpendiculars meet at the point

$$z = \rho(2 - S_1^4 + S_2^4)/\pi_4, \tag{2.5}$$

which point lies on the line

$$z + \bar{z} = 2\rho.$$

The locus of the point (2.5), for varying ρ, is a straight line through the focus, whose equation is

$$(2S_4^4 - 2S_3^4 + S_2^4)z - (2 - S_1^4 + S_2^4)\bar{z} = 0.$$

In fact, the line is the join of R and F.

3. *The Hervey point of four lines*. The line on which lie the points C_4, H_4 and F_4 is the Euler line of the triangle $z_{12}z_{23}z_{31}$. Its equation is

$$S_2^3 z + S_1^3 \bar{z} = 2(S_2^3 - S_1^3 S_3^3)/\pi_3. \tag{3.1}$$

The line through F_4, perpendicular to the Euler line (3.1) is

$$S_2^3 z - S_1^3 \bar{z} = 2(S_2^3 - S_1^3 S_3^3 - S_1^3 S_2^3)/\pi_3. \tag{3.2}$$

For four lines of a parabola we shall have four such lines as (3.2) which are all on the point

$$h = 2(1 - S_1^4)/\pi_4. \tag{3.3}$$

Hence we have the theorem:

The perpendiculars erected to the Euler line at the nine-point center for each triangle, formed from three out of four tangents to a parabola, are concurrent [6].

This point of concurrence will be called the *Hervey point of four lines* (Fig. 2).

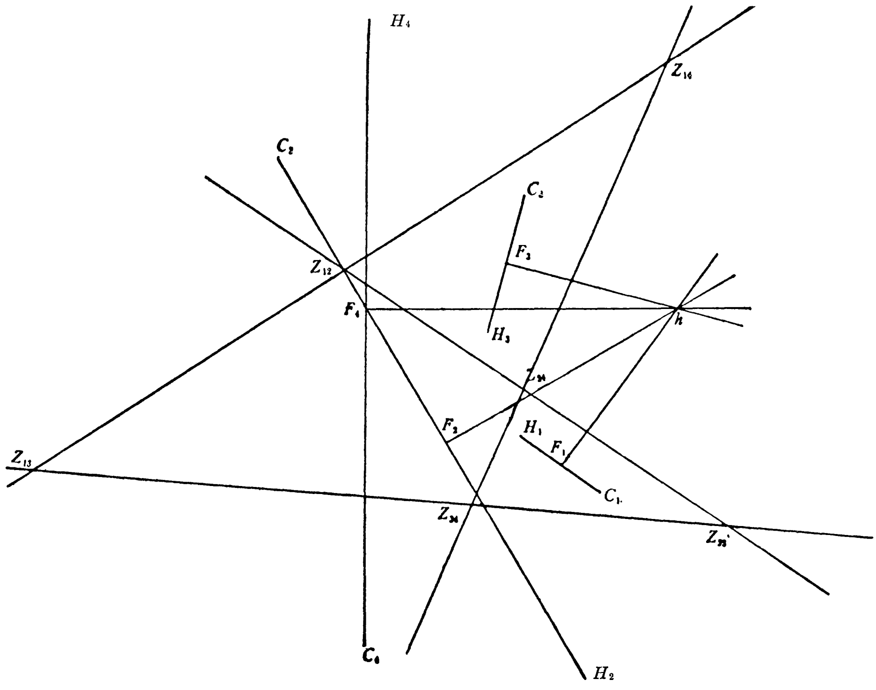

FIG. 2

4. *The centers of certain chains of circles.* The equation of the perpendicular at z_{12} to the line $z_{12}F$, where F is the focus of the parabola, is

$$t_1t_2z + \bar{z} = 4t_1t_2(1 - t_3)/\pi_3.$$

The equation of the perpendicular at z_{31} to the line $z_{31}F$ is

$$t_3t_1z + \bar{z} = 4t_3t_1(1 - t_2)/\pi_3.$$

These two lines meet at

$$z = 4/\pi_3, \tag{4.1}$$

and since this coördinate is symmetric in t_1, t_2, t_3, the perpendicular at z_{23} to the

line $z_{23}F$ likewise is on this point B_4. Now B_4 lies on the line joining C_4, the circumcenter of $z_{12}z_{23}z_{31}$, to F in such a way that $B_4C_4 = C_4F$. Now four lines of a parabola will form four triangles and thus four points B_i which lie on the circle

$$z = 4(1 - T)/\pi_4, \quad |T| = 1.$$

This circle is also on the focus. Its center is

$$z = 4/\pi_4.$$

Evidently we have here a chain of theorems which parallel the chain in §1, with the radii of the circles of this chain twice the radii of the corresponding circles of §1. The centers of the circles of §1 are the midpoints of the segments joining the centers of this chain to the focus of the parabola.

5. *The notable point N.* If we drop perpendiculars from any point of the circumcircle of a triangle to the three sides, the feet of these perpendiculars lie on a line—the Simson line of the point. The feet of the perpendiculars from the focus F to L_1, L_2 and L_3 are respectively the points $(1-t_1)^{-1}$, $(1-t_2)^{-1}$, $(1-t_3)^{-1}$ which lie on the line $z+\bar{z}=1$. Hence we have the theorem:

The Simson line of the focus as to any triangle of tangents is the tangent at the vertex of the parabola.

Associated with any four concyclic points T_i is a notable point N [7]. It is the point of intersection of the Simson line of T_i as to the triangle $T_jT_kT_l$, likewise the common point of the four nine-point circles of the four triangles taking the points T_i three at a time. If the points T_i have as coördinates the turns t_i, then the coördinate of the point N is $S_1^4/2$.

The homology

$$z = 2(1 - y)/\pi_4 \tag{5.1}$$

sends the base circle $y=t$ into the centric circle (1.6) since the point $y=t_4$ becomes

$$z = 2(1 - t_4)/\pi_4 = C_4.$$

Then the point $S_1^4/2$ is sent into the point

$$z = (2 - S_1^4)/\pi_4, \tag{5.2}$$

a point midway between the centric center (1.7) and the Hervey point (3.3). Hence we have the theorem:

The point (5.2) *is the notable point N for the four circumcenters C_i of any four tangent lines of a parabola. It is likewise the point N for the four orthocenters of the points C_i taken three at a time.*

We shall see another property of this point (5.2) in §6.

The orthocenters of the four triangles formed from any four concyclic points

T_i lie on a circle whose center has the coördinate S_1^4. The homology (5.1) sends S_1^4 into

$$z = 2(1 - S_1^4)/\pi_4,$$

the Hervey point (3.3). Hence we have the theorem:

The Hervey point of any four tangent lines of a parabola is likewise the center of the circle passing through the orthocenters of the triangles formed from the four points C_i.

6. *Orthopole relations.* If perpendiculars are dropped on any line from the vertices of a triangle, the perpendiculars to the opposite sides from their feet are concurrent at a point called the orthopole of the line as to the triangle. The equation of the line through C_4 parallel to L_4 is

$$d_4: \quad t_4 z - \bar{z} = 2(S_3^3 + t_4)/\pi_3. \tag{6.1}$$

The orthopole of the line d_4 as to the triangle formed by the three tangents of the parabola L_1, L_2 and L_3 is

$$P_4 = (2 - S_1^4)(1 - t_4)/\pi_4. \tag{6.2}$$

If we draw through each C_i a line parallel to L_i, we obtain four lines d_i and four orthopoles such as (6.2) which lie on the circle

$$z = (2 - S_1^4)(1 - T)/\pi_4, \; |T| = 1; \tag{6.3}$$

when $T=1$, $z=0$, whence this circle passes through the focus. Its center is

$$p = (2 - S_1^4)/\pi_4,$$

the point (5.2). We shall call this point the orthopole center [8]. Hence we have the theorem:

The orthopole center lies midway between the centric center and the Hervey point of any four lines (Fig. 1).

The lines d_4 and d_3 meet at the point

$$y_{34} = 2(1 - t_3 - t_4 - t_1 t_2)/\pi_4.$$

The circumcircle of $y_{34}y_{42}y_{23}$ is then

$$y = 2\{1 - (t_2 + t_3 + t_4) - (1 + t_1)T\}/\pi_4, \quad |T| = 1,$$

whose center is

$$y_1 = 2\{1 - (t_2 + t_3 + t_4)\}/\pi_4,$$

Hence the centric circle of the four lines d_i is

$$y = 2(1 - S_1^4 + T)/\pi_4, \quad |T| = 1,$$

whose center is the point

$$y = 2(1 - S_1^4)/\pi_4.$$

Hence we have the theorem:

The centric center of the four lines d_i is the Hervey point of the four lines L_i of the parabola.

Similarly for the lines d_i the circle through the orthocenters of the four points y_i is the centric circle for the lines L_i.

The orthopole of the line L_4 as to the triangle formed by the lines d_1, d_2, and d_3 is

$$P_4' = (2 - S_1^4)(1 + t_4)/\pi_4$$

which is the point on the circle (6.3) diametrically opposite to the point (6.2). Therefore, the eight orthopoles P_i' and P_i lie on the same circle. The homology which sends the lines L_i into the lines d_i has (5.2) as a fixed point and the line joining (1.7) and (3.3) as a fixed line. The line (7.1) is also a fixed line. Hence we have the theorem:

The parabola which touches the four lines d_i is equal to the one touching the four lines L_i and their foci are at opposite ends of a diameter of the circle (6.3).

7. *The lines N and M.* Given any four lines we have shown the existence of three coaxial circles: (a) the centric circle, (b) the circle through the orthocenters of the triangles formed from the four points C_i, and (c) the orthopole circle. Besides the axis and directrix of the parabola there are two other lines intimately connected with four given lines, which we shall designate as line N and line M.

The line N is one on which lie the midpoints of the diagonals of the four lines, the centroid of the six intersections of the four lines, and the centroid of the four points at which the parabola touches the four lines. This line is parallel to the axis of the parabola, for its equation is

$$z - \bar{z} = (2 - S_1^4 + S_3^4 - 2S_4^4)/\pi_4. \tag{7.1}$$

Since (5.2) satisfies (7.1) we note that *the orthopole center* **[9]** *lies on the line N* (Fig. 1).

The second line M, connected with any four given lines, has a role quite analogous to the Euler line of three given lines. On M lie the centric center (1.7), the Hervey point (3.3), the orthopole center (5.2), and the centroid g of the four circumcenters C_i, in such a way that

$$\overline{cg} = \overline{gp} = \overline{ph}/2.$$

Its equation is

$$S_3^4 z - S_1^4 \bar{z} = 2(S_3^4 - S_1^4 S_4^4)/\pi_4. \tag{7.2}$$

The intersection of the lines M and N is the center of the orthopole circle.

8. *Sets of lines related to a parabola.* Consider five lines L_i tangent to a parabola. For every four out of five lines we have a Hervey point h. The perpendicular dropped from h (3.3) to L_5 has the equation

$$t_5 z + \bar{z} = 2\{t_5(1 - S_1^4) + S_4^4 - S_3^4\}/\pi_4.$$

Five such perpendiculars, one from each Hervey point to the omitted line, meet at the point

$$z = 2(1 - S_1^5 + S_2^5)/\pi_5, \tag{8.1}$$

where $\pi_5 = (1-t_1)(1-t_2)(1-t_3)(1-t_4)(1-t_5)$. Since (8.1) satisfies $z+\bar{z}=2$, we have the following theorem:

For five lines of a parabola, the five perpendiculars dropped from each Hervey point, formed from four out of the five lines L_i, to the omitted line, meet at a point on the directrix of the parabola.

For every four out of the five lines L_i we have an orthopole center p, which lies on the line M midway between the points c and h. The perpendicular erected at p to the line M has the equation

$$S_3^4 z + S_1^4 \bar{z} = 2(S_3^4 + S_1^4 S_4^4 - S_1^4 S_3^4)/\pi_4.$$

We have five such lines all meeting at the point

$$z = 2(1 - S_1^5)/\pi_5. \tag{8.2}$$

For the five lines we have a centric center, whose coördinate is

$$z = 2/\pi_5. \tag{8.3}$$

Now the points (8.2) and (8.3) determine the line

$$S_4^5 z + S_1^5 \bar{z} = 2(S_4^5 - S_1^5 S_5^5)/\pi_5, \tag{8.4}$$

and a midpoint

$$z = (2 - S_1^5)/\pi_5. \tag{8.5}$$

The perpendicular erected at (8.5) to the line (8.4) has the equation

$$S_4^5 z - S_1^5 \bar{z} = 2(S_4^5 + S_1^5 S_5^5 - S_1^5 S_4^5)/\pi_5. \tag{8.6}$$

For six lines tangent to a parabola, there are six such points as (8.5), and six such lines as (8.6) which all meet at the point

$$z = 2(1 - S_1^6)/\pi_6. \tag{8.7}$$

The centric center for the six lines is

$$z = 2/\pi_6. \tag{8.8}$$

The points (8.7) and (8.8) determine a line λ and a midpoint,

$$z = (2 - S_1^6)/\pi_6. \tag{8.9}$$

For seven lines tangent to a parabola there are seven such points as (8.9) and seven such lines as λ, and the seven perpendiculars erected at each point (8.9) to its line λ will meet at a point. This process will go on indefinitely for tangent lines of a parabola.

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MEANING AND FUNCTION OF A PICTURE*

By W. H. ROEVER, Washington University

Definition. Vaguely expressed, a picture of a space object is a resembling representation of this object made upon a plane surface.

In order to clarify this definition, it becomes necessary to define precisely the notions of resemblance and of representation.

Artists' criterion for resemblance. As to the first of these notions, it may be said that the artist has almost unconsciously taken for his criterion of resemblance the requirement that his picture of a space object should, when properly placed, produce upon the retinal surface of the eye an image differing but little from that produced by the object itself.

Projection satisfies criterion. It follows from the optical properties of the eye, which are similar to those of the camera, that this requirement is satisfied, at least approximately, by a central or parallel (orthographic or oblique) projection of the object upon a plane.†

Representation or unambiguous correspondence. The notion of representation, better expressed by the term correspondence, presents itself in nearly all branches of modern mathematics. As an example, we cite the case of cartography which has for its object the representation of the earth's surface by means of a plane map. In this representation, there corresponds to each point P on the earth's surface a definite point, or image, P' on the map, and conversely to a point P' on the map, there corresponds on the earth's surface the point P having

* Read before the American Mathematical Society at Chicago, April 9, 1937.

† For a proof of this see the author's book, The Mongean Method of Descriptive Geometry pp. 2 and 3, published by The Macmillan Company.

this image. Otherwise expressed, there exists between the points of the earth's surface and those of the map, an unambiguous correspondence, i.e., the operation which enables us to pass uniquely from a point on the surface to its image on the map possesses an inverse operation which enables us to pass back uniquely from the image on the map to the point on the surface.

In this case the surface of the earth and the plane of the map are both domains of two dimensions. However, in the representation of a space object by means of a plane picture, we are dealing with a relationship between domains of different numbers of dimensions. If, in particular, we take the process of projection as the operation for obtaining a point P' of the plane from a point P of space, we can easily see that this operation does not possess an inverse. For the point P', besides being the projection of P is also the projection of any other point on the ray of projection of P. Thus it is evident that a single projection is not sufficient to establish an unambiguous correspondence between space and the plane.

It is interesting to note at this point that binocular vision enables us to perceive three dimensional space, i.e., to see objects stand out in relief. This suggests the fact that two projections are, in general, sufficient for the purpose of obtaining an unambiguous correspondence between space and the plane. An illustration of this fact is afforded by the *stereoscope*, an instrument which enables the user thereof, to combine two photographs (i.e., central projections) of an object taken from slightly different points of view, and thus obtain the impression of solidity or relief. The two photographs together, generally on the same card, form an unambiguous representation of the object photographed. This device furnishes an example of an unambiguous correspondence between space and the plane. For, to each point P of space there is a pair of (related) points P', P'' on the card, one in each photograph, and to each such pair of points on the card there is a definite point in space, namely the one to which this pair corresponds.*

In this example of stereoscopic representation, the projections used for the purpose of setting up a correspondence between space and the plane are both central projections. However, the two projections required for this purpose need not be central and they need not be of the same type. Thus, for instance, if in the photograph (i.e., central projection) of an object, the shadow (i.e., parallel projection due to the sun's rays) of this object upon a known plane, such as the ground, is also shown, then again we have two projections; one is that portion of the photograph which comes only from the object itself, and the other is that portion of the photograph which comes from the shadow (this shadow being a projection of a projection, namely, the projection of the shadow), these two projections again being in the same plane. This type of double pro-

* The reason that a single projection, say a photograph, may convey an adequate notion of the form of a space object which is of the type of a building may be explained by the circumstance that we have some information concerning such an object, as, for instance, the perpendicularity of its edges. If, however, such information is lacking, one projection is not sufficient.

jection serves as well the purpose of representing the object as does the stereoscopic picture. However, the photograph of the object alone without that of its shadow would, in general, not be sufficient for this purpose.

In particular, the source of light which casts the shadow, such as the sun or moon, may be in the zenith. If, in addition to this, the object is a house, or some more or less rectangular object of technology, set upon the ground, then the shadow may coincide with the base of this object on the ground and consequently one might not be aware of the fact that the picture is a double projection, since a single point of the photograph might at the same time be the projection of one point of the object and also the projection of the shadow of another point of the object.* But, nevertheless, to each point of the object, there are two points on the photograph which together fully represent this point.

Finally, in order to see that such a double projection of a point of space is sufficient for the determination of the position of this point in space, let us introduce a set of rectangular axes in space, one vertical, the other two in a horizontal plane, and let us lay off on these axes unit segments having for common origin the origin of the axes.

If now we have on the projection of our object also the projection of these axes and of their unit segments, then from the projection of a point of this object and that of its shadow on the ground (due to the sun in the zenith) we can read off from the picture the co rdinates of this point in space.

It might be well to add in closing that due to the famous theorem of Pohlke, one may choose at random in the plane any three segments (or vectors) of common origin as the parallel projection of three mutually perpendicular space vectors of common origin and common length. The direction of the rays of (parallel) projection is determined by the lengths and directions of the segments which are chosen in the plane. However, in order to insure, in particular, that this direction shall be perpendicular to the plane of projection (i.e., that the projection shall be orthographic) there must exist a certain relation between the lengths and directions of the chosen segments in the plane.†

Having chosen these segments in the plane and drawn the lines on which they lie, called the *axonometric axes*, one can thus say for oblique as well as for orthographic projection, that a point of space is represented by two points in the plane which lie on a line parallel to one of these axonometric axes.

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* This is apt to be the case in the *isometric projection* of machinery, which projection is often employed by the engineer for making pictures.

† This relation may be expressed by means of the Theorem of Schwarz or by the following elegant theorem due to Gauss: If the plane of orthographic projection be supposed to be that of the complex variable $z=x+iy$ and the three given segments in this plane denoted by z_1, z_2, z_3 then the condition is that $z_1^2+z_2^2+z_3^2=0$. For isometric projection these vectors are of equal length and are equally spaced.

For more details concerning the subject of this paper see the author's paper in this MONTHLY, vol. 41, 1934, entitled, Some frequently overlooked mathematical principles of Descriptive Geometry.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University Providence, Rhode Island

The department of Questions, Discussions, and Notes in the MONTHLY is open to all forms of activity in collegiate mathematics including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

ON A TRIANGLE CONNECTED WITH ANY TRIANGLE

By J. R. MUSSELMAN, Western Reserve University

Given the triangle $A_1A_2A_3$ with $O_1O_2O_3$ the midpoints of the sides. Let the circumcenter, incenter, ninepoint center, orthocenter, centroid, Nagel point, Fuhrmann center, and Spieker center of $A_1A_2A_3$ be designated by O, I, F, H, M, N, U, and S respectively (Consult Johnson, *Modern Geometry*, pages 225–229 for a discussion of these points). Let the Fuhrmann center of $O_1O_2O_3$ be V and let HI and NO intersect at the point W. Consider the triangle HWN. The points H, F, M, O are harmonic and collinear, so also are N, S, M, I and W, V, M, U; these three lines are the medians of HWN. The midpoints of UI, IO, OU are F, V, S respectively. The centers of similitude of the circumcircle and ninepoint circle are H and M, those of the incircle and Spieker circle are N and M, those of the Fuhrmann circles are W and M. What other properties does the point W possess?

SIMPLE CONSTRUCTIONS FOR THE CONICS

L. S. JOHNSTON, University of Detroit

Let the ordered points $O(0, 0)$, $F(c, 0)$, and $A(a, 0)$ be respectively the center, focus, and vertex of an ellipse, and let FR and FR' be perpendicular lines through F, FR intersecting $x=a$ at $T[a, (a-c)\tan\theta]$ and FR' intersecting $x=-a$ at $T'[-a, (a+c)\cot\theta]$, where $\theta =$ angle AFR. It is well known that TT' is tangent to the ellipse $b^2x^2+a^2y^2=a^2b^2$, but it is not so generally known that the point of tangency may be located by a very simple construction.

The line TT' is given by the equation

$$x(a\cos 2\theta + c) + ay\sin 2\theta = a(a + c\cos 2\theta)$$

and the point of tangency is given parametrically by

$$x = a\,\frac{a\cos 2\theta + c}{a + c\cos 2\theta}, \qquad y = \frac{b^2\sin 2\theta}{a + c\cos 2\theta}\,.$$

These coördinates satisfy the equation $y=(x-c)\tan 2\theta$, and hence the focal radius through the point of tangency makes the angle 2θ with the x axis. From this fact we derive a simple construction for locating the point at which any arbitrary line through F intersects the ellipse, without drawing the ellipse itself. Let FM be such a line, making with the x axis the angles AFM and MFO. Let the bisector of AFM intersect the line $x=a$ at T and let the bisector of MFO intersect $x=-a$ at T'. Then TT' intersects FM on the ellipse, and TT' is tan-

gent to the ellipse at this point. In particular, if FM be the line $x=c$, the intersection of TT' with $x=c$ is $(c, b^2/a)$, the intersection of the latus rectum with the ellipse. This particular case is especially useful for rapid sketching of the ellipse for blackboard demonstration, as well as for cutting templates for elliptical masonry arches.

In particular cases the scale of the figure may be too large to use both $x=a$ and $x=-a$ conveniently. For such cases we note that the line TT' intersects $x=0$ at $(0, a+c \cos 2\theta/\sin 2\theta)$ and $x=c$ at $(c, b^2/a \sin 2\theta)$. For $\theta=45°$ the intersection with $x=0$, viz. $(0, a)$, is particularly convenient; for other values of θ the intersection with $x=c$ is more convenient, and the construction is easily performed.

The theory and constructions just explained are quite easily adapted to the other conics, and need not be discussed in detail here. The writer has found that his students use these constructions habitually in their board work once they have become familiar with them.

A Type of Polygons with a Common Centroid

By J. H. Butchart, Phillips University

In a paper by J. R. Musselman published in this Monthly, 1936, p. 541, attention was called to fifteen triangles having the same centroid. The purpose of this note is to give a theorem suggested by this configuration.

Theorem. *If two positively ordered polygons A_i, B_i, $(i=1, 2, \cdots, n)$, have the same centroid, and if directly similar triangles $A_iB_iC_i$ are constructed, then the centroid of C_i is the same as that for the two given polygons.*

Let the centroid of the given polygons be taken as the origin in the complex plane, and let coördinates be denoted by small letters. Then from the condition for the similarity of two triangles plus the fact that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_i & b_i & c_i \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

we have

$$c_i(a_1 - b_1) = b_i(a_1 - c_1) - a_i(b_1 - c_1), \quad (i = 1, 2, \cdots, n).$$

Then upon summing, we get

$$(a_1 - b_1)\sum_1^n c_i = (a_1 - c_1)\sum_1^n b_i - (b_1 - c_1)\sum_1^n a_i.$$

By hypothesis the summations $\sum a_i$ and $\sum b_i$ vanish and hence $\sum c_i$ must vanish also.

The situation where C_i divides A_iB_i in a constant ratio is clearly a limiting form of this proposition. When B_i is taken as coincident with $A_{i-1}(A_{i+1})$ it also

covers the erection of similar triangles externally (internally) on the sides of a single triangle, and the corresponding limiting form where the sides of a polygon are divided in the same ratio.

It seems to the present writer appropriate to offer some comments and criticisms in connection with the paper by Professor Musselman cited above. For one thing it is slightly misleading to separate sets (2) and (3) of section 8.1, (1) and (3) of 8.2, (1) and (4) of 8.3 and (2) and (3) of the same section, (1) and (4) of 8.4 and (2) and (3) of the same section, for these sets consist of parallelograms of equal areas. Quite a number of parallelograms are repeated under separate categories. Also in 8.5 the statement should be made that the parallelograms of the first set are equal respectively to those of the second. Two obvious omissions were noted, $B'C'A''$ from 5.3 and $B''C''A'$ from 5.4.

Note by the Editor. Professor Musselman says that his results were intended to illustrate a method of proof, and not necessarily to be exhaustive. R. E. G.

Methods for Computing Generalized Euler Numbers

By H. M. Terrill, Institute for Advanced Study

The generalized Euler numbers introduced by Nörlund* are usually denoted by $E_\nu^{(n)}$. They can be defined for positive n by the recurrence formula

$$E_{\nu+2}^{(n)} = n^2E_\nu^{(n)} - n(n+1)E_\nu^{(n+2)}, \qquad n \neq 0, \tag{1}$$

with initial conditions $E_0^{(n)}=1$, $(n=1, 2, \cdots)$.
Thus, for example,

$$E_6^{(2)} = 4\cdot E_4^{(2)} - 6\cdot E_4^{(4)} = 4\cdot 16 - 6\cdot 56 = -272.$$

A small table of the numbers is given below. Since $E_\nu^{(n)}=0$ when ν is odd, the table lists only even values of ν.

ν/n	1	2	3	4	5
0	1	1	1	1	1
2	−1	−2	−3	−4	−5
4	5	16	33	56	85
6	−61	−272	−723	−1504	−2705

Note that the values for $n=1$ are the ordinary Euler numbers.

While the table may be extended indefinitely by means of (1), another method exists which is more convenient for large values of n. Nörlund showed that $E_\nu^{(n)}$ could be expressed as a polynomial in n of degree $\nu/2$. Writing k for $\nu/2$, we can write the polynomial

$$E_\nu^{(n)} = M_\nu^{(1)}n^k + M_\nu^{(2)}n^{k-1} + \cdots + M_\nu^{(k)}n.$$

* Nörlund, Differenzenrechnung, Berlin, 1924; also Milne-Thomson, The Calculus of Finite Differences, London, 1933.

The values of the coefficients, $M_\nu^{(1)}$, $M_\nu^{(2)}$, $\cdots$, $M_\nu^{(r)}$, $\cdots$ etc., were given by Nörlund as follows:

ν/r	1	2	3	4	5
2	-1				
4	3	2			
6	-15	-30	-16		
8	105	420	588	272	
10	-945	-6300	-16380	-18960	-7936

Now it can be shown that a relation exists, by which it is possible to determine the rth M of any row in terms of the first r of the M's in the previous row. Thus,

$$(2) \qquad -M_{\nu+2}^{(r)} = M_\nu^{(1)}\beta_{k-r+1}^{(k)} + M_\nu^{(2)}\beta_{k-r+1}^{(k-1)} + \cdots + M_\nu^{(r-1)}\beta_{k-r+1}^{(k-r+2)} + M_\nu^{(r)}\beta_{k-r+1}^{(k-r+1)},$$

where

$$(3) \qquad \beta_j^{(i)} = \binom{i}{i-j}(i+j+1)\frac{2^{i-j}}{i-j+1}\cdot$$

In the case where r equals $k+1$, the last term in (2) disappears.

The first few values of $\beta_j^{(i)}$ are given in the following table:

4	3	2	1	i/j
16	8	4	2	0
48	20	8	3	1
56	18	5		2
32	7			3
9				4

Instead of computing the values of β directly from (3), they may be determined more easily by the recurrence relation

$$\beta_j^{(i+1)} = 2\beta_j^{(i)} + \beta_{j-1}^{(i)}, \qquad i \neq j.$$

As an example of (2), we have

$$\begin{aligned} M_{10}^{(3)} &= -(M_8^{(1)}\beta_2^{(4)} + M_8^{(2)}\beta_2^{(3)} + M_8^{(3)}\beta_2^{(2)}) \\ &= -(105\cdot 56 + 420\cdot 18 + 588\cdot 5) = -16380. \end{aligned}$$

"Mathematics in general is fundamentally the science of self-evident things." Felix Klein, *Anwendung der Differential-und Integralrechnung auf Geometrie* Leipzig, 1902, p. 26.

RECENT PUBLICATIONS

EDITED BY W. R. LONGLEY, Yale University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Construction, Classification and Census of Magic Squares of Even Order. By A. L. Candy. Ann Arbor, Michigan, Edwards Brothers, Inc., 1937. 6+192 pages. $1.00.

Professor Candy has made an extensive study of the squares of even order obtainable from squares of order half the order. He replaces the elements of a magic square by 2×2 squares and studies the different results that come from writing the four elements of the 2×2 squares in any one of the twelve possible orders. The elements in the 2×2 squares are in the simplest case four consecutive numbers, but this may be generalized so that they are in arithmetical progression.

Studying the squares of even order thus generated Professor Candy has tried to find rules by which the resulting squares shall be symmetric and finds that from any given symmetric square of odd order an even square may be generated which shall be symmetric except for two pairs of numbers in the two middle columns. The author is doubtful as to whether a completely oddly even magic square is obtainable. But until one knows how general the method is and what proportion of the total number of squares of a given order are obtainable by it one should qualify every such statement with the phrase "by this method." Thus it is well known that there are no diabolic squares obtainable by the "uniform step" method when the order is a multiple of 3, but there are plenty of such squares obtainable for such orders by other methods. It may be that this assumption of generality may account for the discrepancy in Professor Candy's census of 4×4 magic squares which are magic in the principal diagonals as compared with the census of Frenicle and others. But as Professor Candy obtains a greater number than Frenicle of such squares it is more probable that he has counted certain "types" more than once.

D. N. LEHMER

A Mathematician Explains. By Mayme I. Logsdon. Chicago, University of Chicago Press, 1935. 12+175 pages. $2.00.

This volume is one of a group produced at the University of Chicago and designed to give the student of non-technical interests a general idea of what modern science is about. It tells briefly the story of elementary mathematics and proceeds thence into the simpler parts of analytical geometry and the calculus.

A rough idea of what the book contains and of how the emphasis is placed may be had by noting the number of pages devoted to each subject. In addition to an introductory chapter and brief appendices there are chapters as follows:

Arithmetic, 35 pp.; Algebra, 13 pp.; Geometry and Trigonometry, 20 pp.; Analytical Geometry, 22 pp.; Differential Calculus, 32 pp.; Integral Calculus, 12 pp.; Mathematical Interpretation of Geometrical and Physical Phenomena (by G. A. Bliss), 13 pp. An augmented later edition has 15 pp. devoted to Mathematics and Life.

This material will be judged according to the use to which the book is to be put. It is hoped (according to the preface) that the volume will be useful (1) preparatory to general science courses; (2) as a text for a short orientation course in college or high school; (3) as a reference book in connection with college or junior college courses; (4) as a supplementary text for courses in the teaching of mathematics in normal schools; or finally (5) as an eye-opener for the adult of scant mathematical training but of healthy curiosity.

The material serves some of these ends better than others. The college teacher will probably think that there is an over-emphasis on arithmetic—that the space devoted to Roman numerals, to scales of notation, and the like, might preferably be used for further work in algebra. Others, notably those giving courses in teachers' colleges, will be satisfied with the present distribution.

The student likely to benefit from the book is, roughly, somewhere between the middle of high school and the middle of college. The high school work itself is very sketchily treated. The algebra is extremely brief (there is, for instance, no mention of logarithms) and there is very little geometry. But the student can look back on his grade school arithmetic and learn something of its history and its theory. And he can look forward to his college mathematics and get an inkling of what the calculus is about.

These more advanced chapters, though quite brief, present a respectable body of material. They show how a student can be led into some quite worthwhile calculus by short and easy stages from extremely modest beginnings. He differentiates polynomials and does problems in maxima and minima. He learns how to find areas and volumes of revolution. The author is at her best, in the reviewer's judgment, in these parts.

These chapters hold a good deal, too, for the curious adult. The elementary historical parts of the book may be to his taste, and he will find them easy to read. But if his algebra can stand the strain, the glimpse of what modern mathematics can do via the calculus is likely to be very illuminating. The reviewer knows adults who have read the book with enthusiasm.

It is perhaps unavoidable that the early editions of the book should contain errata, and a few must be mentioned. There is, for example, the hoary error that in the law of gravitation, $f=kmM/d^2$, the quantity d is the distance between the centers of gravity of the attracting bodies (pp. 3, 4, 9). The beginner is all too ready to believe this without any encouragement from the text, although it could lead a bright student into puzzling paradoxes. Again there is a recurring tendency to consider zero to be a positive number (theorem, p. 25, p. 53).

The rigorist will object to the practice of distinguishing between a maximum and a minimum at x_0 by an investigation of the situation at *one* smaller value

and *one* larger value (pp. 131, 132, 133). The parenthetical ($c\neq 0$) in the differentiation of cx^2 is obviously unintentional (p. 114).

The author's usual pedagogical intuition fails, the reviewer believes, in the matter of the Fundamental Theorem of the Integral Calculus (pp. 139, 140). Here $\lim \sum f(x_i)\Delta x$ is connected with the anti-derivative of $f(x)$ by fiat rather than reason. It is not a difficult matter to connect the primitive with the area under a curve, as most books on the calculus do. This strips the mystery from the Fundamental Theorem and introduces the reader to one of the most striking things in the calculus.

There are very few typographical errors. The printing is well done and the format is pleasing. The illustrations by Mrs. Chichi Lasley are apt and amusing.

The writing of a text which leaves the beaten track, as this one does, is not an easy task, and one should look upon the results with sympathy. There are, it is true, some evidences of hasty writing. But, on the whole, the volume is well conceived, thoughtfully put together, interesting, and readable. It is a book to put in the hands of the thoughtful and enquiring reader.

L. R. FORD.

The Handmaiden of the Sciences. By E. T. Bell. Baltimore, Williams and Wilkins Company, 1937. 8+216 pages. $2.00.

In this book an attempt is made to acquaint the non-mathematician with the services which mathematics has rendered to the physical sciences. The presentation should appeal to a very large circle of readers. The entirely uninformed may well follow the author's suggestion about skipping all formulas. Those readers with high school algebra and geometry will probably be induced to supplement certain brief discussions with further study in "good introductory textbooks." Those with some background in calculus will gain a notion of many ideas not included in most undergraduate mathematics courses. Finally, even professional mathematicians will find new points of view and many stimulating, if challenging, statements. For all there is much amusement.

The author very skillfully uses striking statements and witty similes to engage the reader's attention, and drive home his points. For example, he states that a good definition of invariant is given in the scriptural phrase, "the same yesterday, today and forever."

Among the famous classical examples of the helpfulness of mathematics we have the story of the foundation of celestial mechanics by Kepler and Newton and the development of field physics by Faraday and Maxwell. More briefly, the author sketches some notions of statistical mechanics, the theory of relativity, and the role of groups and probability in modern quantum mechanics.

These historical portions are parts of the sugar coating. The medicine in the pill is properly introduced bit by bit. There are definitions of the trigonometric functions, of derivatives and of integrals. The fundamental idea of analytic geometry is presented with enough indications about n-dimensional manifolds for their later use in connection with statistical mechanics and general relativity. Of course, to compress the central features of a three year course into a few

im kleinen, and gives a poorer rough idea than the slightly inaccurate description as least.

Finally, the reviewer hopes that these minor criticisms, largely questions of taste, will not deter the prospective reader, but rather will convince him that the book was sufficiently entertaining to be carefully read. The book will no doubt prove equally entertaining to other mathematicians and their nontechnical friends.

PHILIP FRANKLIN

Analytical and Applied Mechanics. By G. R. Clements and L. T. Wilson. New York, McGraw-Hill Book Company, 1935. 9+420 pages. $3.75.

In the preface the authors state that the purpose of this text is "to give a simple but rigorous discussion of the mechanical and physical theory necessary for a thorough first course in mechanics, and to present a wide variety of applications, interesting in themselves and of direct interest to the student of engineering." With the second half of the purpose I agree that they have succeeded admirably, but I cannot have this same feeling regarding the first part.

Of the fourteen chapters of the book, the first ten deal with statics and the last four with dynamics. This order of presentation has advantages and it also has disadvantages. The chief advantages perhaps are that the problems of statics are simpler to deal with mathematically after the fundamental principles are obtained, whereas the disadvantages are that so many of the fundamental principles have to be adopted as assumptions or empirical laws. In this text principles that are fundamental are frequently introduced by the authors as assumptions, but in a rather casual way.

To illustrate, in §16 the moment of a force about an axis is defined as the product of the force by the common perpendicular between the axis and the line of action of the force. Then the statement is added: "If the force is thought of as acting on a body pivoted (about the axis) it would tend to turn the body about the axis." Though not specifically stated it is later tacitly implied that the moment of the force is the *measure* of its turning tendency. This will no doubt meet with the *intuitive* approval of a student as being correct. On the other hand intuition is a guide but not an authority, and this cannot be taken as a basis for the truth of the proposition. A prominent featuring of such a principle when introduced by laying it down as an assumption and citing how it may be tested in various ways would, I believe, put a student in a much more confident frame of mind.

But from the point of view of rigor there are other more serious defects.

Thus, the fundamentals of vectors as given in Chapter 1 lack precision and clarity. A vector is defined as a directed line segment, and a vector quantity as one that is completely specified by a vector. That is, a vector is regarded as the geometrical representation of a vector quantity. Then the sum of two vectors is defined as that given by the parallelogram law. The implication seems to be that all quantities which can be represented completely by directed line segments are

compounded by the parallelogram law. But this is not true. Thus, successive finite rotations about axes which intersect can be represented completely by directed line segments which form the coterminal sides of a parallelogram but their resultant is not represented by the coterminal diagonal. Vector addition is one of the characterizing properties of a vector quantity and should be included among its definitional properties.

Again, there is not a clear distinction made between a vector quantity and its magnitude, and there are inconsistent statements made on this account. When a directed segment (arrow) is drawn and labelled with a symbol, that symbol may represent the considered vector quantity completely, or it may represent its magnitude in the direction of the arrow. When the former is intended it is usual to show this by using bold-face type or some other device, and when the latter is meant plain type is used. Of course one could by convention use plain type in the former sense, but it is well to be consistent. The authors use plain letters in the former sense, for when they label the coterminal sides of a parallelogram as u and v they label the coterminal diagonal as $u+v$. They also state on page 6 that if v is a vector quantity and s a scalar then sv is a vector with the same direction and sense as v and s times the magnitude. But on page 7 they state that if the vector makes an angle α with the x-axis then $v \cos \alpha$ is the component of v in the x-direction—that is, a vector in a direction other than the direction of v when $\alpha \neq 0$. But $\cos \alpha$ is a scalar. Hence this is not consistent with the statement on the previous page, and they are here using the symbol in the second sense. This is confusing.

There is likewise no distinction made between a physical quantity and its measure, and there is ambiguity, for example, when v denotes a velocity and the velocity is 15 ft./sec., as to whether v is 15 or 15 ft./sec. The second would seem to be the natural convention, but if this were consistently followed there would be no need to give a warning such as appears on page 201. After writing an equation in general symbols it is there stated that "care must be taken to be consistent in the units used." This neglects the whole possibility of including units in analysis. If an equation is physically homogeneous in dimensions, and if the symbols which appear in it contain their units implicitly, then in any numerical case there can be no inconsistency if units as well as numerical measures are included in the calculations. The trouble is the authors do not use a consistent method. Equations are written down which regard the symbols as numbers (see p. 53, e.g.), then when the equation is solved for the numerical value of the symbol a unit is attached.

This is precisely analogous to what is done in regard to some algebraic quantities with respect to sign. Thus on the first page of Chapter I there are the equations:

$$x=r \cos A, \; y=r \sin A,$$

and the statement is made that these equations determine the magnitudes of x and y while their signs are given from the Figure. But unless in the equations

x and y appear as $|x|$ and $|y|$ then the equations automatically determine the sign as well as the magnitude, when there is uniqueness as there is in this case.

Such practice can hardly be described as rigorous. An algebraic symbol includes both measure and sign implicitly. When an algebraic equation is solved the sign as well as the measure is determined when the solution is unique, and when a physical equation is solved the measure, sign, and unit are all determined when the solution is unique.

The book is, however, well printed. The format of the pages is good and the typographical errors noted are few. Answers to solved examples are given prominently in bold-face type and important formulas and fundamental relations are also exhibited in the same way.

Hyperbolic functions are used freely in the solution of numerical problems where results are most easily determined by their use. The reader is given the impression that these functions may be used as freely and as naturally as circular functions or logarithms and this is as it should be. In the catenary problems, however, much work indicated is needless. It is unnecessary, for example, to solve the equation $\sinh z = 1.25z$ by graphical or other computational methods, for the function $(\sinh z)/z$ is also available in tabulated form. The same is true of the quotients $(\cosh z - 1)/z$ and $(\cosh z)/z$.

J. W. CAMPBELL

An Introduction to Projective Geometry. By L. N. G. Filon. Fourth edition. London, Edward Arnold and Company, 1935. 17+407 pages. 16 sh.

The first edition of this text was published in 1908, followed by second and third editions in 1916 and 1921. The present fourth edition represents a rather extensive revision of the previous work, the number of pages having been increased from 253 to 407. The number of chapters has been increased from fourteen to sixteen; the old chapter on imaginary elements and homography becomes two separate chapters, and a new chapter has been added on projective methods in three dimensions.

There has been one important change in the order of presentation. The discussion of involutions now precedes that of the focal properties of the conic, and this makes possible a more satisfactory treatment of the latter subject.

As the earlier editions of this text were not reviewed either in this MONTHLY or in the *Bulletin of the American Mathematical Society*, it may be useful to our readers to indicate something of the general spirit and point of view of the treatment. In the preface to his monograph on projective geometry, J. W. Young said, "the treatment . . . should keep projective properties sharply distinguished from the metric specializations. . . . Only in this way, I believe, can the reader gain a clear impression of what the word projective implies." Those of us who agree with this point of view find ourselves dissatisfied with a presentation in which euclidean and projective properties are inextricably mixed. In the text under review there is no one chapter, and indeed very few sections, in which one does not find mention of such things as distance, angle, perpendicularity, parallelism, circle, center, diameter, foci, radius of curvature, cylinder, etc. In other

words, there is no part of the text which treats *pure projective geometry* as distinct from metric specializations of projective geometry. And the reviewer concurs in the opinion that from such a treatment the reader is unlikely to gain a clear impression of what the word projective implies.

Another and more serious objection to this kind of presentation of projective geometry is that it makes for logical confusion and leaves one doubtful as to whether or not anything has been proved. Euclidean geometry and projective geometry are quite different logical structures. An axiom or valid theorem in one may be false in the other. The statement that any two distinct lines in a plane have one and only one point in common is true in projective but not in euclidean geometry; while the statement that through any three distinct points not on a line there is one and only one circle is true in euclidean geometry but is either false or meaningless in projective geometry. When logical use is made of both kinds of statements the argument is, to say the least, not clear. Thus, in paragraph 10, the author's proof of the existence of the "pole of perspective" O for two figures in plane perspective is sound only in the logical system of projective geometry (the lines intersecting at O may be parallel); while the proof in paragraph 12 that this same point O moves about a circle when one of the two figures in plane perspective is rotated about the axis of perspective is quite meaningless except in the system of euclidean geometry. Paragraph 10 is sound only in a geometry that includes points at infinity while paragraph 12 is sound only in a geometry which excludes them.

The above example indicates a kind of thing which occurs frequently throughout the text. It creates a particularly bothersome situation when the author comes to a statement of his "principle of duality." We quote:

"It follows from the transformation by reciprocal polars that to *every* (the italics are the reviewer's) theorem concerning a figure made up of points and lines there corresponds another theorem concerning a corresponding figure made up of lines and points respectively, so that geometrical theorems appear in pairs. . . . It should be noticed, however, . . . , that properties of length and angular magnitude (which are termed metrical properties) do not generally reciprocate into like properties. It will be found that the properties to which the principle of duality can be applied successfully are the *projective* properties."

If the author had previously drawn a clear distinction between projective geometry and other geometries, it would be quite simple to say that to every theorem of projective geometry there corresponds a dual theorem; and one could be sure of applying this principle of duality "successfully" in all cases.

For the treatment of imaginary elements in chapter IX, "appeal is made, as in the original edition, to algebraic considerations." This would seem to be perfectly sound procedure calling for no apology. But it is rather unfortunate that the author should have used the non-homogeneous algebra suitable for euclidean geometry rather than the homogeneous algebra which corresponds to projective geometry. His non-homogeneous algebra leads him into such dubious kind of statements as the following (page 170):

"In this system of coördinates the coördinates of the points of any straight

line satisfy an equation of the first degree $Ax+By+C=0$. If we divide this equation by C it takes the form $lx+my+1=0$. A straight line is therefore completely defined when we know the two coefficients l, m. These may be spoken of as the coördinates of the line. . . . If $l=0$, $m=0$, x or y or both must be infinite if $lx+my$ is to be equal to the finite quantity -1. Hence $l=0$, $m=0$ are the coördinates of the line at infinity. Similarly if $x=0$, $y=0$ the lines through the origin must have l or m or both infinite."

For a clear and simple criticism of this kind of algebra I would refer the reader to a note by W. L. G. Williams in this MONTHLY, vol. 30, 1923, p. 384.

Most teachers will agree with the author's emphasis upon "drawing-board constructions." It is a very unusual student who can get any adequate understanding of projective geometry from a first course without the actual drawing of numerous figures. The interesting collections of constructions and other exercises found throughout the text constitute one of its most excellent features.

The book appears to be unusually free from typographical errors, and the publishers have done their part in producing a volume more attractive in form and appearance than the earlier editions.

W. B. CARVER

Differential Equations in Applied Chemistry. By F. L. Hitchcock and C. S. Robinson. Second edition, revised and enlarged. New York, John Wiley and Sons, 1936. 8+120 pages. $1.50.

This book replaces the first edition published in 1923 and retains the unique position of usefulness of its predecessor—that is, a position somewhere between differential equations and applied chemistry. It is definitely not a book on the subject of differential equations, but on those parts of chemistry (and like applications) wherein the computations involve the setting up and solution of a certain few types of differential equations. Thus most of the first four chapters are concerned with problems leading to the equation $dy=f(y)dx$. While these problems are chosen from many fields, most of them are from chemistry. The arrangement is also determined by the chemical rather than the mathematical processes under consideration, except that the Picard process is treated separately in the last chapter. If the title were to be enlarged it might be made "Problems of applied chemistry which contain differential equations."

The main changes from the first edition are: first—the introductory article has been entirely rewritten; second—the treatment of the diffusion equation in article 18 has been amplified; third—Chapter VI, which was formerly devoted to graphical integration, now contains instead an elementary treatment of Numerical Solution of Differential Equations (8 pages). This last change is in accord with the present trend in computing methods and is to be commended. The lists of problems are all slightly extended, and there is a new set for the new chapter.

The Table of Contents is retained without change from the old edition and hence gives incorrect page references.

H. C. HICKS

MATHEMATICS CLUBS

EDITED BY F. W. OWENS AND HELEN B. OWENS, State College, Pa.

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to F. W. Owens, 462 East Foster Ave., State College, Pa.

TOPICS

From a general survey of club activities in 1936–37, a few items are worthy of note. Sixty-two clubs had sent reports to this department before July first. The range of topics in these reports is wider than ever before. While certain old time standbys are not entirely forgotten they no longer dominate the programs. The trisection of an angle; The duplication of a cube; The transcendence of π; The constructions possible with straight edge alone; and similar topics have lost the lead to such new favorites as Relativity; Numerology; and Certain topics in the calculus of variations. The tendency to build a program with some unity is gaining favor. One leader of a club which appeals to less advanced students writes:

"The chief topic at each meeting was drawn from the field of number theory. The rest of the program was planned with the idea of attracting many who would not come if the subject seemed too advanced."

Probably the most popular field of subjects in programs so far received is that of applications of mathematics to other arts and sciences.

EXHIBITS AND CONTESTS

The number of mathematical exhibits and contests for secondary school pupils carried on by clubs is increasing. Plans made in the autumn for an exhibit to be held in the late spring help to keep up attendance and interest in the year's program. Models made for these exhibits are frequently worthy of a place in the permanent collection of the school. From one school which has sponsored such an exhibit each year for several years comes the following:

"The first year our models were very simple and there were not many of them; but each year interest increases and additions are made. Also, if a model does not seem the best possible, some one gets busy and tries to make a better one. The attendance at our exhibit for the benefit of secondary school pupils has grown from a mere handful to over three hundred in a year."

Competitive examinations for high school juniors and seniors bring the subject of mathematics to the attention of hundreds of young people. The Frumveller Competition held at Marquette University and the Pi Mu Epsilon Interscholastic Contest at New York University are outstanding examples. There are many others, some of which are not for secondary school but for college students, such as the Integration Contests held at Brooklyn College and the Pi Mu Epsilon Competitive Examination at The Pennsylvania State College.

CLUB REPORTS
1936–37

Day Session Mathematics Club, Brooklyn College

Fall Semester: President, S. Smallberg; Vice-President, S. Ranz; Secretary, M. Cohen; Social Director, M. Kistenberg; Faculty Adviser, J. Wolfe. Spring Semester: President, S. Ranz; Vice-President, M. Kistenberg; Secretary, M. Cohen; Publicity Directory, H. Williams; Social Director, S. Lopata; Faculty Adviser, Professor S. Borofsky. Meetings were held weekly. The *Math Mirror* was published in enlarged and improved form. The Sixth Annual Mathematics Medal Contest was held. Questions in this contest are drawn from ten undergraduate courses, all with calculus as a prerequisite. Contestants must answer at least four questions, one each from four different courses. The mathematics faculty act as judges. This year gold medals were awarded to M. Abramowitz and S. Ranz and silver medals given M. Cohen, S. Lopata and F. Supnick.

The semiannual Integration Contests were won in the autumn by Professor M. L. Bishop's

team, with S. Minsker, high scorer; in the spring by Professor Bishop's team with B. Lax and J. Peltz tied for high scorer. For this unique contest six integral calculus problems are submitted each week for three weeks. Each class in integral calculus in the college enters a first and second team of five students each. There are both team and individual awards, a dinner for the winning team, a book for high scorer.

At the weekly meetings topics discussed included: Dimensional analysis; Theory of relativity; Perfect numbers; Primalty of numbers composed of a succession of ones; Perfect squares and cubes; Certain indefinite integrals; Mathematical logic; Criteria for summability of series; Osculating circles; Mapping; Finite projective geometry; and Galois field theory.

Intercollegiate Mathematics Association of Milwaukee

President, Norma Fedders, Milwaukee-Downer College; Vice-President, Dolores Elshoff, Mount Mary College; Secretary, H. Schaeffer, Milwaukee State Teachers' College; Treasurer, H. Hibscher, University of Wisconsin, Extension Division; Corresponding Secretary, Frederick Adler, Marquette University. At the five meetings of this lively organization these topics were presented: Rhind papyrus; Omar Khayham; Mathematics and the development of the human race; Non-euclidean geometry; Finding a place for mathematics in the curriculum; and Demonstration of Napier's rods, the abacus, and a modern computing machine. An impromptu presentation of "Discord in Mathematics Land," by members selected at random, and a banquet gave social interest, and the publication of volume 3 of *The Circle* completed the year's work.

Mathematics Club, University of Alberta

President, M. Wyman; Secretary, J. R. Munn; Faculty Adviser, Professor J. W. Campbell. Topics at meetings included: Mathematical certainty; Differences and differentials; Mortality tables; Forerunners of Descartes; Group theory; Hyperbolic functions; Orbit of a spectroscopic binary; Vectors; The Oslo Mathematical Congress. Criticism of student papers was given and the club prize awarded to M. Wyman for his paper on "Mortality tables."

Pi Mu Epsilon, University of Kentucky

Director, Professor Flora E. LeStourgeon; Vice-Director, W. H. Pell; Secretary, O. B. Ader; Treasurer, Pauline Thompson; Librarian, Professor H. H. Downing. Topics discussed at regular meetings included: Problems in infinite series; Generalization of the Hamilton-Cayley theorem; An inheritance tax problem; Ideals in a quaternion algebra. A banquet in December and a picnic in May featured the initiation of new members.

Delta Chi Mathematics Club, University of Kansas City

President, Margaret Hopper; Vice-President, H. Brooks; Secretary, R. Magovern; Treasurer, R. Grafrath; Faculty Advisers, Professor W. A. Luby, Dr. D. T. Sigley. The club each autumn issues a printed program for the current year. Topics presented included: Constructions with compasses alone; Neglected operations with series; Geometrical forms in nature; Representation and interpretation of physical data by mathematical curves; Some problems not in Euclid's geometry; Astronomer's yardsticks; Conception and simplification of mathematical notation; Prismoidal formula applied to areas and volumes.

Phi Chi Mu, Washington and Jefferson College

President, J. R. Bukey; Secretary, E. P. Allright. This honorary society in mathematics, chemistry, and physics has an enviable record of members going forth to postgraduate work in science. The talks on mathematical topics included: Non-involutorial de Jonquières transformations leaving curve-wise invariant a pencil of cuspidal cubics; Some unusual types of Dirichlet functions; Projective generalization of plane quartic curves; Certain types of non-perspective de Jonquières transformations; Classification of plane monoidal involutions of order three.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 294. *Proposed by V. Thébault, Le Mans, France.*

Show that the difference between the cubes of two successive positive integers can never be either the double or the triple of a perfect square.

E 295. *Proposed by P. W. Lonergan, DeWitt Clinton High School.*

In preparing a board for Chinese checkers, a conventional six-pointed star is formed by placing an equilateral triangle upon another of equal size, so that the trisection points of the sides of one coincide with those of the other. Each side of each "point" is divided into four parts, and the complete figure is cut into small equilateral triangles by drawing through each marked point, lines parallel to the other sides of the figure. Find the number of small equilateral triangles, and the total number of equilateral triangles of all sizes, in the complete figure.

E 296. *Proposed by D. L. MacKay, Evander Childs High School, N.Y.*

D, E, and F are the centers of the semicircles constructed on the sides BC, CA, and AB as diameters, and exterior to the triangle ABC. If d and e are the lengths of the common external tangents between the points of contact for the semicircles D and E, and D and F, construct triangle ABC, given d, e, and angle A.

E 297. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

When the proper fraction, WPA/USA, is reduced to lowest terms, it becomes UV/VU. Here each letter represents a digit in the decimal system. Determine the original fraction and show that the solution is unique.

E 298. *Proposed by C. T. Holmes, Brunswick, Me.*

As a first step in making a string model of the hyperboloid, $x^2/a^2+y^2/b^2-z^2/c^2=1$, we drill holes through two pieces of board along the curve in which the surface is cut by the plane $z=Z$. That is, the ellipse,

$$x^2/(ka)^2 + y^2/(kb)^2 = 1,\ z = Z, \quad \text{where} \quad k = \sqrt{(c^2+Z^2)}/c > 1. \tag{1}$$

The rulings must cut the plane $z=0$ in the ellipse

$$x^2/a^2 + y^2/b^2 = 1,\ z = 0, \tag{2}$$

which is similar to (1). Hence the projections of these rulings on the plane $z=0$ are chords of (1) which are tangent to (2) at their midpoints.

If through a point P_1 on the ellipse (1) we draw the chord P_1P_2 tangent to (2), then through P_2 draw the chord P_2P_3 tangent to (2), then through P_3 the chord P_3P_4 tangent to (2), and so on, what is the condition that we return ultimately to the point P_1, so that we shall not be under the tiresome necessity of drilling infinitely many holes?

SOLUTIONS

E 257 [1937, 49]. *Proposed by Mannis Charosh, New Utrecht High School, Brooklyn, N. Y.*

Construct the triangle ABC, given the altitude and median from A, and the difference, $b-c$, of the adjacent sides.

Solution by L. M. Kelly, Northeastern University

Given the altitude h_a, the median m_a, and the difference $b-c$. We may first construct the triangle AHM, having the given altitude and median as one leg and hypothenuse. If I_a is the point of tangency of the inscribed circle to side a, $\overline{MI_a}=(b-c)/2$, and we may thus locate point I_a within the segment HM. But $\overline{MI_a^2}=MH\cdot MX$, where X is the point where the internal bisector of angle A meets side a. The point X may thus be located, and the angle bisector AX drawn. The perpendicular to a at I_a meets AX at the incenter, whose location enables us to draw the incircle, whose tangents from A are the missing sides of the triangle ABC, interesecting HM produced at B and C. For reference see Johnson's *Modern Geometry*, pages 184 and 186.

Also solved by W. B. Clarke, Samuel Kramer, Alice E. Gibson, W. R. Hardman, J. W. Kitchens, D. L. MacKay, K. B. Patterson, E. P. Starke, Althéod Tremblay, C. W. Trigg, Simon Vatriquant, and the proposer.

E 258 [1937, 49]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

The sum and difference of $S\ L\ A\ P$ and $D\ E\ B$ are respectively $D\ U\ D\ E$ and $P\ I\ P\ S$, where each digit is replaced by a code letter. Determine the digit represented by each letter, and show that the solution is unique.

Solution by W. R. Ransom, Tufts College

Let I, II, III, and IV represent the columns of the sum, and V, VI, VII, and VIII the columns of the difference, in order from the left. Since only 1 can be carried, we have (I) $D=S+1$ and (V) $S-P=1$. P is not more than B or we should have (VIII) $P-B=S$, whence $S-P=-B$, contradicting (V). So P is less than B, and from (VIII) $10+P-B=S$, or $S-P=10-B$, whence $B=9$. Then (IV) $P+B=10+E$, or $P=E+1$. The numbers $E\ P\ S\ D$ are therefore consecutive digits in increasing order.

A cannot be more than E for then (VII) $A-1-E=P$ and $A=P+1+E=2P$. But if $E=0$ this gives $A=2=S$; if $E=1$, $A=4=D$; if $E=2$, (III) gives $6+2+1=5$ or 15; if $E=3$, (III) gives $8+3+1=6$ or 16; and if E exceeds 3, A exceeds 9.

So (VII) gives $10+A-1-E=P$, or $A=2(P-5)$. Hence $P=6$, 7 or 8. But if $P=7$, $D=9=B$; and if $P=8$, $S=9=B$; so that $P=6$ and $A=2$.

We have only 0, 1, 3, and 4 left for U and L, and (II) gives $L+8=10+U$, or $U=L-2$, so $U=1$ and $L=3$. Then (VI) gives $I=4$, and the unique solution is $SLAP=7326$, $DEB=859$, $DUDE=8185$ and $PIPS=6467$.

Also solved by W. E. Buker, Mary L. Constable, Fred Discepoli, William Douglas, S. E. Field, Daniel Finkel, Elmer Latshaw, Yetta V. Maizlish, C. W. Trigg, E. P. Starke, Simon Vatriquant, and the proposer.

E 259 [1937, 104]. *Proposed by Mannis Charosh, New Utrecht High School, Brooklyn, N. Y.*

If the tangents of the angles of a plane triangle form an arithmetic progression, prove that the Euler Line is parallel to a side of the triangle.

Solution by Robert Gaskell, Brooklyn, Michigan.

Let O and H be the circumcenter and orthocenter, respectively, of triangle ABC; and also let A' be the midpoint of BC, and D the foot of the altitude on BC. Since it is given that

$$\tan A = \tan B - d = \tan C + d,$$

$$\text{we have } \tan A = \tfrac{1}{2}(\tan B + \tan C).$$

$$\text{also } \tan B = \tan \angle DHC = CD/DH,$$

$$\text{and } \tan C = \tan \angle BHD = DB/DH.$$

$$\text{Hence } \tan A = \tfrac{1}{2}CB/DH = A'B/DH.$$

But $\tan A = \tan A'OB = A'B/A'O$, so $A'O = DH$; and since $A'O$ is parallel to DH, $A'OHD$ is a parallelogram, and the Euler Line, OH, is parallel to BC.

Also solved by W. B. Clarke, K. W. Crain, L. M. Kelly, D. L. MacKay, C. E. Springer, E. P. Starke, C. W. Trigg, Simon Vatriquant, and the proposer.

E 260 [1937, 104]. *Proposed by C. E. Springer, University of Oklahoma.*

Two lines AB and CD of given lengths slide independently along two fixed skew lines. Show that the locus of the center of the sphere through A, B, C, and D is a hyperbolic paraboloid.

Solution by E. P. Starke, Rutgers University.

Let X-, Y-, and Z-axes be chosen so that the skew lines are $z=a$, $y=mx$, and $z=-a$, $y=-mx$, respectively, and let points A, B, C, and D have coördinates (x_1, mx_1, a), (x_2, mx_2, a), $(x_3, -mx_3, -a)$, and $(x_4, -mx_4, -a)$ respectively. Then if l_1 and l_2 are the given lengths, we have

$$(1) \qquad (x_2 - x_1)\sqrt{1+m^2} = l_1, \qquad\qquad (2) \qquad (x_4 - x_3)\sqrt{1+m^2} = l_2.$$

The planes which are the perpendicular bisectors of lines AB, CD, and AC respectively are easily determined as

$$2x + 2my = (x_1 + x_2)(1 + m^2), \tag{3}$$

$$2x - 2my = (x_3 + x_4)(1 + m^2), \tag{4}$$

$$2(x_1 - x_3)x + 2(x_1 + x_3)my + 4az = (x_1^2 - x_3^2)(1 + m^2). \tag{5}$$

If the value of x_1 determined from (1) and (3), and the value of x_3 from (2) and (4) are put in (5), we have at once the equation of the locus of the center (x, y, z) of the required sphere. It comes out as

$$mxy + a(1 + m^2)z + rx + sy + t = 0,$$

where r, s, and t are constants depending on m, l_1, and l_2. Since the original lines are not parallel $m \neq 0$. Similarly $a \neq 0$. Thus regardless of the values of r, s, and t, this is a hyperbolic paraboloid. Note that only its position, and not its shape, depends on the choice of l_1 and l_2.

Also solved by the proposer.

E 261 [1937, 104]. *Proposed by V. Thébault, Le Mans, France.*

Find the smallest possible base for a system of enumeration which contains three-digit squares of the form *aaa*, and six-digit squares of the form *bcbcbc*.

Solution by C. W. Trigg, Cumnock College, Los Angeles.

When the square numbers in the scale of r are expressed in the decimal scale, $N^2 = a(r^2+r+1)$ and $M^2 = br^5+cr^4+br^3+cr^2+br+c = (br+c)(r^4+r^2+1) = (br+c)(r^2+r+1)(r^2-r+1)$, where a, b, and c are less than r. Now (r^2+r+1) and (r^2-r+1) are relatively prime, and since neither can be a square number, each must contain a square factor. Set $r^2+r+1 = kd^2$, then $k \leqq a < r$. Put $r^2-r+1 = mg^2$, then $mk \leqq (br+c) \leqq (r^2-1)$. The only values of r less than 100 for which r^2+r+1 contains a repeated factor with $k < r$, are 18, 22, 30 and 68. The last is the only one for which $r^2-r+1 = 93 \cdot 7^2$ contains a repeated factor. Since $r^2+r+1 = 13 \cdot 19^2$, $a = 13$ or 52, and $(br+c) = 93 \cdot 13 = 17 \cdot 68 + 53$, so $b = 17$ and $c = 53$. Thus the smallest possible base for such a system of enumeration is 68, in which there are two squares of the form *aaa*, and but one of the form *bcbcbc*.

In examining the numbers r^2+r+1 for repeated factors, it is convenient to note that if a non-square number Q has a repeated factor, then it has a factor less than, or equal to the cube root of Q, so in none of the cases examined was it necessary to test divisibility with a prime greater than 19.

Also solved by E. P. Starke, who finds the above solution and points out that it does not quite comply with the requirements, since it furnishes but one square of the form *bcbcbc*. He determines that the smallest base which supplies squares (plural) of both the desired forms, is 313. In this case $a = 3 \cdot 1^2$, $3 \cdot 2^2$, $3 \cdot 3^2$, $\cdots$, $3 \cdot 10^2$, and $br+c = 5979 \cdot 1^2$, $5979 \cdot 2^2$, $5979 \cdot 3^2$, and $5979 \cdot 4^2$. Also solved by Simon Vatriquant and the proposer.

Advanced Problems

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3841. *Proposed by V. Thébault, Le Mans, France.*

Consider the orthocentric tetrahedron $ABCD$, any point P, and the inverse points A', B', C', D' of A, B, C, D in an inversion with the pole P and with the arbitrary modulus k. The planes perpendicular to the lines PA, PB, PC, PD passing through A', B', C', D' determine a tetrahedron A_1, B_1, C_1, D_1. (a) Show that the pedal tetrahedron $A_1'B_1'C_1'D_1'$ of a point P' with respect to $A_1B_1C_1D_1$ is orthocentric. (b) Show that the planes through P parallel to the faces of $A_1'B_1'C_1'D_1'$ cut the planes of the faces of $A_1B_1C_1D_1$ in four straight lines lying in the same plane perpendicular to the line joining P' to the orthocenter of $A_1'B_1'C_1'D_1'$.

3842. *Proposed by V. Thébault, Le Mans, France.*

A given plane (P) cuts the edges A_2A_3, A_3A_1, A_1A_2, A_4A_1, A_4A_2, A_4A_3, of a tetrahedron $A_1A_2A_3A_4$ in a_1, a_2, a_3, a_1', a_2', a_3', respectively. Parallel lines, with arbitrary direction, through A_1, A_2, A_3, A_4 cut respectively the spheres circumscribing the tetrahedrons $A_1a_2a_3a_1'$, $A_2a_3a_1a_2'$, $A_3a_1a_2a_3'$, $A_4a_1'a_2'a_3'$ in the points α_1, α_2, α_3, α_4. Show that these points lie in a plane (Q) which passes through a fixed point in the plane (P) when the direction of the parallels varies.

3843. *Proposed by V. Thébault, Le Mans, France.*

Given a tetrahedron $ABCD$, with an orthocenter H, inscribed in a sphere with center O. (a) Show that the tetrahedron with vertices O_a, O_b, O_c, O_d at the centers of the spheres $HBCD$, $HCDA$, $HDAB$, $HABC$ is orthocentric. (b) The center of the sphere $O_aO_bO_cO_d$ is the symmetric of the centroid with respect to H. (c) The points O and H are isogonal conjugates with respect to $O_aO_bO_cO_d$.

3780 [1936, 245]. *Proposed by J. M. Feld, New York City.*

In triangle $A_1A_2A_3$ the transversal A_iD_i divides A_jA_k in the ratio A_jD_i: $D_iA_k = p_i:q_i$, where ijk is a cyclic permutation of 123. The transversals A_iD_i and A_jD_j intersect in P_k. Find the value of the cross ratio

$$\frac{P_3P_2}{P_2A_1} \Big/ \frac{P_3D_1}{D_1A_1}$$

in terms of the p's and q's. Show that Ceva's theorem is a special case. (*An omission supplied.*)

SOLUTIONS

3754 [1935, 571]. *Proposed by Warren Jones, Maryville College, Tennessee.*

Find the minimum area of the segment cut from a parabola by a chord passing through a given point in its interior not on the curve or on its axis.

Solution by E. P. Starke, Rutgers University.

Take $y^2=ax$ as the equation of the parabola and let (x_0, y_0) be the given point. If we represent the ends of the chord by (x_1, y_1) and (x_2, y_2) we may employ the solution of E 128 [1935, 323] to give the area bounded by the chord and parabola,

$$\text{Area} = (y_1 - y_2)^3/6a. \tag{1}$$

The condition that the three points shall be collinear is easily reduced to

$$y_1y_2 - y_0(y_1 + y_2) + ax_0 = 0. \tag{2}$$

Since (1) is to be a minimum, we differentiate with respect to y_1 and equate to zero, obtaining $(y_1-y_2)^2(1-dy_2/dy_1)/2a=0$. So then $dy_2/dy_1=1$. Using this result in the derivative of (2) we have at once

$$y_1 + y_2 - 2y_0 = 0, \tag{3}$$

so that (x_0, y_0) is the midpoint of the chord.

Solving (2) and (3) for y_1 and y_2 and substituting these values in (1), the minimum area is given as

$$4(ax_0 - y_0^2)^{3/2}/3a.$$

Solved also by W. B. Campbell, J. W. Clawson, E. J. Scott, C. E. Springer, C. W. Trigg, F. Underwood, and M. Y. Woodbridge.

Editorial Note. The reasoning by R. E. Gaines, quoted in the note on the solution of 3713 [1936, 316], may be easily modified to prove in this problem that the minimum is attained when the given point bisects the chord, without using the expression for the area. With this fact it is easily found that the breadth of the minimum segment is $x_0-y_0^2/a$: its area is the same as that of a segment with this breadth and a vertical chord. The above result then follows.

3755 [1935, 571]. *Proposed by J. Rosenbaum, Hartford Federal College, Connecticut.*

The volume V of an orthocentric tetrahedron $ABCD$ is given by

$$288V^2 = f(a, b, c) + f(a, y, z) + f(x, b, z) + f(x, y, c),$$

where a, b, c, x, y, z are the lengths of the edges BC, CA, AB, DA, DB, DC, and

$$f(a, b, c) = (a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2).$$

Show that the converse is not true.

An instance showing that the converse is not true is found in a non-regular isosceles tetrahedron (a tetrahedron is isosceles when its opposite edges are equal). It is easily proved that such a tetrahedron is not orthocentric, and it also can be proved that its volume is given by

$$72V^2 = f(a, b, c), \tag{3}$$

where a, b, c are three edges forming a face. Multiplying (3) by 4, it becomes

$$288V^2 = f(a, b, c) + f(a, b, c) + f(a, b, c) + f(a, b, c), \tag{4}$$

and writing x for a, y for b, and z for c in the second, third, and fourth terms of the right hand member of (4) respectively, it becomes the equation of the problem. The question whether the converse is true for *non*-isosceles tetrahedrons has of course not been answered here.

3756 [1935, 572]. *Proposed by J. M. Feld, New York City.*

If

$$S_p = \sum_{k=1}^{n} (2k - 1)^p,$$

prove that

$$\sum_{i=0}^{k} \binom{2k+1}{2i} S_{2i} = 2^{2k} n^{2k+1}.$$

Solution by F. Underwood, University College, Nottingham.

We have

$$\begin{aligned} (x+1)^{2k+1} - (x-1)^{2k+1} &= \sum_{j=0}^{2k+1} [1 + (-1)^j]\,{}_{2k+1}C_j x^j, \\ &= 2\sum_{i=0}^{k} {}_{2k+1}C_{2i} x^{2i}. \end{aligned} \tag{1}$$

In this identity set in succession $x = 1, 3, \cdots, 2n-1$, and add the results. After removing from each side the factor 2, we get

$$2^{2k} n^{2k+1} = \sum_{i=0}^{k} {}_{2k+1}C_{2i} S_{2i},$$

and this is the desired result.

Note. This problem seems to be a particular case, with a slightly different notation, of Theorem VI of the article, *The sums of powers of integers*, E. E. Witmer, in this MONTHLY, 1935, p. 547.

Solved also by Frank Ayres, Jr., E. P. Starke, and the proposer.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items of interest to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

A "Colloque consecré à la Théorie des Probabilités" was held at Geneva, Switzerland, October 11–16, 1937, under the auspices of the University of Geneva. Professor Aurel Wintner of Johns Hopkins University spoke on "Distribution functions of sums of independent random variables," and Professor E. L. Dodd, of the University of Texas, spoke on "Certain coefficients of regression or trend associated with largest likelihood."

Appointments to the faculty of Queens College, the new city college just opened at Flushing, N. Y., include the following mathematicians, as announced by Dr. Paul Klapper, president of the college: Dr. T. F. Cope, professor and head of the department at Marietta College, to be assistant professor; Dr. H. W. Raudenbush, Jr., assistant professor at Yale University, to be instructor; Dr. Lulu H. von Bechtelsheim, instructor at Barnard College, to be instructor; Dr. Arthur Sard, formerly instructor at Harvard University, member of the actuarial staff of the Prudential Life Insurance Company, to be instructor.

Associate Professor E. F. Allen of the Oklahoma A. and M. College has been promoted to a professorship.

Assistant Professor Edith I. Atkin of the Illinois State Normal University has been promoted to an associate professorship.

Associate Professor J. P. Ballantine of the University of Washington has been promoted to a professorship.

Assistant Professor S. F. Bibb of Armour Institute of Technology has been promoted to an associate professorship.

Associate Professor J. H. Binney of the A. and M. College of Texas has been promoted to a professorship.

Dr. Scott Buchanan of the University of Chicago has been appointed dean at St. John's College, Annapolis, Md.

Assistant Professor J. Hobart Bushey of Hunter College has been promoted to an associate professorship.

Associate Professor W. E. Byrne of Virginia Military Institute has been promoted to a professorship.

Assistant Professor Iris Callaway of the University of Georgia has been promoted to an associate professorship.

Dr. E. J. Camp has been appointed to an associate professorship at Macalester College.

Professor W. B. Carver of Cornell University is on leave of absence for the first semester and is traveling in Europe.

Assistant Professor B. G. Clark of the University of Alabama has been promoted to an associate professorship.

J. A. Duerksen of the U. S. Coast and Geodetic Survey has been promoted from assistant to associate mathematician.

L. A. Fair of the Morehead (Kentucky) State Teachers College has been promoted to an assistant professorship.

Assistant Professor L. R. Ford of Rice Institute has been appointed professor of mathematics and chairman of the department at Armour Institute of Technology.

Dr. A. H. Fox of Union College has been promoted to an assistant professorship.

E. T. Frankel, who has been director of research for the New York State temporary emergency relief administration in New York City, is now assistant director of the Bureau of Research and Statistics, State Department of Social Welfare, at Albany.

Dr. F. C. Gentry of the University of Illinois has been appointed assistant professor at the University of Oklahoma.

Dr. S. G. Hacker of Indiana University has been appointed assistant professor at the State College of Washington.

Associate Professor Cora B. Hennel of Indiana University has been promoted to a professorship.

Associate Professor R. C. Huffer of Beloit College has been promoted to a professorship.

Dr. Ralph Hull of the University of Michigan has been appointed an assistant professor at the University of Illinois.

Professor E. V. Huntington of Harvard University delivered an invited address on "The method of postulates" at the meeting of the Institute of Philosophy of Bowdoin College held during the second week in April.

Dr. C. M. Jensen is now an associate professor at Kansas Wesleyan University.

Associate Professor R. A. Johnson of Brooklyn College has been promoted to a professorship.

Dr. H. S. Kaltenborn of the University of Michigan has been appointed assistant professor of mathematics at the Philadelphia College of Pharmacy and Science.

P. V. Kunkel of the Kutztown (Pennsylvania) State Teachers College has been appointed to a professor at Cedar Crest College.

Associate Professor A. J. Lewis of the University of Denver has been promoted to a professorship.

Assistant Professor C. I. Lubin of the University of Cincinnati has been promoted to an associate professorship.

Associate Professor C. F. Luther of Willamette University has been promoted to a professorship.

Professor C. C. MacDuffee of the University of Wisconsin is on leave for the year and will be at the Institute for Advanced Study, Princeton, N. J.

After a year's leave of absence spent at the University of Michigan, Assistant Professor Roy MacKay of Eastern New Mexico Junior College has been promoted to an associate professorship.

Assistant Professor Anna E. Many of Sophie Newcomb College has been promoted to an associate professorship.

W. R. Murray of Franklin and Marshall College is now an assistant professor.

Dr. O. K. Sagen of Iowa State College has been appointed to an assistant professorship at the University of Maryland.

Dr. Gerhard Tintner of Cowles Institute has been appointed assistant professor of mathematics and economics at Iowa State College.

Dr. P. L. Trump of the University of Wisconsin has been appointed assistant professor in the teaching of mathematics in the University of Wisconsin High School.

Dr. H. L. Turrittin of the College of Mines and Metallurgy, El Paso, has been promoted to an assistant professorship.

Professor F. E. Wood of Northwestern University is on leave of absence for the academic year 1937–38.

Dr. G. A. Baker has been appointed instructor in mathematics and statistician for the Experiment Station of the Branch of the College of Agriculture, University of California, at Davis, Calif.

Dr. R. P. Boas, Jr. of Harvard University will be a National Research Fellow at Princeton University for 1937–38.

The following appointments to instructorships for the year 1937–38 are announced:

Brown University: Dr. J. V. Wehausen
California Institute of Technology: Dr. E. W. Paxson

University of California at Los Angeles: Dr. F. A. Valentine
Campbell College, Buies Creek, North Carolina: R. E. Smith
University of Delaware: Dr. G. C. Webber
Gettysburg College: Dr. R. H. Wilson, Jr.
Harvard University, part-time instructors: R. F. Clippinger, M. P. Fobes, A. D. Hestenes, R. F. Jackson, D. T. McClay, H. E. Robbins, A. Spitzbart
University of Illinois: Dr. C. W. Mendel
University of Indiana: Dr. Louis Green
Iowa State College: R. H. Cook, H. C. Fryer
University of Montana: Dr. Harold Chatland
New Mexico College of A. and M.A.: Dr. Gordon Fuller
Rensselaer Polytechnic Institute: R. E. Street
Rice Institute: Dr. Walter Leighton
South Dakota School of Mines: A. W. Davis.
University of Tennessee: Dr. J. W. Blincoe, M. J. Turner
Utah State Agricultural College: Dr. M. T. Bird
University of Utah: Dr. Harriet Rees, Dr. Anna A. Stafford
Western Illinois State Teachers College: Dr. H. G. Ayre

E. D. Rainville of the U. S. Bureau of Reclamations, Denver, is a teaching fellow for this year at the University of Michigan.

Professor R. P. Baker of the University of Iowa died on August 13, 1937, at the age of seventy-one years. He had been a member of the department of mathematics there since 1905. He served on the editorial board of the MONTHLY for a number of years beginning in 1913, and was a charter member of the Association.

Professor R. D. Beetle of Dartmouth College died on July 9, 1937, at the age of fifty-one. He was a charter member of the Mathematical Association.

Professor L. M. Hoskins, emeritus professor of applied mathematics at Stanford University since 1925, died on September 8, 1937, at the age of seventy-seven. He joined the faculty of the university as assistant professor in 1892, a year after David Starr Jordan became president of the university. Professor Hoskins was a charter member of the Mathematical Association. At the time of the first summer meeting of the Association, Professor Hoskins made the trip to Cambridge, Massachusetts in order to lead the discussion on Professor E. V. Huntington's paper "The teaching of elementary dynamics."

Members or librarians who no longer need the copy of the MONTHLY for January 1937 will confer a favor on the Association if they will send such copy to the Secretary, W. D. Cairns, Oberlin, Ohio, who will give credit on future dues or subscriptions.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, E. J. MOULTON, Northwestern University, Evanston, Ill.

BOOKS FOR REVIEW should be addressed to REVIEW EDITOR, American Mathematical Monthly, 531 West 116th Street, New York, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of addresss to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-first Summer Meeting, Pennsylvania State College, Sept. 6–7, 1937.
Twenty-second Annual Meeting, Indianapolis, Ind., December 30–31, 1937.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1937 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Waynesburg, Pa., May 1; Pittsburgh, October 23.
ILLINOIS, DeKalb, May 14–15.
INDIANA, Greencastle, April 30–May 1.
IOWA, Dubuque, April 16–17.
KANSAS, Wichita, April 3.
KENTUCKY, Louisville, May 1.
LOUISIANA-MISSISSIPPI, Hammond, La., March 5–6.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Lynchburg, Va., May 8.
MICHIGAN, Ann Arbor, March 20.
MINNESOTA, St. Paul, May 15.
MISSOURI.
NEBRASKA, Lincoln, May 7.
OHIO, Columbus, April 1.
OKLAHOMA, Tulsa, February 5.
PHILADELPHIA, Haverford, Nov. 27.
ROCKY MOUNTAIN, Greeley, Colo., April 16–17.
SOUTHEASTERN, Nashville, Tenn., April 16–17.
SOUTHERN CALIFORNIA, Los Angeles, March 6.
SOUTHWESTERN, State College, N.M., April 2–3.
TEXAS, Houston, April 23–24.
WISCONSIN, Milwaukee, May 8.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

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(INCORPORATED)

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THE TWENTY-FIRST SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION

The twenty-first summer meeting of the Mathematical Association of America was held at the Pennsylvania State College, State College, Pennsylvania, on Monday and Tuesday, September 6–7, 1937, in conjunction with the summer meeting and colloquium of the American Mathematical Society. Four hundred fifty-seven were in attendance at the meetings, including the following two hundred nine members of the Association:

C. R. ADAMS, Brown University
V. W. ADKISSON, University of Arkansas
R. P. AGNEW, Cornell University
E. B. ALLEN, Rensselaer Polytechnic Institute
R. LUCILE ANDERSON, Hunter College.
R. C. ARCHIBALD, Brown University
H. T. R. AUDE, Colgate University
FRANK AYRES, JR., Dickinson College
W. L. AYRES, University of Michigan

CLARA L. BACON, Goucher College
H. M. BACON, Stanford University
I. A. BARNETT, University of Cincinnati
HELEN BARTON, Woman's College, University of North Carolina
ETHELWYNN R. BECKWITH, Milwaukee-Downer College
J. A. BENNER, Lafayette College
G. D. BIRKHOFF, Harvard University
A. H. BLACK, Lebanon Valley College
H. L. BLACK, Westminster College
H. F. BLICHFELDT, Stanford University
G. A. BLISS, University of Chicago
HENRY BLUMBERG, Ohio State University
R. P. BOAS, JR., Princeton University
C. C. BRAMBLE, U. S. Naval Academy, Postgraduate School
FOSTER BROOKS, Kent State University
MYRTLE C. BROWN, North Texas State Teachers College
E. T. BROWNE, University of North Carolina
J. A. BULLARD, University of Vermont
J. H. BUSHEY, Hunter College
JEWELL HUGHES BUSHEY, Hunter College
W. H. BUSSEY, University of Minnesota

W. D. CAIRNS, Oberlin College
C. C. CAMP, University of Nebraska
W. B. CAMPBELL, Drexel Institute
I. S. CARROLL, Syracuse University
E. W. CHITTENDEN, University of Iowa
R. V. CHURCHILL, University of Michigan
A. B. COBLE, University of Illinois
TERESA COHEN, Pennsylvania State College
J. B. COLEMAN, University of South Carolina
LENNIE P. COPELAND, Wellesley College
RICHARD COURANT, New York University
H. B. CURRY, Pennsylvania State College
D. R. CURTISS, Northwestern University
J. H. CURTISS, Cornell University
E. H. CUTLER, Lehigh University

WAYNE DANCER, University of Toledo
P. H. DAUS, University of California at Los Angeles
H. A. DAVIS, West Virginia University
H. T. DAVIS, Northwestern University
J. E. DAVIS, Drexel Institute
F. F. DECKER, Syracuse University
L. L. DINES, Carnegie Institute of Technology
C. H. DIX, Humble Oil and Refining Company, Houston, Texas
H. L. DORWART, Washington and Jefferson College
ARNOLD DRESDEN, Swarthmore College
D. M. DRIBIN, Yale University
L. T. DUNLAP, Pennsylvania State College

J. A. EIESLAND, West Virginia University

E. J. FINAN, Catholic University
M. M. FLOOD, Princeton Local Government Survey
W. B. FORD, University of Michigan
TOMLINSON FORT, Lehigh University
R. M. FOSTER, Bell Telephone Laboratories
J. S. FRAME, Brown University
ORRIN FRINK, JR., Pennsylvania State College
T. C. FRY, Bell Telephone Laboratories

A. S. GALE, University of Rochester
JOY GILDER, New York University
B. P. GILL, College of the City of New York
R. E. GILMAN, Brown University
A. M. GINSBURG, Brooklyn High School of Arts and Trades
J. S. GOLD, Bucknell University
W. O. GORDON, Pennsylvania State College

W. C. GRAUSTEIN, Harvard University
T. E. GRAVATT, Pennsylvania State College
C. H. GRAVES, Pennsylvania State College
W. L. GRAVES, University of Pennsylvania
F. L. GRIFFIN, Reed College
V. G. GROVE, Michigan State College

BEATRICE L. HAGEN, Pennsylvania State College
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E. H. HANSON, North Texas State Teachers College
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M. L. HARTUNG, Ohio State University
L. A. HAZELTINE, Stevens Institute of Technology
G. A. HEDLUND, Bryn Mawr College
E. R. HEDRICK, University of California at Los Angeles
H. C. HICKS, Carnegie Institute of Technology
E. H. C. HILDEBRANDT, State Teachers College, Montclair, New Jersey
T. H. HILDEBRANDT, University of Michigan
T. R. HOLLCROFT, Wells College
CHARLES HOPKINS, Tulane University
W. A. HURWITZ, Cornell University

M. H. INGRAHAM, University of Wisconsin

DUNHAM JACKSON, University of Minnesota
R. L. JEFFERY, Acadia University
FLORENCE A. JEFFRIES, State College Pennsylvania
R. A. JOHNSON, Brooklyn College
L. S. JOHNSTON, University of Detroit
B. W. JONES, Cornell University

H. S. KALTENBORN, Philadelphia College of Pharmacy and Science
A. J. KEMPNER, University of Colorado
EVELYN M. KENNEDY, West Night H. S., Cincinnati, Ohio
L. S. KENNISON, Brooklyn College
P. W. KETCHUM, Institute for Advanced Study
J. R. KLINE, University of Pennsylvania
M. S. KNEBELMAN, Princeton University
G. R. KRAUS, Cathedral College, Erie, Pennsylvania

W. D. LAMBERT, U. S. Coast and Geodetic Survey
A. E. LANDRY, Catholic University
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W. H. MCEWEN, Mount Allison University
J. V. MCKELVEY, Iowa State College
J. S. MCNAIR, Canal Zone High School
A. E. MEDER, JR., New Jersey College for Women
D. D. MILLER, University of Michigan
W. I. MILLER, Bucknell University
W. M. MILLER, Massachusetts State College
U. G. MITCHELL, University of Kansas
E. B. MODE, Boston University
VIRGINIA MODESITT, Randolph-Macon Woman's College
E. C. MOLINA, Bell Telephone Laboratories
ETHEL I. MOODY, Pennsylvania State College
C. N. MOORE, University of Cincinnati
R. L. MOORE, University of Texas
RICHARD MORRIS, Rutgers University
MARSTON MORSE, Institute for Advanced Study
DAVID MOSKOVITZ, Carnegie Institute of Technology
E. J. MOULTON, Northwestern University
W. R. MURRAY, Franklin and Marshall College
J. R. MUSSELMAN, Western Reserve University

MARY W. NEWSON, Eureka College

C. O. OAKLEY, Haverford College
E. G. OLDS, Carnegie Institute of Technology
F. W. OWENS, Pennsylvania State College
HELEN B. OWENS, State College, Pennsylvania

H. A. PERKINS, Hampton Institute
T. S. PETERSON, The Shipley School
A. E. PITCHER, Harvard University

O. J. RAMLER, Catholic University
W. C. RANDELS, Northwestern University
J. F. RANDOLPH, Institute for Advanced Study

MINA S. REES, Hunter College
C. N. REYNOLDS, West Virginia University
C. E. RHODES, Ohio State University
C. H. RICHARDSON, Bucknell University
R. G. D. RICHARDSON, Brown University
H. L. RIETZ, University of Iowa
J. F. RITT, Columbia University
SELBY ROBINSON, College of the City of New York
J. B. ROSENBACH, Carnegie Institute of Technology

I. J. SCHOENBERG, Colby College
WLADIMIR SEIDEL, University of Rochester
R. S. SHAW, College of the City of New York
I. M. SHEFFER, Pennsylvania State College
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L. L. SMAIL, Lehigh University
CLARA E. SMITH, Wellesley College
W. M. SMITH, Lafayette College
VIRGIL SNYDER, Cornell University
P. I. SPEICHER, Albright College
C. E. SPRINGER, University of Oklahoma
E. R. STABLER, State Teachers College, Montclair, New Jersey
G. W. STARCHER, Ohio University
M. H. STONE, Harvard University
J. L. SYNGE, University of Toronto
GABRIEL SZEGÖ, Washington University

J. D. TAMARKIN, Brown University
J. S. TAYLOR, University of Pittsburgh
MILDRED E. TAYLOR, Mary Baldwin College
T. Y. THOMAS, Princeton University
C. C. TORRANCE, Case School of Applied Science
BIRD M. TURNER, University of West Virginia

H. S. VANDIVER, University of Texas
H. E. VAUGHAN, University of Illinois
JOHN VON NEUMANN, Institute for Advanced Study

R. J. WALKER, Cornell University
J. L. WALSH, Harvard University
R. M. WALTER, New Jersey College for Women
WARREN WEAVER, Rockefeller Foundation
J. V. WEHAUSEN, Brown University
ANNA PELL WHEELER, Bryn Mawr College
E. A. WHITMAN, Carnegie Institute of Technology
K. P. WILLIAMS, Indiana University
J. B. WINSLOW, University of Toledo
F. E. WOOD, Northwestern University
F. L. WREN, George Peabody College
FRANCES M. WRIGHT, Elmira College

OSCAR ZARISKI, Johns Hopkins University

The mathematicians were housed comfortably in the dormitories of the State College, enjoyed excellent meals in the large dining room of McAllister Hall and used very constantly the Upper and Lower Lounges of "Old Main" which occupies a central position on the campus. Registration, mail, and the manifold needs of the visitors were cared for very ably by a corps of members of the department of mathematics.

Tea was served Monday and Tuesday afternoons in Old Main by the ladies of the department. An artistic musical program was given Monday evening at the Nittany Lion Inn consisting of songs by Mrs. C. C. Wagner, a violin sonatina by Professor Teresa Cohen and piano selections by Professor T. C. Benton. Tuesday evening President Hetzel of the College and Mrs. Hetzel, assisted by Professor and Mrs. Owens, President Moore of the Society and President Kempner of the Association, received the visitors in the Upper Lounge of Old Main. Wednesday afternoon was set aside for a picnic excursion by buses and private automobiles to the Nature Study Camp, Seven Mountains; three "hikes" of varying difficulty and other milder diversions occupied the afternoon, followed by a picnic supper and by campfire singing. Professor and Mrs. Owens received

at a pleasant "At Home" Thursday afternoon. Visits to the Mineral and Industrial Art Exhibit and to the Textile Chemistry Building were made possible at stated times during the week. Many availed themselves of the very convenient facilities afforded for bridge, tennis, and golf. The cordiality of our reception, the agreeable and varied nature of the entertainment, and the generosity of the College in furnishing accommodations at such reasonable expense, were recognized by a resolution of thanks proposed at the joint dinner by Professor W. L. Hart and adopted with unanimity.

The annual dinner on Thursday evening, attended by three hundred eleven persons, was held in Nittany Lion Inn. Professor Virgil Snyder acted as toastmaster. Dean Hammond, himself a newcomer to the College, greeted the mathematicians on behalf of the President and officers.

President Moore of the Society referred to the surprising information furnished by Professor Richardson in his recent report, indicating how few Ph.D.'s in mathematics have published more than twenty or thirty research papers; only twenty-two up to 1933 had published more than thirty papers. Can this situation be remedied? He maintained that students are started too early on theses for the Ph.D. degree and that if the one in charge never assigns a thesis until he is assured that the student is capable of original work, it will be almost a guarantee that he will continue to publish beyond the Ph.D. thesis.

President Kempner of the Association spoke of his dismay, on coming to the United States twenty-five years ago, in finding the low character of freshman and sophomore courses in college mathematics, not to mention the low ebb today when students without any more than grammar school mathematics may enter college, obtain the bachelor's degree and even the doctor's degree in their chosen fields. Happily we as mathematicians are beginning to realize that the fault is largely our own and that it can be remedied by well-directed efforts on the part of all of us. The mistrust which has existed between high school and college teachers is happily dying down in a common cause. He urged the enrollment of a much larger number of secondary school teachers in the Association and more frequent joint meetings of the Sections of the Association with high-school teachers.

Doctor Fry, as executive secretary of the Semi-Centennial of the Society next September, described the accomplishment of Professor T. S. Fiske who, as a result of his visit as a young man to the London Mathematical Society, brought about the formation of the New York Mathematical Society forty-nine years ago, the first mathematical group ever to get together in America. The secret of the success of the Society was the existence of someone wanting it to go, willing to make it go, working to make it go. Aside from announcing features of the Semi-Centennial which will soon be publicized more fully, he said that he looks for next year to be the beginning of another fifty year period in which the Society will take a place of national influence, such as we do not now have, in determining what the objectives and what the standards in mathematics shall be in the United States.

The American Mathematical Society held sessions for the reading of short

papers on Tuesday and Thursday afternoons and Wednesday, Thursday and Friday mornings. The Colloquium Lectures on "Continuous geometry" were given by Professor John von Neumann, of The Institute for Advanced Study, Tuesday afternoon and Wednesday, Thursday, and Friday mornings. At a general session Thursday afternoon Professor Hassler Whitney of Harvard University gave by invitation an address on "Topological properties of differentiable manifolds."

The Mathematical Association held sessions on Monday afternoon and Tuesday morning. The Association is indebted to the Program Committee consisting of Professors Arnold Dresden, J. A. Shohat, and Tomlinson Fort, chairman, for the organization of the program. This follows, together with abstracts of some of the papers, numbered in accordance with their place on the program.

First Session of the Association

1. "Integration in finite terms" by Professor J. F. Ritt, Columbia University.

2. "Some elementary aspects of topology" by Professor W. L. Ayres, University of Michigan.

3. "Algebra of logic" by Professor Orrin Frink, Jr., Pennsylvania State College.

1. Professor Ritt's paper dealt with the work of Liouville and his followers on the possibility of performing integrations and solving differential equations in terms of elementary functions.

2. Professor Ayres stated that he had encountered difficulty explaining the exact nature of topology and its problems to engineering colleagues and undergraduate students in mathematics, and that he had found more success by connecting elementary topological theory with well-known geometrical puzzles and problems. He discussed the Problem of the Seven Bridges of Königsberg, which led Euler to the general solution of the traversing of a finite graph. The Icosian Game of Sir William Hamilton and the Knight's Tour on the chessboard were used to lead to the Four Color Problem and the general question of joining a given finite set of points in a graph or Peano space by an arc or simple closed curve. The Water, Gas, and Electricity Puzzle was mentioned to connect with Kuratowski's solution of the embedding of a graph in the plane, the embedding of Peano spaces in the plane by Claytor, and embeddings in the torus and projective plane by MacLane and Kagno. These were given as illustrations of the method the speaker found useful in discussing the question of "What is topology?"

3. Professor Frink's paper will appear in an early issue of this Monthly.

Second Session of the Association

1. "Fashions in mathematics" by Professor D. R. Curtiss, Northwestern University, retiring president of the Association.

2. "The Principia and the modern age" by Professor C. S. Slichter, University of Wisconsin.

3. "Mathematical adventures in social science" by Professor H. T. DAVIS, Northwestern University.

4. "Herbert Ellsworth Slaught—editor and organizer" by Professor W. D. CAIRNS, Oberlin College.

5. "Herbert Ellsworth Slaught—teacher and friend" by Professor G. A. BLISS, University of Chicago.

1. Professor Curtiss's retiring presidential address appears in this issue of the MONTHLY.

2. Professor Slichter's address appeared in the August-September issue of this MONTHLY.

3. Professor Davis's paper will appear in an early issue of this MONTHLY.

4, 5. The memorial papers by Professor Cairns on Professor Slaught's career in connection with this MONTHLY and the Association, the Central Association of Teachers of Science and Mathematics and the National Council of Teachers of Mathematics, and by Professor Bliss on his teaching and other activities, will appear in the issue of this MONTHLY for January 1938.

Following these papers, Professor E. R. Hedrick, by request of the Trustees, offered the following resolution, which was adopted by a unanimous rising vote, as a fitting conclusion to the meeting:

> With the death of Herbert Ellsworth Slaught on May 21, 1937 an irreparable loss has been sustained by this Association. He is recognized by all as the prime mover in the establishment of the Mathematical Association. For years before its organization, he labored to maintain and to improve the AMERICAN MATHEMATICAL MONTHLY, later to become the official organ of this Association. After the creation of the Association in December, 1915, he continued his active work for the MONTHLY and for the Association practically up to the end of his days. To him the Association owes, more than to any other, its existence and its success.
>
> To him many students and teachers owe real inspiration toward their careers, and their vision of what unselfish effort may be.
>
> To him the University of Chicago owes a tradition of excellence in instruction in mathematics. Through the guidance and example which he gave to others many institutions have profited in like manner.
>
> His services to mathematics in connection with the Carus Monographs, in connection with the Chicago Section of the Society, in connection with organization and encouragement of sections of this Association in many parts of this country, and in many other ways, are known to all of us. Mathematics in America is better and richer because he lived.
>
> It is futile for us to attempt to honor him now in the sense of transmitting our gratitude. All that we may do is ourselves to realize his unselfish service and his worth, that we may emulate him and per-

haps lead others to do so. Thus we may support and develop the work of this Association, we may improve our own teaching, we may seek to improve the status of mathematics in schools of all grades. By doing so, we shall best show our respect for what he did, and we shall do honor not only to him but also to ourselves.

We, the Mathematical Association of America, therefore now formally acknowledge our indebtedness to him and we do resolve to carry out in this Association, in so far as lies in our power, the ideals and the idealism which were his.

We direct the Secretary of the Association to spread upon the minutes of this meeting these resolutions and to transmit a copy of them to his daughter, Miss Katharine M. Slaught.

Meeting of the Board of Trustees

The Trustees met on Monday evening.

The following twenty-five persons were elected to membership on applications duly certified:

To Individual Membership

Frank Boehm, Life Insurance Manager, New York, N. Y.

L. P. Chebotar, D.P.C. (Univ. of Naples) Research chemist, The Texas Co., New York, N. Y.

Virgil Claudian, Grad. (Univ. of Bucharest) Prof. Liceul Mihai Viteazul, Bucharest, Roumania.

Nancy Cole, Ph.D. (Radcliffe) Instr., Sweet Briar Coll., Sweet Briar, Va.

Richard Courant, Ph.D. (Univ. of Göttingen) Prof., New York Univ., New York, N. Y.

Rev. Hilary Doerfler, A.M. (St. John's Univ., Collegeville, Minn.) Head, Physics and Math. Dept., Saint Gregory's College, Shawnee, Okla.

N. Durairajan, Exec. Engineer, Mylapore, Madras, India.

E. E. Everett, B.S. (Brigham Young Univ.) Instr., Dixie Jr. Coll., St. George, Utah.

E. E. Hagler, Jr., A.B. (Harvard) Capt., U. S. Army, retired, Bartlesville, Okla.

Emma V. Hesse, A.M. (Columbia) Supervising teacher, Univ. High School, Oakland, Calif.

L. O. Kattsoff, Ph.D. (Pennsylvania) Instr., Philos., Univ. of North Carolina, Chapel Hill, N. C.

G. R. Kraus, M.S. (Carnegie Inst. of Tech.) Instr., Cathedral College, Erie, Pa.

V. V. Lavroff, B.S. (Georgia School of Tech.) Prof., Univ. System of Georgia Evening School and Jr. Coll., Atlanta, Ga.

Jack Lorell, A. B. (Brooklyn Coll.) 225 Parkside Ave., Brooklyn, N. Y.

D. D. Miller, A.M. (Wayne Univ.) Teaching fellow, Univ. of Michigan, Ann Arbor, Mich.

Opal L. (Mrs. K. T.) Miller, A.M. (Texas Tech.) Instr., Texas Tech. Coll., Lubbock, Texas.

Virginia Modesitt, Ph.D. (Illinois) Instr., Randolph-Macon Woman's Coll., Lynchburg, Va.

C. D. Olds, A.M. (Stanford) Acting Instr., Stanford Univ., Stanford University, Calif.

C. W. Perry, B.S. (McGill) Clerk, Sun Life Assur. Co. of Canada, Montreal, P. Q.

R. A. Purviance, A.B. (Wabash) Instr., Univ. of Tennessee, Knoxville, Tenn.

M. M. Slotnick, Ph.D. (Harvard) Chief Supervisor of Interpretation, Geophysics Dept., Humble Oil and Refining Co., Houston, Texas.

Joshua Tabatchnik, Architect (Imp. Petrograd Acad.) 44 Court St., Brooklyn, N. Y.

Harriet A. Welch, M.L. (California) Head of Dept., Lowell High School, San Francisco, Calif.

S. W. Woodard, M.S. (Northwestern) Party chief, Schlumberger Well Surv'g Corp., Corpus Christi, Texas.

H. E. Woodward, A. M. (Texas Tech. Coll.) Instr., Texas Tech. Coll., Lubbock, Texas.

It was voted (1) to accept the invitation to meet at the University of Wisconsin in September 1939; (2) to accept the invitation of the Society to attend its Semi-Centennial at New York, N. Y. in September 1938, probably without active participation in the meetings; (3) to empower the Secretary to draw up an amendment removing the word "Manager" from the By-Laws; (4) to appoint W. B. Carver as a member of the Committee on Official Journal in place of Professor Slaught for the remainder of the year 1937; (5) on request of the Secretary-Treasurer, to authorize a formal audit of the accounts of the Association at the close of the present fiscal year; (6) to reaffirm the endorsement of the World Calendar and to request the Department of State that the United States attend the League conference on the reform of the calendar and to voice its approval of the League's efforts. Other actions concerned working details of the offices of the Secretary-Treasurer and the Editor-in-Chief.

W. D. CAIRNS, *Secretary-Treasurer*

PROPOSED AMENDMENTS TO THE ASSOCIATION BY-LAWS

The Trustees of the Mathematical Association of America, Inc., recommend that the following resolutions be adopted at the annual meeting at Indianapolis, Indiana, December 31, 1937.

"Resolved that Article III, Section 3 of the By-Laws of the Mathematical Association of America, Inc., be amended to read:

3. The President shall be elected by the Association's members biennially for a term of two years and shall be ineligible for reëlection. The Vice-Presidents shall be elected by the Association's members annually for a term of one year, and four members of the Board shall be elected by the Association's members annually for a term of three years. They shall be eligible for reëlection, but not for more than two (2) consecutive terms. The Secretary-Treasurer, the Librarian, and the Committee on Official Journal, consisting of the Editor-in-Chief and two other members, shall be appointed by the Board. All Officers and other Trustees shall hold over until their respective successors are elected or appointed and qualify."

The purpose of the amendment is to remove "the Manager" from the By-Laws, inasmuch as Professor H. E. Slaught was so uniquely "the Manager."

"Resolved that Article III, Section 7 of the By-Laws of the Mathematical Association of America, Inc., be amended to read:

7. Approximately two months before the date of the annual meeting all members shall be given an opportunity to nominate by mail a candidate for each office to be filled by the members for the ensuing year. Approximately one month before the annual meeting the Board shall select a nominee for President out of the three persons who received the most votes for this office in the nominations; the Board shall furthermore select two candidates for each other

office to be filled by the members, one being the person who received the highest vote in the nominations and the other being selected from among the nominees next in order. The election shall be by mail and in person and shall close on the day of the annual meeting."

The purpose of the amendment is both to give proper consideration to the expressed preferences of members for presidential nominees and to avoid the embarrassment involved in putting two presidential nominees in seeming competition for this, the most prominent office to be filled.

W. D. CAIRNS, *Secretary-Treasurer*

FASHIONS IN MATHEMATICS*

By D. R. CURTISS, Northwestern University

My title may impress some of you as beneath the dignity of this occasion. Surely a farewell message should deal with things more important than styles or modes. You may feel that the herd instinct does not even exist among mathematicians, who are, by common report, individualistic in the extreme. The Herr Kamerad who wrote the introductory article for *Deutsche Mathematik* felt that this judgment was only too true, and that it was the supreme duty of good nationalists to regiment mathematics and mathematicians, with a different fashion, presumably set by one who knew best, for each nation and race. To him, certainly, my subject would not seem frivolous.

The field is, in fact, far too wide, in all its implications, for one brief address. I might, for example, follow the line suggested by the Herr Kamerad, and discuss whether and why there have been nationalistic preferences in mathematics. For instance, is it true that geometry is especially the mode in Italy, or point sets in Poland? And if so, why? It has been said that Anglo-Danes are better mathematicians than Anglo-Saxons (hence, according to this authority, Cambridge's mathematical superiority over Oxford). Who has not heard that the Jew is superior to the Gentile in mathematical endowment? But I have no time to examine these speculations further.

Manifestations of the herd instinct among students of mathematics, that must also be dismissed with a word, concern the relative popularity of different teachers and institutions. Thus for years it was decidedly the fashion for young Americans to get their doctor's degrees in Germany, and then this ceased to be the style.

Fashions in notations have been seriously studied; see, for example, Cajori's two volume *History of Mathematical Notations*. Here, at least, fashions are really rather important.

I might even have tried to answer the question,—is mathematics itself

* Retiring presidential address presented at the meeting of the Association at Pennsylvania State College, Sept. 7, 1937.

fashionable? When we consider the enormous increase of mathematical literature (we have been told that the universe will be packed solid with it at some alarmingly near date if the present rate of increase continues; as an editor, I sometimes feel that this is only too probable), mathematics seems very much the mode. But our friends in Education (with a capital E) do not appear to believe this at all, and see mathematics as a subject that must disappear from the schools except for the few who may insist on electing it.

After these preliminary suggestions as to what I might discuss, but will not, let me narrow my subject to fashions in fields of research in mathematics. If this were a sermon, I might take as my text the remark made about ten years ago by an ambitious young producer of articles in not quite sufficiently fashionable fields of analysis, "Must I go into non-Riemannian geometry to get anybody to read my stuff?"* Topology and abstract spaces would fit even better into the frame of this remark. Someone strikes gold, and the rush is on until the diggings become less productive, then on to the next rich prospect! Is there any truth in this metaphor? This is what I propose to consider.

I am certainly not the first who has noted such phenomena. Bôcher, in his presidential address to the American Mathematical Society (*Bulletin of the American Mathematical Society*, vol. 18, 1911, page 7) says that Sturm in 1829 chose to formulate problems regarding differential and difference equations in terms of the flow of heat rather than of vibrating strings because the former subject was more "up to date," largely because of the influence of Fourier's treatise. He notes that for nearly fifty years a problem (equivalent to Sturm's) on vibrating strings had gone untouched, after initial work by Daniel Bernoulli, Euler, and Lagrange, but the latest mode in 1829 was flow of heat, and vibrating strings were out of style. It seems that fashions in mathematics not only change, but sometimes even reverse themselves, for Bôcher in 1911 preferred vibrating strings.

In his fascinating book, *Men of Mathematics*, Bell several times suggests my subject. For instance, he notes how the invariant theory of Cayley and Sylvester passed out of style. Coolidge only four years ago addressed this Association on *The Rise and Fall of Projective Geometry* (see this MONTHLY, vol. 41, pp. 217–228) and gave a striking picture of a field of research, synthetic projective geometry, once in vogue, that has now become as unfashionable as side whiskers. Darboux, in his St. Louis address (1904), referred, apparently with some regret, to the fact that analysis was becoming more popular than geometry.

Even great mathematicians may prefer to be fashionable. Newton used synthetic methods in his *Principia* because he feared that his fluxions would seem outlandish to those who still felt the authority of the ancient Greeks. Nevertheless, his fluxions themselves set a fashion in England that lasted long. Gauss suppressed his new ideas on non-euclidean geometry and on complex numbers, preferring to publish on subjects that appealed to his probable readers.

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* Note the fashionable term "stuff," almost as much in style as "play around with," so dear to "researchers" in science and pseudo-science. Here also are fashions that deserve investigation.

Although the existence of these tendencies in mathematics has often been noted, there has been very little study of any kind of mathematical fashions as such. I mean by this that although historians of mathematics have traced the development of the subject and have noted the succession of great contributions, almost no one has recorded for us how the volume of papers, distinguished and undistinguished, in a special field has increased, reached its peak, and then fallen off. Even Coolidge has not ventured any quantitative estimates in the paper to which I have referred. Perhaps I should except Smith and Ginsberg's Carus Monograph on the history of American mathematics, where articles by Americans in different fields are actually counted.

However, before I pass to some modest attempts at measurement of the popularity of various fields at various times, I wish to make some remarks about styles through the ages. The vogue of synthetic methods with the Greeks has often been noted. The seventeenth century has been called the age of discovery, the eighteenth that of applications, the nineteenth that of criticism, while the dominant spirit now is that of generalization (in the sense of E. H. Moore). Attempts to predict future trends have been none too successful. See, for example, the second chapter of Poincaré's *Science and Method*.

In any attempt to measure even roughly changes in emphasis on various parts of mathematics, and the growth of new disciplines in comparatively recent times, the chronicler must rely largely on journals devoted to reviews. The oldest, and most useful for this purpose, is the *Jahrbuch über die Fortschritte der Mathematik* whose first volume appeared seventy years ago. Smith and Ginsberg employed this periodical, in their history of American mathematics, to make a count of articles in various fields by American authors. Their purpose was to show how this work was distributed, or, as we might say in the terminology of this address, to demonstrate American styles before 1900. Inevitably I am drawn to this same source of information in my attempt to cover a broader field. I shall first record some comparisons between the contributions recorded in the volume for 1869–70 (the second volume, in which classifications had taken on a relatively fixed form), in the 1910 volume, and in that for 1930. In these comparisons, much depends on the scheme of classification, and upon the classifier. Even the *Fortschritte* changes its schemes and its judgments from year to year. Who shall say whether Abstract Spaces belongs under Analysis or Geometry, and where shall we place Mengenlehre? (After moving it from one compartment to another, the *Fortschritte* finally gave it a room of its own.) Such considerations make comparisons difficult. I hope that none of you will try to check on some of my counts; I shall do the best I can according to my lights, and not all that I say must be taken too seriously.

You will note in what follows that I make no attempt at estimates relative to applied mathematics. Here the very first question,—is a given paper to be classified as mathematical, physical, or astronomical?—is too often unanswerable. I have given it up, and shall restrict myself to what is often termed pure mathematics.

About ten per cent of the part of the 1869–70 volume relating to pure mathematics was devoted to History and Philosophy. We may compare this with fifteen per cent in 1910 (but now including Mengenlehre and Logic), and fourteen per cent in 1930, including Pädagogik. In view of these later additions to the original classification, we may conclude that these subjects have neither gained much relatively, nor lost much. Of course, there has been a considerable shifting of interest within these fields.

Arithmetic (Theory of Numbers) and Algebra seem to have gained somewhat from 1869–70 to 1910, and to have lost from 1910 to 1930. Perhaps this needs further investigation. At any rate my figures, after adjustments made to reduce classifications to something like the same basis, are:

	1869–70	1910	1930
Arithmetic	6%	8%	$5\frac{1}{2}$%
Algebra	11%	11%	7 %

These figures are based in part upon count of articles and in part on number of pages of reviews as tending to show the relative importance of articles. My conclusions must therefore be taken as strictly tentative. However, as we shall presently see, the growing popularity of Analysis has inevitably decreased the vogue of other fields, and the influence of this factor is probably recorded in the figures above.

There can be no question as to the growth of interest in Analysis. My methods of computation show that the percentages of contributions in pure mathematics which would be classed under Analysis according to the scheme of 1930 are 26 for 1869–70, 38 for 1910, and over 46 for 1930. I doubt if the last six years have shown any change in this trend. Darboux was right in 1904 when he pointed out that Analysis had usurped the place of Geometry as the center of mathematical interest, but he would hardly have ventured to predict that in thirty more years the trend in this direction would be still stronger.

Geometry, of course, is another story. In 1869–70, its measure of popularity is indicated by the fact that 47% of all the recorded papers in pure mathematics were classified under geometry. In 1910 this had shrunk (on an adjusted basis of reckoning) to 38%, and in 1930 to 26%; this in spite of the fact that some branches of geometry have become decidedly fashionable.

It is perhaps not without interest to note that the German *Encyclopedia of Mathematics*, with twenty-three volumes of not far from the same size, devotes two to Arithmetic and Algebra, five to Analysis, six to Geometry, and ten to Applied Mathematics. Unless, indeed, geometers are more long-winded than analysts, this may show that Geometry, owing to its earlier supremacy, still has a greater literature than Analysis.

As we compare 1870 with 1930, we note a marked increase of sophistication. For example seven per cent of the pure mathematics of the former year was elementary geometry, as contrasted with two per cent in 1930. Something of the same sort has occurred in other fields also.

The rise of new subjects is recorded by corresponding changes in the scheme

of classification. Thus in Algebra we note the addition of group theory and abstract algebra. In Analysis, the classifier has so far modified his structure that the modernistic edifice of 1930 is a long cry from its predecessor of 1870. Separate sections or chapters have been added for newer theories of real functions, trigonometric and related series, conformal representation, integral equations, functions of infinitely many variables, functional analysis, continuous groups, differential and integral invariants, boundary value problems and associated developments, difference equations and the analytic theory of continued fractions. In Geometry we find new developments also. There is a rush to topology and the newer varieties of differential geometry. Vector and tensor calculus have a chapter to themselves.

A mere catalogue of new names, however, gives a very inadequate idea of the changes in mathematical fashions in the last seventy years. The titles of the articles themselves convey a further impression of advanced technique, greater power, increased specialization, and at the same time a trend toward generalization in the sense of abstract developments. We read of abstract groups, abstract algebra, abstract spaces. In 1869–70, analysis situs listed three papers; in 1930, there were over one hundred and twenty papers in topology. Differential geometry, not classified as such in 1870, included 10% of all pure mathematics produced in 1930; half of this related to spaces of more than three dimensions and general spaces. One notes also the growth of mathematical logic and of statistics.

When were these new fashions set? The more I have pondered on this, the more I have been struck with the extraordinary fertility in new mathematical ideas that characterized the years from about 1900 to 1905. I am, of course, not the first to have made this remark. Let us begin with Hilbert's *Festschrift* on the foundations of geometry, which appeared in 1899. Systems of postulates had been considered before, but Hilbert's publication concentrated attention on the subject. A large part of the impetus toward modern abstract mathematics can be traced to this paper. Mathematicians of all degrees of proficiency began to turn out postulates for this, that, and the other mathematical system. It was a real "gold rush." Hilbert also added new interest to the study of the calculus of variations at about this time, and Bolza's book appeared in 1904.

In 1903 appeared the great paper of Fredholm on integral equations. In a later part of my address I shall give some details regarding the subsequent rush of workers to this field, and the rise and development of the subject, as affording an especially good example of mathematical fashions. Again it is interesting to note how this movement led E. H. Moore toward his general analysis, and others on to the consideration of abstract spaces. Fréchet's thesis also falls late in this period, contributing another powerful impulse toward the study of function spaces.

Lebesgue's first expositions of his new theory of integration come within this period.

Another epoch-making paper was that of Einstein, setting forth his first

theory of relativity, in 1905. Though published as a paper in physics, it had profound mathematical consequences. At that time it was scarcely recognized that there would be any connection between relativity and a paper that had been published in the *Mathematische Annalen*, in 1901, by Ricci and Levi-Civita,—their now famous article on the absolute calculus. Modern differential geometry has been recast by these influences.

Wilczynski's book on projective differential geometry was published in 1906, —we may stretch our period a little to include this, since it had a considerable subsequent influence, particularly among American mathematicians.

To this period we may trace some beginnings of modern topology, and significant developments in the theory of groups. Boundary value problems took on a new interest, and pioneering work was done in the theory of analytic functions of several complex variables. Finally, this was the period when the great *Encyclopedia of Mathematics* was launched, an almost superhuman undertaking, but not too great for such an era of inspiration. Some of us can say that we had our training in that period. I wonder how many of us realized in what sort of an age we were living.

Your patience would be exhausted if I attempted to give a detailed account of the rise, development, and in some cases the fall, of activity in all the fields of mathematics, a task far beyond my ability in any event. You will surely be satisfied with a few instances. As I have noted before, counts of articles listed in certain fields must be taken with more than a grain of salt, since so much depends on the classifier and the counter.

I have already referred to Coolidge's account of the rise and fall of (synthetic) projective geometry. As is the fashion with authors of similar chronicles, he first works out an impressive pedigree for his subject, going back, of course, to the ancient Greeks. Then comes the Messiah of this new faith in the person of Poncelet. After him appear the apostles and expounders, such as Chasles, Steiner, and von Staudt. The subject becomes, and remains for a time, very popular; then it must either develop or decay. Coolidge believes that the latter conclusion prevailed because the methods of analytic geometry are more simple and powerful. I have attempted some counts of papers to find how fashionable this subject became, and how low it has fallen. The trouble again is to be sure of one's classification of a paper. I venture to estimate that in 1870 about four per cent of all contributions in pure mathematics were in this field, as compared with two per cent in 1910. By 1930 it had practically disappeared as a subject for research, if one may judge by the *Fortschritte*.

I have also been interested in tracing the rise and fall of elliptic and abelian functions. The difficulties here are two-fold. In the first place, the classifier may or may not include the general theory of algebraic functions and their integrals, and secondly, much of the literature of these functions is devoted to applications and may be classed elsewhere. After various forerunners, the theory of these functions took shape under Abel and Jacobi. In the hands of Riemann, Weierstrass, and others it profoundly influenced, and was influenced by, the general

theory of functions of a complex variable. Hardly less important were its contributions to geometry, and even to number theory. Yet here also we seem to have a subject that is becoming less fashionable. I suspect that many of our young analysts have only a casual acquaintance with elliptic functions. To descend to percentages, I find that this field included a little over three per cent of all contributions in pure mathematics in 1870, about two and two thirds per cent in 1910, and only about one per cent in 1930.

A more modern instance is afforded by summation methods for infinite series. In an article in the 1932 number of this MONTHLY, C. N. Moore has given an account of the development of this essentially modern subject whose recent history begins with Frobenius (1880) and goes on with Cesàro (1890), Borel (1899), Fejér, and others. I have made no counts here, but hazard the guess that the climax was reached before 1925, perhaps even by 1920, and that there has been a marked falling off since then.

These examples have not been chosen quite at random; they all exhibit mathematical fashions that have waxed and then waned. The same is in one sense true of my concluding illustration, but here (and the same remark holds for the theory of summation also) we shall see that the subject is not dying but is rather passing on into new forms. I refer to the history of integral equations, which is so recent and so well documented that it lends itself admirably to my methods of measurement, crude as they are. The reader who may wish other points of view can consult Bôcher's Cambridge tract, Bateman's report to the British Association (1910), or the *Encyclopedia* article by Hellinger and Toeplitz.

One finds a highly respectable pedigree in Bôcher's tract, with Fourier, Abel, Liouville, and Neumann as progenitors. Then comes the forerunner, Volterra, and next Fredholm with his great paper of 1903. Hilbert is the St. Paul, the expounder and missionary of the new movement, and zealous followers are Schmidt and Hellinger and Toeplitz. More and more sheep come into the fold until the names in the *Encyclopedia* article read like a directory of analysts.

The rapid growth of this new subject is shown by the year-by-year increase in the number of articles listed in the *Fortschritte*. In 1905 there were three papers (two by Hilbert), next year twelve, and the following year twenty, increasing to thirty-four, forty-two, and in 1910 to fifty-seven, which is perhaps a high-water mark in papers chiefly concerned with the restricted subject rather than its developments. These fifty-seven papers were reviewed in about one seventh of all the space given to analysis, and about five per cent of that devoted to all of pure mathematics. For the next three years this output fell off very slightly (49, 49, 53), then came the war, with a reduction to from thirty to thirty-five papers annually.

The field had now broadened so that comparisons become more difficult. Before 1906 the classifier had put the papers under the heading "Funktionentheorie. Allgemeines." For the next ten years he listed them under "Funktionalgleichungen." In 1916 he used the caption, "Integralgleichungen und verwandte

Funktionalgleichungen." Three years later he included orthogonal functions. In 1923 this last part of the title was dropped; in 1927 our subject was listed under "Funktionalanalysis"; in 1935 there was a separate chapter entitled "Integralgleichungen und Funktionalanalysis." This succession of titles suggests how the field has developed and made contacts with other fields, but these changes make comparisons more difficult. In 1921–22 about three per cent of work in pure mathematics was here listed. Five years later it had fallen to a little over two per cent. Then the classifier added functional analysis, and the percentage jumped to five. In 1935 it was over six per cent, but with the emphasis now on functional analysis.

Here the picture is of a fashion which has brought in its train new ones that are still very much alive and promise to remain so for some time to come.

Let me conclude with another figure of speech, perhaps unfortunately borrowed from military science, but in some ways more apt than the comparison I have made to a gold rush. I think of mathematicians as behind lines that separate them from the territory not yet acquired. Some genius marks out a great salient to be added to the conquered regions. Other mathematicians occupy it and push it out still further. Finally it merges with other salients, which tend to disappear as such, and new wedges are extended from the consolidated territory into regions previously unconquered. These salients correspond to what I have called fashions. They are the landmarks of progress.

THE CLASSIFICATION OF CORRELATIONS IN THE PLANE

By E. T. BROWNE and C. A. DENSON, University of North Carolina

1. *Introduction.* A correlation in a plane is a projective correspondence between the points and the lines of the plane such that all the lines through a point correspond to all the points on a line. Analytically, a correlation in a plane is defined by equations of the type

$$\rho u_i = \sum a_{ij}x_j, \quad (i = 1, 2, 3; \ |A| = |a_{ij}| \neq 0), \tag{1}$$

where (x_1, x_2, x_3) are the homogeneous coördinates of a point x, (u_1, u_2, u_3) are the coördinates of the corresponding line u, and the frame of reference is thought of as being so chosen that the condition for united position of u and x is

$$\sum u_i x_i = 0.$$

On solving the equations (1) for the x's in terms of the u's, we obtain

$$\sigma x_i = \sum A_{ji}u_j, \qquad (i = 1, 2, 3), \tag{1'}$$

where A_{ij} is the cofactor of a_{ij} in $|A|$. The equations (1′) also determine the correlation.

Suppose now that the line u turns about a fixed point $y = (y_1, y_2, y_3)$, so that we have from (1)

$$0 = \rho\sum u_i y_i = \sum a_{ij}y_i x_j.$$

From this it is clear that the corresponding point x will move along the line v, where

$$\tau v_j = \sum a_{ij} y_i,$$

or

$$(2) \qquad \tau v_i = \sum a_{ji} y_j.$$

This line v will coincide with the line which corresponds to y under (1) if, and only if, there exists a number $\lambda \neq 0$ such that $\sum a_{ij} y_j = \lambda \sum a_{ji} y_j$, or

$$(3) \qquad \sum (a_{ij} - \lambda a_{ji}) y_j = 0, \qquad (i = 1, 2, 3).$$

In order that such a point y may arise it is necessary and sufficient that λ be so chosen that the determinant $D(\lambda) = |A - \lambda A'|$ vanish; i.e., that λ be a root of the equation

$$(4) \qquad D(\lambda) = |A - \lambda A'| = \begin{vmatrix} a_{11} - \lambda a_{11}, & \cdots, & a_{13} - \lambda a_{31} \\ \cdots & \cdots & \cdots \\ a_{31} - \lambda a_{13}, & \cdots, & a_{33} - \lambda a_{33} \end{vmatrix} = 0.$$

Similarly, as a point x moves along a line v, the line u corresponding to x will turn about the point y corresponding to v under (1′) if, and only if, there exists a number $\lambda \neq 0$ such that

$$(5) \qquad \sum (A_{ij} - \lambda A_{ji}) v_j = 0, \qquad (i = 1, 2, 3);$$

that is, λ must be a root of the equation

$$(6) \qquad \bar{D}(\lambda) = |(A^{-1})' - \lambda A^{-1}| = 0,$$

where $(A^{-1})'$ is the transpose of the inverse of A.

The equations (4) and (6) are equivalent. For from the matricial relation

$$(7) \qquad A - \lambda A' = A'[(A^{-1})' - \lambda A^{-1}]A,$$

we have on taking determinants

$$D(\lambda) = |A|^2 \bar{D}(\lambda).$$

Moreover, it is obvious from (7) that the matrices $A - \lambda A'$ and $(A^{-1})' - \lambda A^{-1}$ are of the same rank, so that there are just as many linearly independent lines v satisfying (5) as there are points y satisfying (3).

Suppose now that y is a point satisfying (3) and that v is the line corresponding to y under (1); i.e.,

$$(3) \qquad \sum a_{ij} y_j = \lambda \sum a_{ji} y_j,$$

$$(8) \qquad \sum a_{ij} y_j = \rho v_i.$$

On substituting ρv_i for the left member of (3), multiplying through by $A_{\tau i}$ and summing as to i, we get

$$\rho \sum A_{\tau i} v_i = \lambda \sum \sum A_{\tau i} a_{ji} y_i = \lambda |A| y_\tau.$$

But from (8) on multiplying through by $A_{i\tau}$ and summing as to i, we have

$$|A|\, y_\tau = \rho \sum A_{i\tau} v_i,$$

whence

$$\sum (A_{\tau i} - \lambda A_{i\tau}) v_i = 0.$$

That is, *if y is a point satisfying* (3), *its corresponding line v satisfies* (5) *with the same value of* λ. Such a pair (y, v) we call a *double pair* of the correlation.*

2. *Projectively equivalent correlations.* Suppose now that the points and lines of the plane be referred to a different frame of reference. The coördinates x and the coördinates x' of a typical point referred to the two different frames of reference are connected by relations of the type

$$\sigma x_i = \sum p_{ij} x_j' , \qquad |P| = |p_{ij}| \neq 0. \tag{9}$$

Also, the relations between the coördinates u and the coördinates u' of a line referred to the two frames are

$$\sigma' u_i = \sum P_{ij} u_j' , \tag{10}$$

where P_{ij} is the cofactor of p_{ij} in the determinant $|P|$. On eliminating the u's and the x's from (1), (9), and (10) we obtain

$$\rho' u_i' = \sum q_{ij} x_j' , \tag{11}$$

where the matrix Q is equal to $P'AP$.

The correlations (1) and (11) are projectively equivalent; and, conversely, a correlation (1) of matrix A and a correlation (11) of matrix Q will be projectively equivalent if there exists a non-singular matrix P such that $Q = P'AP$, i.e., if A and Q are *congruent.*

3. *Purpose of this paper.* Kronecker has shown that two non-singular square matrices A and Q of the same order will be congruent if, and only if, the two matrices $A - \lambda A'$, $Q - \lambda Q'$ have the same elementary divisors.** The problem of a projective classification of correlations therefore reduces algebraically to a problem in the theory of elementary divisors. However, many students of projective geometry are unfamiliar with that theory, so that various writers† have

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* It is clear that the equations (2) also determine a correlation which has as its matrix the transpose matrix A' of A, and which will therefore be identical with (1) if, and only if, A is symmetric. We call (2) the correlation *induced* by (1). By reference to (3) and (5) it is easy to see that a double pair (y, v) of (1) will be a double pair of (2) also. Moreover, by the symmetry of the formulas and equations with respect to A and A', it will follow that the two conics, shown subsequently to be associated with (1), will coincide with those associated with (2), and that the properties of the two correlations are identical. We shall therefore confine our attention to (1) only.

** Muth, Theorie und Anwendung der Elementartheiler, Leipzig, 1899, p. 143; also L. Kronecker, Über die congruenten Transformationen der bilinearen Formen, Werke, vol. I, pp. 424–483.

† Cf., for example, F. S. Woods, Higher Geometry, Boston, 1922, pp. 88–94.

been led to set up classifications without invoking the theory of elementary divisors at all. In the opinion of the authors of this paper, there is a logical criticism to be levelled at some of these classifications in the sense that those writers define the correlation algebraically and then immediately proceed to classify them geometrically, bringing into play certain suppositions concerning the mutual relationships of conics, while it is not at all obvious that such relationships need arise. It is the purpose of this paper to give an algebraic classification of correlations without making use of the theory of elementary divisors. The discussion finally leads to criteria whereby the different types of correlations can be distinguished from one another by means of certain simple invariants.

4. *The conics C and K.* A point x will lie on the line u corresponding to it under (1) if, and only if,

$$\sum u_i x_i = \sum a_{ij} x_i x_j = 0, \tag{12}$$

i.e., if, and only if, x lie on a conic C whose equation is given by (12). Since the matrix of a conic is taken in symmetric form and since A is not in general symmetric, we take as the matrix of C the matrix $\frac{1}{2}(A+A')$. Similarly, using (1′) it will follow that the line u will pass through the point x corresponding to it if, and only if, u is tangent to the conic K

$$\sum A_{ij} u_i u_j = 0,$$

whose matrix may be taken to be $\frac{1}{2}[A^{-1}+(A^{-1})']$.

The conics C and K do not in general coincide. However, they always have the same rank as follows directly from (7) with $\lambda = -1$.

THEOREM I. *A point x and the line u corresponding to it under* (1) *will be in united position if, and only if, x is a point of a conic C of matrix $\frac{1}{2}(A+A')$ and u is a line of a conic K of matrix $\frac{1}{2}[A^{-1}+(A^{-1})']$. The conics C and K are of the same rank.*

5. *Canonical form of a non-singular correlation.* Let A be the matrix of the correlation (1) and write

$$S = \tfrac{1}{2}(A + A'), \qquad T = \tfrac{1}{2}(A - A'),$$

so that $A = S+T$. Obviously S is a symmetric and T a skew-symmetric matrix, the former being the matrix of C. It is well known that there exists a non-singular matrix P such that $P'SP = N$, where N consists of zeros except in the diagonal.* The elements in the diagonal of N are the coefficients a, b, c in the canonical equation of C:

$$ax_1^2 + bx_2^2 + cx_3^2 = 0.$$

* M. Bôcher, Introduction to Higher Algebra, New York, 1935, pp. 129–132; cf. also, F. S. Woods, loc. cit., pp. 65–66.

Since T, and therefore $P'TP$, is skew-symmetric, then

$$P'AP = P'SP + P'TP$$

is of the form

$$P'AP = \begin{pmatrix} a & h & -g \\ -h & b & f \\ g & -f & c \end{pmatrix}, \tag{13}$$

where we suppose that

$$abc + af^2 + bg^2 + ch^2 \neq 0. \tag{14}$$

Hence, (13) is a canonical form to which, by proper choice of the frame of reference, the matrix of any non-singular correlation can be reduced. With A in the form (13), the matrix $A - \lambda A'$ assumes the simple expression:

$$A - \lambda A' = \begin{pmatrix} a(1-\lambda) & h(1+\lambda) & -g(1+\lambda) \\ -h(1+\lambda) & b(1-\lambda) & f(1+\lambda) \\ g(1+\lambda) & -f(1+\lambda) & c(1-\lambda) \end{pmatrix} \tag{15}$$

so that

$$D(\lambda) = |A - \lambda A'| = (1-\lambda)[abc(1-\lambda)^2 + (af^2 + bg^2 + ch^2)(1+\lambda)^2]. \tag{16}$$

From (16) we can draw at once the following theorem:

THEOREM II. *The equation $D(\lambda) = 0$ is a reciprocal equation, having at least one root at $\lambda = 1$. Moreover, $\lambda = -1$ is a root of $D(\lambda) = 0$ if, and only if, $abc = 0$, i.e., the conics C and K are singular.*

6. *The classification.* The classification of correlations is now easy. In fact, it is clear that in so far as concerns the roots of $D(\lambda) = 0$ and the ranks of the auxiliary matrices $A - \lambda A'$, the following five cases and these only can arise.

First, suppose that $D(\lambda) = 0$ has the roots 1, 1, 1. Then $af^2 + bg^2 + ch^2 = 0$, but $abc \neq 0$ since $|A| \neq 0$. The conics C and K are proper. However, there is a separation of cases according as the matrix $A - A'$ is of rank 0 or 2, i.e., according as $(f, g, h) = (0, 0, 0)$ or $\neq (0, 0, 0)$.

Case I. $D(\lambda) = 0$ has the roots 1, 1, 1; $abc \neq 0$; $f = g = h = 0$.

Case II. $D(\lambda) = 0$ has the roots 1, 1, 1; $abc \neq 0$; $af^2 + bg^2 + ch^2 = 0$, but $(f, g, h) \neq (0, 0, 0)$.

Second, suppose that $D(\lambda) = 0$ has the roots 1, -1, -1. From (16) it then follows that $abc = 0$, but $(a, b, c) \neq (0, 0, 0)$ since $|A| \neq 0$. There is therefore a separation of cases according as one or two of the quantities a, b, c are zero.

Case III. $D(\lambda) = 0$ has the roots 1, -1, -1; $c = 0$, $ab \neq 0$, $af^2 + bg^2 \neq 0$. The conics C and K are of rank 2, i.e., C degenerates into a pair of lines while K reduces to a pair of points.

Case IV. $D(\lambda) = 0$ has the roots 1, -1, -1; $b = c = 0$, $af^2 \neq 0$. C and K are

of rank 1, i.e., the former reduces to a repeated line while the latter reduces to a repeated point. In both cases III and IV the matrix $A - A'$ is of rank 2.

Case V. $D(\lambda) = 0$ has the roots $1, \alpha, 1/\alpha (\alpha \neq \pm 1)$. Then $abc \neq 0$, $af^2 + bg^2 + ch^2 \neq 0$, $|A| \neq 0$. Here C and K are proper conics while each of the matrices $A - A'$, $A - \alpha A'$, $A - 1/\alpha(A')$ is easily shown to be of rank 2.

7. *Condition that C and K coincide*. In order to study the relationship between the conics C and K we might take A in the form (13). It would then follow that

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} bc + f^2 & fg - ch & fh + bg \\ fg + ch & ac + g^2 & gh - af \\ fh - bg & gh + af & ab + h^2 \end{pmatrix}.$$

The matrix $\frac{1}{2}[A^{-1} + (A^{-1})']$ of K in line coördinates is then a multiple of

$$K = \begin{pmatrix} bc + f^2 & fg & fh \\ fg & ac + g^2 & gh \\ fh & gh & ab + h^2 \end{pmatrix}.$$

Since for C non-singular the line equation is

$$C: \quad bcu_1^2 + acu_2^2 + abu_3^2 = 0,$$

it is obvious that C will coincide with K if and only if $f = g = h = 0$, i.e., if and only if A is symmetric. We have then the so-called *polarity*. (Cf. section 9 below).

However, instead of proceeding further with this actual calculation, it seems more elegant to obtain our results by first establishing more general theorems. We content ourselves here then merely with stating the following theorem:

THEOREM III. *The conics C and K are coincident if and only if the matrix A of the correlation is symmetric.*

8. *Theorems on correlations*. Let y be a point of a double pair (y, v). Then from (3) we have on multiplying through by y_i and summing as to i,

$$\sum a_{ij}y_iy_j - \lambda \sum a_{ji}y_iy_j = 0.$$

Hence, if $\lambda \neq 1$, $\sum a_{ij}y_iy_j = 0$, so that y is a point of C. In a similar manner, it follows from (5) that v is a line of K. These results, in connection with Theorem I give

THEOREM IV. *A point y and a line v of a double pair (y, v) arising from a root $\lambda \neq 1$ of $D(\lambda) = 0$ are always in united position; y is therefore a point of C and v is a line of K.*

As in section 5, let us denote by S the symmetric matrix $\frac{1}{2}(A + A')$ and by T the skew-symmetric matrix $\frac{1}{2}(A - A')$. From the fact that $A = S + T$, $A' = S - T$, we have

$$a_{ij} = s_{ij} + t_{ij}, \qquad a_{ji} = s_{ij} - t_{ij}.$$

If now y is a point of a double pair, (3) may be written

$$\sum [(s_{ij} + t_{ij}) - \lambda(s_{ij} - t_{ij})]y_j = 0,$$

or

$$\sum [(1 - \lambda)s_{ij} + (1 + \lambda)t_{ij}]y_j = 0. \tag{17}$$

This last equation with $\lambda = -1$ gives

$$\sum s_{ij}y_j = 0.$$

Since the equation of the conic C can be written $\sum s_{ij}x_ix_j = 0$, it follows from this that y is a *singular* point of the *degenerate* conic C.

THEOREM V. *If* -1 *is a root of the equation* $D(\lambda) = 0$, *the conic* C (K) *is degenerate and the point* y (*line* v) *of the double pair* (y, v) *arising from the root* -1 *is a singular point* (*line*) *of the conic.*

Again let y be a point of a double pair arising from the root λ and let v be the line corresponding to y under the correlation, so that we have

$$\sum a_{ij}y_j = \rho v_i.$$

This last may be written

$$\sum s_{ij}y_j + \sum t_{ij}y_j = \rho v_i.$$

If now this equation be multiplied through by $1+\lambda$ and (17) be subtracted from the resulting equation, we have

$$2\lambda \sum s_{ij}y_j = \rho(1 + \lambda)v_i.$$

For $\lambda \neq -1$, this yields:

THEOREM VI. *If* y *and* v *are point and line of a double pair* (y, v) *arising from the root* $\lambda \neq -1$ *of* $D(\lambda) = 0$, *then* y *and* v *are pole and polar as to the conics* C *and* K. *In particular, if* y *is a point of* C, *and therefore* v *a line of* K, *then* v *is a common tangent to* C *and* K *at the point* y.

As a consequence of Theorems IV and VI, we may state also

THEOREM VII. *If* y *and* v *are point and line of a double pair* (y, v) *arising from a root* $\lambda \neq \pm 1$ *of* $D(\lambda) = 0$, *then* v *is a common tangent to* C *and* K *at* y.

9. *Mutual relationship of the conics* C *and* K *and of the double pairs in the various cases.*

Case I. Here $f = g = h = 0$. Hence by Theorem III the conics C and K coincide. Since the matrix $A - A'$ is zero, every point y in the plane is a point of a double pair (y, v) the corresponding line v being by Theorem VI the polar of y as to C. This is the so-called polarity.

Case II. Since $A-A'$ is of rank 2 there is a single double pair (y, v) the coördinates of which, if we employ the canonical form (13) of A, are found to be

$$(18) \qquad y: \quad (f, g, h); \qquad v: \quad (af, bg, ch).$$

Since, in this case, $af^2+bg^2+ch^2=0$, y is a point of C, so that by Theorem VI v is a common tangent to C and K at y.

Case III. Since $A-A'$ is of rank 2, there is a single double pair (y, v) corresponding to the root 1, the coördinates of which are given by (18) with $c=0$. The matrix $A+A'$ also is of rank 2. Hence C degenerates into two lines l_1, l_2, while K degenerates into two points P_1, P_2. Moreover, corresponding to the root -1 there is a single double pair (x, u), x being the intersection of the lines l_1, l_2, and u being the line P_1P_2. By Theorem IV u passes through x, and by use of the canonical form of A, it is easy to show that y lies on u and that the pair x, y separate the pair P_1, P_2 harmonically. Moreover, v passes through x and the pair u, v separates the pair l_1, l_2 harmonically.

Case IV. Here C reduces to a repeated line l while K reduces to a repeated point P, which, as shown by the canonical form, does not lie on l. The single double pair corresponding to the root 1 is the pair (P, l). Since $A+A'$ is of rank 1, there corresponds to the root -1 an entire line l of points x and a pencil P of lines u of double pairs (x, u), where, by Theorem IV, u is precisely the line Px.

Case V. C and K are distinct proper conics. Corresponding to each of the roots α, $1/\alpha$ there is a single double pair (y, v), (y', v'), respectively, and by Theorem VII, v is a common tangent to C and K at y, while v' is a common tangent to the conics at y'. The single double pair (x, u) arising from the root 1 is such that x is the point of intersection of v and v', while u is the polar of x as to C or K, i.e., the chord of contact yy'.

10. *Classification of correlations by means of invariants.* The five cases above can be characterized completely by means of invariants. If as in section 5 we write $S=\frac{1}{2}(A+A')$, $T=\frac{1}{2}(A-A')$, it is clear that under the transformation $P'AP$ the quantities $|A|$ and $|S|$ are algebraic invariants while the ranks of S and T are arithmetic invariants. Evidently in the canonical form the expression $af^2+bg^2+ch^2$ is the invariant $|A|-|S|$. We have therefore in the respective cases, it being always understood that $|A|\neq 0$:

Case I. $T=0$, i.e., the matrix A is symmetric.
Case II. S of rank 3, $T\neq 0$, $|A|=|S|$.
Case III. S of rank 2, $T\neq 0$.
Case IV. S of rank 1, $T\neq 0$.
Case V. S of rank 3, $T\neq 0$, $|A|\neq|S|$.

Example. If $A=\begin{pmatrix}1 & 1 & 2\\ 3 & -1 & -1\\ 0 & 1 & -6\end{pmatrix}$ then $S=\begin{pmatrix}1 & 2 & 1\\ 2 & -1 & 0\\ 1 & 0 & -6\end{pmatrix}$.

Here $T\neq 0$, and $|S|=|A|=31$. The correlation $u=A(x)$ comes under case II.

A NEW SIMPLIFICATION OF KRONECKER'S METHOD OF FACTORIZATION OF POLYNOMIALS

By B. A. HAUSMANN, University of Detroit

The first systematic procedure for the factorization of polynomials with rational integral coefficients in a finite number of steps was given by Kronecker and is known by his name. The method briefly is the following. Let $f(x)$ be a polynomial of degree n whose factors we wish to find. If $f(x)$ has a factor, then one factor $g(x)$ must be of degree $\leqq n/2$. Let s be the largest integer $\leqq n/2$, and form the numbers $f(d_0)$, $f(d_1)$, $\cdots$, $f(d_s)$ by substituting arbitrary integers $d_0, d_1, \cdots, d_s$ in $f(x)$. If $f(x)$ has a factor $g(x)$, we see that $f(d_0)$ must be divisible by $g(d_0)$, $f(d_1)$ by $g(d_1)$, etc. Since each $f(d_i)$ has only a finite number of divisors, there are only a finite number of possible values for each $g(d_i)$. Hence there are only a finite number of possible combinations of these possible values. To each combination corresponds a unique polynomial $k(x)$ which can be found by the Lagrange or the Newton interpolation formula. After a $k(x)$ has been calculated in this way, we determine whether or not it is a factor of $f(x)$ by ordinary division.

The method clearly may be exceedingly long and tedious, and many improvements on the method have been found which considerably shorten the work. However, in all of these improvements a certain number of polynomials $g(x)$ have to be calculated and tested by division. The present improved method eliminates this work, for it gives a criterion by which we can tell from the numbers $g(d_0)$, $g(d_1)$, $\cdots$, $g(d_s)$ themselves whether or not the corresponding polynomial $g(x)$ is a divisor of $f(x)$ without calculating $g(x)$.

We shall need the following property of polynomials. Let $f(x)$ be a polynomial of degree n with rational, integral coefficients. If we substitute any $n+1$ integers separated by equal intervals for x in $f(x)$, which, for the sake of definiteness, we shall suppose to be the integers $1, 2, \cdots, n+1$, we get $n+1$ numbers $a_0, a_1, \cdots, a_n$. If we now form the difference table for these numbers a_i, we find that the difference of the nth order is a constant not equal to zero. If we extend the difference table to the right, we obtain further values $a_{n+1}, a_{n+2}, \cdots$ for the set a_i which are precisely the values of $f(x)$ for $x=n+2$, $n+3, \cdots$.

Consider the special class of polynomials $f(x)$ of degree n with rational integral coefficient which have the property that *all their real zeros are negative and all their complex zeros have a negative real part.* Such polynomials obviously can have only positive coefficients none of which is zero. If they are factorable, each factor is a polynomial of the same kind and has the same properties. Let $f(x)$ be such a polynomial and suppose it factorable. Then $f(x)=g_1(x)\cdot g_2(x)$ where $g_1(x)$ is of degree r and $g_2(x)$ is of degree s. Substitute for x the numbers $1, 2, \cdots, n+1$. We obtain the set of values

$$[a_0, \cdots, a_n] = [b_0, \cdots, b_n]\cdot[c_0, \cdots, c_n]$$

where $a_i=f(i+1)$, $b_i=g_1(i+1)$, $c_i=g_2(i+1)$, $i=0, 1, \cdots, n$. The numbers b_i

and c_i have the following properties:

1) $a_i = b_i \cdot c_i$.
2) $b_0 \geqq 2$ and $c_0 \geqq 2$. [For the smallest possible factors of $f(x)$ are $(x+1)(x+1)$ which make b_0 and c_0 equal to 2.]
3) $b_0 < b_1 < \cdots < b_n$ and $c_0 < c_1 < \cdots < c_n$. [For the coefficients are all positive.]
4) The differences of the rth and sth orders of the numbers b_i and c_i respectively are constants not equal to zero.
5) $$r + s = n.$$

These properties give us a criterion for factorability and a new method of factorization. We summarize it in the

THEOREM. *If* $[a_0, \cdots, a_n] = [b_0, \cdots, b_n] \cdot [c_0, \cdots, c_n]$, *where the* a_i *were obtained from some polynomial* $f(x)$ *of the type under consideration by substituting for* x *the* $n+1$ *integers* $1, 2, \cdots, n+1$, *and the* b_i *and* c_i *satisfy the relations*

1) $a_i = b_i \cdot c_i$;
2) $b_0 \geqq 2$ *and* $b_0 < b_1 < \cdots < b_n$;
3) $c_0 \geqq 2$ *and* $c_0 < c_1 < \cdots < c_n$;

then the b_i *define a polynomial* $g_1(x)$ *and the* c_i *define a polynomial* $g_2(x)$ *such that* $f(x) = g_1(x) \cdot g_2(x)$ *if and only if the difference table for the* b_i*'s has the difference of the* r*th order constant*; *the difference table for the* c_i*'s has the difference of the* s*th order constant*; *and* $r+s=n$.

Assuming that the differences of the rth and sth orders are constants, the theorem follows from the fact that there is only one polynomial of degree r which assumes the values $b_0, b_1, \cdots$ at the points $1, 2, \cdots$ and only one of degree s which assumes the values $c_0, c_1, \cdots$ at the same points. Since their product is unique and determines precisely the numbers $a_0, a_1, \cdots$ which in turn uniquely determine $f(x)$, they must be the factors of $f(x)$.

In this method then instead of calculating a polynomial $g(x)$ from a set of factors b_i of the a_i, we form the difference tables of these numbers b_i and of the corresponding numbers c_i. If the differences of the rth and sth orders respectively are constant and if $r+s=n$, we have found factors; if not, we try other values for the b_i and c_i. The calculation of these differences seems to be easier than the calculation of the corresponding polynomials and the testing of these polynomials by division.

The method as outlined above applies to a very restricted class of polynomials. We must still show how the above procedure applies to an arbitrary polynomial $P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$, where the a_i's are rational integers. It can be shown that

$$|P(z)| > 0 \qquad\qquad |z| \geqq 1 + m,$$

where $m = \text{maximum } |a_k/a_0|$, $(k=1, 2, \cdots, n)$. Hence $P(z)$ has no zeros outside of the circle $|z| = 1+m$. Therefore if we make the substitution $z = x+m+1$, $P(z)$ goes over into a polynomial $f(x)$ of the type considered above. The transformation obviously does not affect the reducibility or irreducibility of $P(z)$.

Practically we can apply the following procedure and eliminate a direct calculation of $f(x)$. Let $P(z)$ be a given polynomial of even degree n. (A similar procedure may be followed if n is odd.) Substitute for z the $n+1$ integers $-n/2, \cdots, 0, 1, \cdots, n/2$ and form the difference table for these values. Then extend the difference table to the right until a point is reached where we have obtained the equivalent of substituting for z the value $n+m+2$. Use the last $n+1$ values of the extended table for the numbers $a_0, a_1, \cdots, a_n$. They are the values for $x=1, 2, \cdots, n+1$ of the polynomial $f(x)$ obtained from $P(z)$ by substituting for z the value $z=x+m+1$; and $f(x)$ is of the type considered above. When we have found the factors of $f(x)$, we get those of $P(z)$ by the inverse substitution $x=z-m-1$.

We illustrate the method by solving a problem given by Van der Waerden. Find the factors of $f(z)=2z^5-z^3+3z^2+8z-4$. Substituting for z the numbers $-3, -2, -1, 0, 1, 2$, we obtain the values $-460, -64, -10, -4, 8, 80$. Forming the difference table for these numbers, extending it to the right until we have equivalently substituted for z the number 11, and retaining the last 6 numbers, we get for a_i the numbers

$$15{,}488 \qquad 33{,}470 \qquad 65{,}276 \qquad 117{,}680 \qquad 199{,}376 \qquad 371{,}218.$$

Factoring these numbers with the aid of a factor table, we have

$$2^7\cdot 11^2 \qquad 2\cdot 5\cdot 3347 \qquad 2^2\cdot 16319 \qquad 2^4\cdot 5\cdot 1471 \qquad 2^4\cdot 17\cdot 733 \qquad 2\cdot 23\cdot 6983.$$

Consider $a_2=2^2\cdot 16319=b_2\cdot c_2$. By our theorem $b_2\geqq 4$. Here therefore $b_2=4$. Since $b_1<b_2$ and $a_1=2\cdot 5\cdot 3347$, b_1 must be 2. But this is impossible since by the same theorem $b_1\geqq 3$. Therefore no factors of the a_i exist with the required properties, and $f(z)$ is irreducible.

It is sometimes of advantage to apply the following criterion in choosing factors b_i of the a_i. Since $f(x)$ has integral coefficients, its factors, if it has any, must, by a theorem of Gauss, have integral coefficients. Now any set of $r+1$ numbers $b_0, b_1, \cdots, b_r$ represents a polynomial $g(x)$ of degree r at the points $x=1, 2, \cdots, r+1$ if the difference of the rth order of these numbers is not zero. However $g(x)$ need not have integral coefficients. A necessary though not a sufficient condition that $g(x)$ have integral coefficients is

$$b_0 - C_r^1\cdot b_1 + C_r^2\cdot b_2 - \cdots b_r \equiv 0 \pmod{r!},$$

where the $C_r{}^i$ are the binomial coefficients, the signs are alternately plus and minus, and the modulus is factorial r. The proof follows from the fact that the left hand side of the above congruence if divided by factorial r is the coefficient of x^r in the Lagrange expansion of $g(x)$, where $g(x)$ assumes the values $b_0, b_1, \cdots$ for $x=1, 2, \cdots, r+1$.

Finally, the essential part of the above method can be used in conjunction with any other method of factorization which evaluates any polynomial $f(x)$ at equal intervals (e.g. the method of Runge, *Crelle*, vol. 99, p. 89) if we calculate the value of $f(x)$ at $n+1$ instead of merely at s intervals where s is the greatest integer $\leqq n/2$.

ON THE POLYNOMIAL DERIVATIVE CONSTANT FOR AN ELLIPSE

By W. E. SEWELL, Georgia School of Technology

Let $P_n(z)$ be a polynomial* of degree n in $z=x+iy$ and let $|P_n(z)| \leqq M$ on a set E, where M is a constant independent of n and z. The author has shown† that if the set E is bounded by an analytic Jordan curve C then $|P_n'(z)| \leqq K(C)Mn$, where $K(C)$ is a constant depending only on C. If C is the unit circle we know by a theorem of M. Riesz‡ that $K(C)=1$. Here we will show that *if C is an ellipse with semi-axes a and b, $a \geqq b$, then $K(C) \leqq 1/b$.*

Let us suppose that the ellipse C has its vertices in the points a and $-a$ on the real axis. Then for a point $z=x+iy$ on C we have $x=a\cos\theta$, $y=b\sin\theta$, and

$$z = x + iy = a\cos\theta + ib\sin\theta. \tag{1}$$

From (1) it follows§ that $P_n(z)$ for z on C is a trigonometric polynomial of order n in θ,

$$P_n(z) = \sum_{k=0}^{n} a_k z^k = \sum_{k=0}^{n} (A_k \cos k\theta + B_k \sin k\theta) = Q_n(\theta).$$

Thus

$$P_n'(z) = \frac{dP_n(z)}{dz} = \frac{dQ_n(\theta)}{d\theta}\,\frac{d\theta}{dz} = Q_n'(\theta)\,\frac{d\theta}{dz}. \tag{2}$$

If $|P_n(z)| \leqq M$ for z on C, then $|Q_n(\theta)| \leqq M$; and by a theorem of S. Bernstein‖ we know that

$$|Q_n'(\theta)| \leqq Mn. \tag{3}$$

Also from (1) we have

$$\frac{dz}{d\theta} = -a\sin\theta + ib\cos\theta,$$

whence

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* An expression of the form $a_nz^n+a_{n-1}z^{n-1}+\cdots+a_0$ is called a polynomial of degree n; we do not assume $a_n\neq 0$.

† W. E. Sewell, Generalized derivatives and approximation by polynomials, Transactions of the American Mathematical Society, vol. 41, 1937, pp. 84–123.

‡ M. Riesz, Eine trigonometrische Interpolationsformel und einige Ungleichungen für Polynome, Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 23, 1914, pp. 354–368; see Theorem I′, p. 357.

§ See e.g. Ch. J. de la Vallée Poussin, Leçons sur l'approximation des fonctions d'une variable réelle, Paris, 1919; see pp. 1–2.

‖ S. Bernstein, Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réelle, Paris, 1926; see p. 39.

$$\left| \frac{dz}{d\theta} \right|^2 = a^2 \sin^2 \theta + b^2 \cos^2 \theta = a^2 - (a^2 - b^2) \cos^2 \theta \geqq b^2.$$

Therefore $|dz/d\theta| \geqq b$, and hence

$$\left| \frac{d\theta}{dz} \right| \leqq \frac{1}{b}. \tag{4}$$

An application of inequalities (3) and (4) to equation (2) yields

$$| P_n'(z) | \leqq \frac{Mn}{b}, \qquad z \text{ on } C. \tag{5}$$

Now suppose γ is an arbitrary ellipse in the w-plane with semi-axes a and b, $a \geqq b$. By the transformation

$$z = e^{i\alpha} w + w_0, \qquad \alpha \text{ real}, \tag{6}$$

γ is carried into the ellipse C with vertices a and $-a$ on the real axis. Also if $p_n(w)$ is a polynomial of degree n in w, we see by (6) that $p_n(w) = P_n(z)$, where $P_n(z)$ is a polynomial of degree n in z. Furthermore $|p_n(w)| \leqq M$, w on γ, implies $|P_n(z)| \leqq M$, z on C. Thus (5) is valid for z on C. But

$$| p_n'(w) | = \left| \frac{dp_n(w)}{dw} \right| = \left| \frac{dP_n(z)}{dz} \frac{dz}{dw} \right| = | P_n'(z) |,$$

since it is clear from (6) that $|dz/dw| = 1$. Consequently

$$| p_n'(w) | \leqq \frac{Mn}{b}, \qquad w \text{ on } \gamma,$$

and we have established the following result:

THEOREM. *Let C be an ellipse in the z-plane with semi-axes a and b, $a \geqq b$. Let $P_n(z)$ be a polynomial of degree n in z and let $|P_n(z)| \leqq M$ for z on C. Then we have*

$$| P_n'(z) | \leqq \frac{Mn}{b}, \qquad z \text{ on } C.$$

An interesting special case is that of the ellipse with foci 1 and -1 on the axis of reals and with semi-axes

$$R + \frac{1}{R} \qquad \text{and} \qquad R - \frac{1}{R}, \qquad R > 1.$$

By the theorem we see that $|P_n(z)| \leqq M$ on C implies $|P_n'(z)| \leqq (R/R^2 - 1) Mn$ on C.

Finally, if $a = b = 1$ the ellipse C becomes the unit circle and for $P_n(z) = \alpha z^n$, $(|\alpha| = 1)$, the constant $K(C) = 1/b = 1$.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions, and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

NOTE CONCERNING SOME TRIGONOMETRICAL INEQUALITIES

By K. P. WILLIAMS, Indiana University

In a paper *On mathematical life in Hungary*,* Professor Radó in alluding to the Eötvös prize refers to the problem of proving that, if A, B, C are angles of a triangle, then

$$\sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C < \tfrac{1}{4},$$

and states that one contestant, who has later acquired international mathematical fame, showed that the true maximum of the product is $\frac{1}{8}$. The proof made use of the radii of the inscribed and circumscribed circles. Professor Radó gives a demonstration which does not employ such ideas, which are essentially foreign to the problem.

One would expect from considerations of symmetry that the maximum is attained for an equilateral triangle. A conclusive treatment of the problem by elementary means is, however, interesting. We shall give here a proof based on ideas that can be extended to related problems, including some where the triangle which is obtained is isosceles and the final result cannot be predicted at the start. The proofs in some cases are very simple, making the method of maximizing by the calculus quite superfluous; in other cases they are more involved and show the advantage of the procedure which the calculus affords.

(1) Let

$$P = \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C, \qquad A + B + C = 180°;$$

then

$$\begin{aligned} P &= \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \cos \tfrac{1}{2}(A+B) \\ &= \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B(\cos \tfrac{1}{2}A \cos \tfrac{1}{2}B - \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B) \\ &= \tfrac{1}{4}(\cos A + \cos B - \cos (A+B) - 1). \end{aligned}$$

Put

$$A = x + d, \qquad B = x - d;$$

then

$$\begin{aligned} \cos A + \cos B &= 2 \cos d \cos x = 2(1-k) \cos x, \qquad 0 \leqq k \leqq 1, \\ \cos (A+B) &= 2\cos^2 x - 1. \end{aligned}$$

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* This MONTHLY, vol. 39, 1932, p. 87.

Hence

$$P = \tfrac{1}{8} - \tfrac{1}{2}(\cos x - \tfrac{1}{2})^2 - \tfrac{1}{2}k \cos x.$$

Since $x = \frac{1}{2}(A+B) < 90°$, we have $\cos x > 0$, and it is seen that the maximum of P is $\frac{1}{8}$, this value being attained for $\cos x = \frac{1}{2}$, $k=0$, that is, when $x = 60°$, $d=0$. Therefore $A = B = C = 60°$.

(2) Let

$$P = \sin A \sin B \sin C, \qquad A + B + C = 180°.$$

We find readily

$$P = \tfrac{1}{4}(\sin 2A + \sin 2B - \sin 2(A + B)).$$

Letting

$$2A = x + d, \qquad 2B = x - d, \qquad \cos d = 1 - k,$$

we can write

$$\begin{aligned} P &= \tfrac{1}{2}(\sin x(1 - \cos x) - k \sin x) \\ &= \tfrac{1}{2}\left[((1 - \cos^2 x)(1 - \cos x)^2)^{1/2} - k \sin x\right. \\ &= \tfrac{1}{2}\left[(\tfrac{27}{16} - (\cos x + \tfrac{1}{2})^2(\cos^2 x - 3 \cos x + \tfrac{11}{4}))^{1/2} - k \sin x\right]. \end{aligned}$$

Since $\cos^2 x - 3 \cos x + 11/4$ is positive for all values of $\cos x$, we see that the maximum of P is $3\sqrt{3}/8$, this value being attained for $\cos x = -\frac{1}{2}$, $k=0$. We again have the equilateral triangle.

(3) Let us next consider

$$P = \sin A \sin B \sin \tfrac{1}{2}C, \qquad A + B + C = 180°.$$

Here we expect that P is a maximum for an isosceles triangle.

We have

$$\begin{aligned} P &= \tfrac{1}{2}(\cos (A - B) - \cos (A + B)) \cos \tfrac{1}{2}(A + B) \\ &= (\cos^2 \tfrac{1}{2}(A - B) - \cos^2 \tfrac{1}{2}(A + B)) \cos \tfrac{1}{2}(A + B). \end{aligned}$$

Let

$$A = x + d, \qquad B = x - d, \qquad \cos^2 d = 1 - k;$$

then

$$\begin{aligned} P &= (1 - k - \cos^2 x) \cos x \\ &= \frac{2}{3\sqrt{3}} - \left(\cos x + \frac{2}{\sqrt{3}}\right)\left(\cos x - \frac{1}{\sqrt{3}}\right)^2 - k \cos x; \end{aligned}$$

and it is seen that the maximum of P is $2\sqrt{3}/9$, attained for $\cos x = 1/\sqrt{3}$, $k=0$. The angles are $A = B = 54°44'$, $C = 70°32'$.

(4) Let us consider

$$P = \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin C, \qquad A + B + C = 180°.$$

In this case some difficulty appears, and one sees the advantage of the calculus. We have

$$P = (\cos \tfrac{1}{2}(A - B) - \cos \tfrac{1}{2}(A + B)) \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A + B).$$

Put

$$A = x + d, \qquad B = x - d, \qquad \cos x = y \geqq 0, \qquad \cos d = 1 - k.$$

Then

$$P = (1 - k - y)y\sqrt{1 - y^2},$$

and we can write

$$P = [(-(y^6 - 2y^5 + 2y^3 - y^2))^{1/2} - ky\sqrt{1 - y^2}].$$

We have the identity

$$\begin{aligned}(y^2 - 2ay + a^2)[y^4 + 2(a - 1)y^3 + a(3a - 4)y^2 \\ + 2(a - 1)^2(2a + 1)y + a(a - 1)^2(2a + 1)] \\ = y^6 - 2y^5 + 2y^3 - (3a^4 - 5a^3 + 3a)y^2 + a^3(a - 1)^2(2a + 1).\end{aligned}$$

The coefficient of y^2 will reduce to -1, which is the coefficient of y^2 in the last expression for P, if

$$3a^4 - 5a^3 + 3a - 1 = 0,$$

that is, if

$$(a^2 - 2a + 1)(3a^2 + a - 1) = 0,$$

which gives $a = 1$, $a = (-1 \pm \sqrt{13})/6$. The value $a = (-1 + \sqrt{13})/6$ is easily seen to be the one that should be used. We have $a = .4343 < 1/2$.

Let $c = 2(a-1)^2(2a+1)$, $h = a(a-1)^2(2a+1)$; then we have

$$P = [a^2h - (y - a)^2(y^4 + 2(a - 1)y^3 + a(3a - 4)y^2 + cy + h)]^{1/2} - ky\sqrt{1 - y^2}.$$

We find

$$\sqrt{a^2h} = a(1 - a)\sqrt{2a^2 + a} = \frac{(-5 + 2\sqrt{13})\sqrt{22 + 2\sqrt{13}}}{54},$$

and it follows that this is the maximum of P, and that the maximum will be attained for

$$y = \cos x = \frac{-1 + \sqrt{13}}{6}, \qquad k = 0,$$

provided the quartic that appears in P is positive for $0 \leqq y \leqq 1$. As the quartic has two negative coefficients for the value of a that is to be used, a discussion of it is necessary.

Let

$$f(a, y) = y^4 + 2(a-1)y^3 + a(3a-4)y^2 + 2(a-1)^2(2a+1)y + a(a-1)^2(2a+1).$$

We have

$$f(\tfrac{1}{2}, y) = y^4 - y^3 - \tfrac{5}{4}y^2 + y + \tfrac{1}{4} = (y-1)(y+1)(y^2 - y - \tfrac{1}{4}) > 0, \quad \text{if} \quad 0 < y < 1.$$

Further

$$f(1, y) = y^4 - y^2 < 0, \quad \text{if} \quad 0 < y < 1.$$

Since all non-vanishing coefficients are positive, it is seen that

$$f(\tfrac{4}{3}, y) > 0, \qquad y \geqq 0.$$

Writing the quartic according to powers of a we have

$$f(a, y) = 2a^4 + (4y-3)a^3 + 3y(y-2)a^2 + (2y^3 - 4y^2 + 1)a + y^4 - 2y^3 + 2y.$$

As regards the coefficients we observe that:

(i) $2y^3 - 4y^2 + 1$ has a zero between $\frac{1}{2}$ and $\frac{3}{4}$, and one between 1 and 2; and

(ii) $y^4 - 2y^3 + 2y > 0$ for $y > 0$.*

It is accordingly seen that there are two changes of sign in the equation

$$f(a, y) = 0$$

if $y < 1$, so there are not more than two real roots in a if $y < 1$. But we have noted that

$$f(\tfrac{1}{2}, y) > 0, \quad f(1, y) < 0, \quad f(\tfrac{4}{3}, y) > 0, \quad \text{if} \quad y < 1,$$

so that for $y < 1$ there is one root in a between $\frac{1}{2}$ and 1 and one root between 1 and $\frac{4}{3}$. It follows that for $a < \frac{1}{2}$ the equation has no root y which is less than 1.

As a result of what has been proved it is seen that for the value of a in which we are interested the quartic does not vanish for $0 \leqq y \leqq 1$,† and it is evidently positive.

* $y^4 - 2y^3 + 2y = y(y^3 - 2y^2 + 2)$. The cubic $y^3 - 2y^2 + 2$ is found to have a zero α, $-6/7 < \alpha < -5/6$. Writing $y^3 - 2y^2 + 2 = (y - \alpha)(y^2 - (2 - \alpha)y - 2/\alpha)$, one can show that the discriminant of the quadratic factor is negative since $-6/7 < \alpha < 0$. Hence the cubic has no positive zeros.

† The value $y = 1$ has not been examined. We find $f(a, 1) = (a-1)(a+1)^2(2a-1) > 0$ for $a < \frac{1}{2}$.

The graph of $z = f(a, y)$ as a changes behaves as follows. For $a = 0$, we have $z = y^4 - 2y^3 + 2y$; the curve goes through the origin but does not intersect the positive y-axis. For $a = \frac{1}{2}$ we have $z = (y-1)(y+1)(y^2 - y - \frac{1}{4})$, and the curve has one intercept at $y = 1$ and another at $y = y_1$, $9/8 < y_1 < 5/4$, the curve being above the axis except for $1 < y < y_1$. Thus as a increases from 0, the curve dips down, while its z-intercept becomes positive, touching the y-axis for a value of y not much in excess of 1, for a value of a a little less than 1/2. From $f(a, 1) = (a-1)(a+1)^2(2a-1)$ it is seen that for $1/2 < a_1 < 1$, the curve has one y-intercept between 0 and 1 and one intercept > 1.

Accordingly P has the maximum given above. When simplified the maximum is found to be .2213 and the triangle in question is $A=B=64°15'.5$, $C=51°29'.0$.

(5) Another case where one angle coefficient is twice the others occurs in

$$P = \sin \tfrac{1}{4}A \sin \tfrac{1}{4}B \sin \tfrac{1}{2}C.$$

The maximum is easily found and is $(-14+5\sqrt{10})/54=.0335$; this value is given by $A=B=2\cos^{-1}(2+\sqrt{10})/6=61°16'$, $C=57°28'$.

Annuity Formulas for Payments Made between Conversion Dates

By A. H. Diamond, University of California

1. *Introduction.* In the theoretical treatment of annuities presented in textbooks, the formulas for annuities having payments between conversion dates are derived on the assumption that each payment receives compound interest for the fraction of the conversion period for which it is invested. But in actual practice banks and other financial houses compute simple interest on payments made between conversion dates. The theoretical results, therefore, are not applicable to bank practice. This was recognized by H. W. Sibert in an article* where he proposed formulas which would take into account the current practice of banks and other financial institutions. The prevailing practices of banks, however, differ in several respects from those assumed by Sibert in his paper. It is the purpose of the present paper to point out these differences and to give alternative formulas wherever necessary.

2. *Formulas for the amount of an annuity.* Sibert derives his results on the assumption that banks credit simple interest from date of payment to the next conversion date on each payment made between conversion dates. He obtains the following formulas for the amounts of an ordinary annuity and of an annuity due respectively

$$S=R\left[1+\frac{p-1}{2p}\right](s_{\overline{n}|}\text{ at } i) \qquad \text{and} \qquad S=R\left[1+\frac{p+1}{2p}\right](s_{\overline{n}|}\text{ at } i).$$

These results agree with the practice of some banks. But in many banks the interest on sums deposited within less than a certain period (three months in some instances and six months in others) before a conversion date is not credited on that date but on the conversion date six months later if the sums are not withdrawn in the interim. The interest from these deposits is thus credited in

For $a=1$, we have $z=y^4-y^2$, and intercepts are at $y=0$ and $y=1$. Thus as a increases from 1/2 to 1 the intercepts move towards the left. As a continues to increase the intercepts approach each other, remaining between 0 and 1; the part of the curve below the y-axis shrinking, the curve becoming tangent to the axis for a value $a=a_2$, where $1<a_2<4/3$. For $a>a_2$ the curve does not cut the positive y-axis.

* New annuity formulas for payments made between conversion dates, this Monthly, 1926, pp. 139–142.

place of the deferred interest on corresponding deposits of the following conversion period. Consequently, the total interest credited on every interest date after the first is virtually the same as that assumed by Sibert; but the interest credited on the first interest date is less than that assumed by Sibert. Therefore, at the end of n periods the amount on hand in these banks will be less than the amount on hand in banks following Sibert's method by the accumulation for $n-1$ periods at i percent of the deficiency in interest credited on the first interest date.

To illustrate, consider the case of monthly deposits made in a bank converting interest semiannually and not crediting interest on sums deposited within less than three months of a conversion date until six months after that date. Interest on the last three deposits of the first conversion period of an ordinary annuity, or on the last two deposits of an annuity due, is not credited at the end of the period. This interest, which amounts to $Ri/12$ in both cases, is credited six months later, however, thus exactly compensating for the interest deferred on the corresponding deposits of the second period, and so on. Sibert's formulas in this case, therefore, must be diminished by $Ri/12(1+i)^{n-1}$.

3. *Formulas for present value.* In the case of the present value, A, of an annuity, we require the amount which invested now at i percent would provide for p equal withdrawals, R/p, at equal intervals during each conversion period for n periods and become exactly exhausted by the last withdrawal. Now, the procedure of banks in figuring interest on sums upon which withdrawals are made between conversion dates differs materially from that used in figuring interest on sums deposited between conversion dates. Since Sibert assumes that the procedure is the same whether the payments are deposits or withdrawals, his formulas for present value are not in accordance with bank practice.

It will be sufficient to consider the effect of making withdrawals upon a sum which is invested at the beginning of the conversion period. Assume, for definiteness, the conversion period is 6 months and begins on January 1. If a sum, X, deposited on January 1 were left undisturbed it would be credited with interest equal to Xi on July 1. But if withdrawals are made upon this sum between January 1 and July 1, it is the custom of many banks to deduct from Xi interest for 6 months on amounts withdrawn in the first quarter, from January 1 to March 31 inclusive, and to deduct interest for three months on amounts withdrawn in the second quarter, from April 1 to June 30 inclusive.

To illustrate, consider the present value on January 1 of an ordinary annuity of monthly payments for n conversion periods. It will be necessary to assume that the withdrawals are made on February 1, March 1, etc., instead of at the end of each month, in order to have a result distinct from that for an annuity due. Let X be the amount which invested on January 1 at i percent would provide for withdrawing $R/6$ every month beginning February 1 and ending July 1 and become exactly exhausted on July 1. The required formula is clearly the present value of an annuity due of X every 6 months for n periods. The problem resolves itself then into finding the expression for X in terms of R and i.

If X were left undisturbed, interest equal to Xi would be credited on July 1. But interest is deducted for 6 months on the first two withdrawals and for 3 months on the next three. Consequently, on July 1 the interest credited will be $Xi-7Ri/12$. The total amount on hand July 1, therefore, before the withdrawal of that date, will be $X-5R/6+Xi-7Ri/12$ or $X(1+i)-R(5/6+7i/12)$. This must be equal to $R/6$ since the original amount X is to become exactly exhausted with the last withdrawal. Thus

$$\frac{R}{6} = X(1+i) - R\left(\frac{5}{6}+\frac{7}{12}i\right).$$

Solving for X, we obtain

$$X = R\left(1+\frac{7}{12}i\right)\cdot\frac{1}{1+i}\cdot$$

Now, the present value of an annuity due of X for n periods may be written

$$A = X(a_{\overline{n|}}\text{ at }i)(1+i)$$

Whence, replacing X by its value above,

$$A = R\left(1+\frac{7}{12}i\right)a_{\overline{n|}}\text{ at }i$$

which is the required formula. The conversion factor given by Sibert is $1+5i/12$ instead of $1+7i/12$. His formula for present value of an ordinary annuity of monthly payments is therefore too small by $Ria_{\overline{n|}}/6$, a difference which may be considerable if R and n are sufficiently large.

Suppose now the payments are made at the beginning of each month. If the first payment and the last five were omitted, the other payments would constitute an ordinary annuity of monthly payments for $n-1$ periods. The amount which must be available immediately after the payment at the beginning of the last period must be $5R/6$, since it must provide for five payments and become exhausted one month before the end of the period, at which time no interest is paid. Consequently, the formula for the present value of an annuity due of monthly payments of $R/6$ for n periods is

$$\frac{R}{6} + R\left(1+\frac{7}{12}i\right)a_{\overline{n|}}\text{ at }i + \frac{5R}{6}\cdot\frac{1}{(1+i)^{n-1}}\cdot$$

The formula given by Sibert in this case is actually the formula for an ordinary annuity derived above.

If the formulas for present value given by Sibert are examined for other than monthly payments, corresponding differences will be found to exist in all but one case. The exceptional formula is the one for present value of an ordinary annuity of quarterly payments which remains inviolate.

The Major in Mathematics

By Charles Wexler, Arizona State Teachers College

At present most colleges define an undergraduate major in mathematics in terms of number of units, with perhaps a certain minimum number of units in upper division courses. This would be good enough, if an upper division course in one college were also an upper division course in all colleges. But the definitions vary so with colleges, and enforcement is so elastic, that it has been possible in some places for a student majoring in mathematics to graduate without having had a course more advanced than one in differential calculus.

It appears desirable, to the writer at least, that a mathematics major should be defined, not by number of units in mathematics, but by the subject matter studied; and that this subject matter should include not only a course in advanced calculus (making at least three semesters in calculus or the equivalent required) but also a one or two semester course in the theory of functions of a complex variable. The latter course is important, in the opinion of the writer, for two reasons. First, it is possible for a student to have had three semesters of the calculus without having been exposed to any mathematical rigor whatever in the development of the theory. Secondly, such a course is of a fundamental nature in mathematics, paving the way towards a more nearly complete understanding of elementary mathematics as well as to the study of almost any branch of higher mathematics. In a sense the study of higher mathematics may be said to begin with such a course.

Such a requirement would also tend to diminish the variation in quality of work towards the Master's degree in mathematics. In some universities, for example, a graduate student who has prepared at a small college can take towards his Master's degree courses in mathematics that another student has had during his sophomore and junior years.

Moreover, the high school must soon begin to offer analytic geometry and some calculus, and one cannot hope to be a good teacher in these subjects without having absorbed concepts developed in a course like the theory of functions of a complex variable. With such a course as part of his background, a teacher of mathematics will more nearly be worthy of the name, mathematician.

A Theorem on Subsequences

By Hugh J. Hamilton, Pomona College

It is the purpose of this note to establish the nearly obvious

Theorem. *If $\{a_k\}$ is any sequence (finite or infinite) of complex numbers for which $\sum_k |a_k| < \infty$, and if t is any number for which $0 \leqq t \leqq 1$, then there exists a sector of aperture $t \cdot 2\pi$ (closed on one side and open on the other), within which lies some argument of each element of a subsequence $\{a_{k_j}\}$ for which $\sum_j |a_{k_j}| \geqq t\sum_k |a_k|$.*

The symbol (ψ, Ψ) will signify the (closed) sector with initial angle ψ and terminal angle Ψ. In this notation the replacing of ψ by $\psi+0$ or of Ψ by $\Psi-0$

will indicate openness of the sector at ψ or at Ψ, respectively. The proof of the theorem consists of two parts, in the first of which t is rational.

Part I. $t=p/q$, *where* p *and* q *are positive integers, with* $0\leqq p\leqq q$. The cases $p=0$ and $p=q$ being trivial, let the discussion be restricted to the remaining case. Denote by $\{a_{k_j}^{(\nu)}\}$ the sequence of those elements of $\{a_k\}$ for which some argument lies within the sector $2\pi(\nu-1)p/q,\ 2\pi\nu p/q-0)$. Now $\sum_{\nu=1}^{q}\sum_j|a_{k_j}^{(\nu)}|=p\sum_k|a_k|$. Hence, for some μ, $\sum_j|a_{k_j}^{(\mu)}|\geqq(p/q)\sum_k|a_k|$, and the conclusion follows.

Part II. t *is any number for which* $0<t<1$. Let $\{r_n\}$ be any sequence of rational numbers for which $r_n<r_{n+1}\to t$. Corresponding to each n there exists, by Part I, a sector $(\psi_n, \psi_n+r_n 2\pi-0)$ within which lies some argument of each vector of a subsequence of $\{a_k\}$, the sum of the moduli of these vectors being not less than $r_n\sum_k|a_k|$. Let ψ be the monotonic limit (mod 2π) of some subsequence $\{\psi_{n_i}\}$ of $\{\psi_n\}$ for which $\psi+t\cdot 2\pi$ is the monotonic limit (mod 2π) of $\psi_{n_i}+r_{n_i}2\pi$. Depending upon whether $\psi_{n_i}>\psi$ or $\psi_{n_i}<\psi$, the sector $(\psi+0,\ \psi+2\pi t)$ or $(\psi, \psi+2\pi t-0)$ satisfies the conclusion of the theorem.

Remarks on the Decomposition of $4(x^p-1)/(x-1)$

By A. A. Krishnaswami Ayyangar, Mysore, India

In the May, 1936* issue of this Monthly, we read a list of decompositions of $4(x^p-1)/(x-1)$ for odd prime values of p from 67 to 97. W. E. H. Berwick has pointed out that such decompositions have been tabulated up to 101, in Dr. Hermann Teege's *Inaugural Dissertation* 1901.† Before seeing this remark, I published a note on the following formulas:

$$2nP_n-(2n-1)P_{n-1}=(-1)^{(p-1)/2}p\{\phi_{n-1}-2\sum\phi_{n-r}+2\sum\phi_{n-r-1}\},$$
$$2n\phi_n-(2n-1)\phi_{n-1}=P_{n-1}+2\{e_n-e_{n-1}-\sum P_{n-r}+\sum P_{n-r-1}\},$$
$$e_n=(h/p)\ \text{in the notation of the theory of numbers,}$$

where the summation on the right-hand side is extended over all the values of $r<n$ and quadratic non-residues of p, and where the P's and Q's are the coefficients of the polynomials Y and Z of the decomposition.‡ With the aid of this, I listed the coefficients for $p=(67, 71, 73$, and $79)$ with which the results now published agree. It may be of interest to add that the values of 83, 89, 97 of Cornelius Gouwens also agree with mine, which I did not publish after seeing Berwick's remark. As Teege's dissertation may not be accessible to the interested reader, I may add my result for 101: Y: 2, 1, 26, −12, 40, −65, 33, 9, 40, 49, −110, −4, 13, 63, 86, −166, −3, 6, 90, 77, −185, 19, 53, 76, 66, −206, the coefficients hereafter occurring in the reverse order. Z: 1, 0, 4, −4, 3, −5, 5, 6, −3, 0, −10, 5, 11, −4, 0, −15, 8, 12, −5, −1, −13, 11, 14, −10, 0, the coefficients hereafter occurring in the reverse order.

* This Monthly, vol. 43, 1936, pp. 283–4.

† Nature, July 1921, vol. 107, p. 652.

‡ The Journal of the Indian Mathematical Society, December 1921, vol. 13, pp. 225–229. This paper, unfortunately, contains a number of printer's slips.

RECENT PUBLICATIONS

EDITED BY W. R. LONGLEY, Yale University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Mechanics, Molecular Physics, Heat, and Sound. R. A. Millikan, D. Roller and E. C. Watson. 14+498 pages and 54 plates. Boston, Ginn and Company, 1937. $4.00.

Most of the elementary physics texts that have appeared in recent years in this country have been written for the student who is not majoring in any of the sciences. Because they employ the very minimum amount of mathematics, their treatments of many subjects are far from satisfactory for the serious student. The present book, however, is designed for the science or engineering major who has already had a good high school course in physics, has mastered trigonometry and is probably concurrently beginning the study of the calculus in his first college mathematics course. For the authors introduce considerable analysis of important points and do not hesitate to use the calculus where it is obviously desirable.

This textbook is far more than just a revision of the senior author's *Mechanics, Molecular Physics and Heat.* Again the emphasis is on fundamental principles, but the treatment is more detailed and comprehensive. As in the earlier book, the close relationship between class and laboratory work is stressed. Experiments, together with optional laboratory problems, are described at the end of each chapter. Commendable is the emphasis placed on problems. This reviewer agrees heartily with the authors' statement that "it is only by practice in applying physical principles to many different situations that any real understanding of these principles can be obtained."

A novel feature for a book of this type is the introduction of considerable historical and humanistic material in chapter headings and in the descriptions accompanying an interesting set of plates. Wherever possible, the references, quotations, and historical material have been taken from original sources. The student is urged to consult source material, and a lengthy bibliography of histories, biographies, and collections of original papers is included. These references are on too high a plane for any but the most thorough and interested student, but they should be of great value to advanced students of the subject. Most of the illustrations are either instructive or entertaining, although a few would seem to be either irrelevant or too far removed from general principles.

This book sets a high standard and merits a good reception. Many physicists will welcome this reversal of the all too general tendency to over-simplify and dilute our introductory physics courses. We will look forward to the companion volume covering the remaining subdivisions of physics, which the authors are no doubt planning to publish.

W. W. WATSON

treatment of the law of gravitation, and then treats the problem of motion under central forces and in particular the problem of the motion of the planets.

The authors solve a goodly number of problems, and give an unusually large and varied assortment of problems to be worked. Answers are given to all problems. Since this book seems very valuable as a problem source, it may be worth while to note the following criticisms. On page 88 the upper figure shows 13 tons load on the bridge truss where 3 tons is intended, and the lower part of the force diagram on the same page is incorrect. On page 133, a problem is solved in which a locomotive is supposed to do work at a constant rate as it accelerates a train from rest; it is not noted that these assumptions require infinite acceleration and tractive force initially. On page 162, a particle initially at rest on a rotating spherical earth is considered, and the authors reach the absurd conclusion that the particle would start to move northward toward the pole. On page 205 an example is worked to illustrate an experimental method of determining moment of inertia. An object is suspended by a string and supposed to swing rigidly in a plane like an ordinary compound pendulum. Actually it could not swing thus as it constitutes a system of two degrees of freedom and the general solution is a linear combination of two simple modes of oscillation, with different characteristic frequencies. In the text and at the end of the last chapter, the metric gravitational unit of power (which the authors call activity) is given as the watt, a kilogram meter per second. This unit is quite different from the watt of the electrical engineer, and it would be better omitted.

The book seems unusually free from misprints and errors, and a large number of answers have been checked and found correct. The authors restrict attention to problems in one and two dimensions, and do not use the more general methods such as Lagrangian equations or Hamiltonian equations of motion or the principle of least action. There are so many short paragraphs with headings and conclusions emphasized by heavy print and italics, that the main theorems do not stand out as highlights. There is no index. The theory is well done, but conventional save in the early use of vectors. The great wealth of problems makes it a valuable book for the teacher.

T. L. SMITH

A School Algebra (Parts 1 and 2). By R. M. Carey. London and New York, Longmans, Green and Company, 1936. 288 pages. $1.50.

This book is a first course in algebra intended for use at the secondary level in England and written by an assistant master at Rugby School. The underlying algebraic and graphical content is of essentially conventional type as compared to typical ninth grade American texts of conservative type. However, the book carries its applications and problem material to a level of difficulty which, in many respects, would not be reached in American texts on algebra below the college level. The book is concise in its explanations and well written from a mathematical standpoint. One is impressed by the relatively large proportion of word problems in the exercises, a feature which could well be copied by

American texts. The emphasis given to the problem material is shown by the fact that, out of 288 pages, the author devotes the last 81 pages to miscellaneous exercises. Although it appears obvious to the reviewer that this book would not fit any of the typical groups to whom we offer a first course in algebra in the United States at the present time, it would be of interest to any person who desires to compare English and American pedagogy in mathematics or who wishes suggestions for original exercises in third or fourth semester algebra, as taught in the United States.

W. L. HART

Mathematics for the Million. By L. Hogben. Illustrations by J. F. Horrabin. New York, W. W. Norton & Company, 1937. 647 pages. $3.75.

Despite the first appeal of this very entertaining and instructive book, "the million" will probably not find it easy reading. By the million the author means the many who have tried to master elementary mathematics through the calculus and have got little out of it. Their experience with mathematics has been one long baffling and bewildering nightmare; they have become more and more discouraged and finally have quit. The author blames their teachers for this. It is the teachers' insistence on technical detail and their failure to reveal the social implications of mathematics which, according to him, have made difficult and obscure that which is obvious and simple. To some extent he is correct in this; but much of the author's material and many of his methods are used by good teachers and still there are pupils who do not understand. Ancient priests and modern teachers are alike, says Hogben, in their reluctance to divulge the mysteries of mathematics. He would emancipate the millions; he would democratize mathematics; the advance of western civilization, he says, depends upon it (pp. 20, 27). So he offers the downtrodden this fresh treatment of the mathematics they have tried to learn, and gives them a fresh chance to prove themselves. They will appreciate his friendly interest but will probably not really appreciate his book.

The million will be delighted with the author's engaging explanations, set in a fascinating historical background, and will catch some of the spirit of the actual mathematics, just as the duller students in any class in mathematics perk up when the teacher mentions how a subject came into being and what some of its interesting applications are. The million will be grateful to the author for this new light, and their mathematical morale will be greatly improved. They will step with a firmer tread and look the world squarely in the eye, for they will be convinced that whenever they can take a little time to attend to it they can now master the mathematics which formerly baffled them. They won't do it, of course, and no one is going to ask them embarrassing questions to test their genuine comprehension of some of the details of the actual mathematics. But after all, the measure of genuine appreciation is not the reader's gratitude to the author for making his subject so alluring: it is the reader's actual grasp of the mathematics with which sooner or later he and Mr. Hogben must come to grips.

Unfortunately, few teachers bring to their subject as wide a knowledge of the history of mathematics as does the author. There are many teachers, however, who already know how to enliven their instruction with material of the sort which the author uses and they actually do so; only they do it without sacrificing certain little details of mathematical truth which are a definite help to thorough understanding. For in tossing aside the "technical details" so dear to the professional mathematician the author has cast out also certain nice distinctions which are of genuine significance even for the million. The reviewer will, of course, be delighted if this book really leads many of the million to a better understanding and appreciation of mathematics. This will indeed be a boon to civilization and will redound ultimately to the benefit of all teaching of mathematics. He is inclined to believe, however, that the ratio of "followers of the sport" to those actually engaged in the rough and tumble of the arena will remain about what it has always been.

In the opinion of the reviewer, the most appreciative readers of this book will be, not the million, but the thousands of teachers in secondary schools and colleges. They are the ones who will derive the most enjoyment and the greatest stimulation from the wealth of material it contains. Indirectly, therefore, the author may be helping later millions in their effort to grasp the significance of mathematics. Teachers will require one warning, however: the author is not only not interested in rigor; he feels that the mathematicians' interest in rigor has contributed largely to the downfall of the million. Consequently his treatment of mathematics is discursive and casual, and relies heavily upon intuition. Teachers, therefore, must read charitably if their reading is to be properly rewarded.

Just what does this book cover and why will the million not appreciate it? The publisher's advertisement calls it "an illustrated history of mathematical discovery portrayed against the social background." Illustrated it certainly is, profusely and well; but it is hardly a history of mathematics. It might better be called socialized mathematics. To be sure, the order of the topics is suggested by the history of mathematics. For example, the chapter entitled "Euclid without Tears *or* What you can do with Geometry" is taken up before "The Dawn of Nothing *or* How Algebra Began," and spherical trigonometry comes before logarithms. Also, the development of certain subjects is derived from the history of those subjects, but not always; for the author has his own ideas, and many of them are good. Often, however, it is not clear to the reader precisely what of the treatment is history and what is Hogben. The citation of a few details will give further hint of the contents and serve also to support the reviewer's opinion that this book is better suited to teachers than to the million.

In the introductory chapter is exhibited a 4-unit beaker with siphon shrewdly attached to show that $2+1$ equals 3 while $2+2$ equals only 2. In this way the reader is warned of the frequent discrepancy between mathematics and the world of reality. In the second chapter, entitled "Mathematics in Prehistory," appear diagrams of the sun's apparent diurnal motion and its annual motion in

the celestial sphere. On page 65 we read that Babylonian arithmetic employed a zero, "which was intercalated to represent a gap in the sexagesimal series." In Chapter III, on the "Grammar of Size, Order, and Number," there is great emphasis on the analogy between number and noun, operator and verb, and so forth until on page 99 we learn that "-3 is not simply a number. . . . It is a size gerund." Immediately below this we read that imaginary numbers "are not really numbers." There is much that is ingenious in all this, though from the point of view of the priests some of it requires correction and from the point of view of the million there is need of further interpretation. Indeed, the million in their distress might do worse than to listen again to the simple, straightforward, and accurate explanation by a professional mathematician.

Chapter IV, on geometry, contains frequent mention of Thales, Pythagoras, the Phoenicians, the Chinese, the Alexandrians, the Great Pyramid, and early methods of determining latitude and longitude; Euclid wins only grudging reference. Mr. Hogben offers here his own geometry; it begins as follows. A casual description, without diagram, of the construction of a single triangle when three sides are given—a description which does not mention the second point of intersection of the two circles—leads immediately to Triangle Rule One, which states that "The sizes of two triangles are equivalent if they have equivalent sides." This is evidently intended as postulate rather than theorem, but the author seems to prefer not to raise questions of this sort. There are two other triangle postulates; then two postulates on angles concerning the equality of "vertically opposite" angles and the sum to 180° of certain properly chosen, but poorly described, angles; two postulates concerning corresponding angles and alternate angles formed by two parallel lines cut by a transversal, the first of which is confused with what he regards as his definition of parallel lines, if indeed he has a definition of parallels—one is not sure on this point, which is lost in the excitement of summoning modern astronomy to discredit Euclid's definition of parallels; and lastly, two postulates called Line Rules One and Two. Line Rule One postulates the equality of radii of a circle. Line Rule Two is worth quoting: "If you draw a line joining two corners of a straight-sided figure, you divide it into two figures having one side in common, the line you have drawn. So there is at least one side of one figure equivalent to one side in the other." Nowhere is this qualified to rule out three-cornered figures or polygons which are not convex. Based on these nine postulates are ten "demonstrations" which purport to establish the remaining essentials of elementary geometry. The reviewer is in sympathy with the idea of making freer use of postulates to avoid proofs of the obvious which distress beginners. Mr. Hogben might have gone further. His "Demonstration 2," with no previous definition of area, requires two pages to prove that "if one side of a rectangle is divided into separate segments of any length, its whole area is equivalent to the sum of the areas of the rectangles formed by the undivided side and each segment of the divided side." His "Demonstration 7" requires three pages to prove that mutually equiangular triangles have their corresponding sides in proportion.

Chapter V, on arithmetic, lends significance to the figurate numbers of the Pythagoreans by linking them adroitly with permutations and combinations. Chapter VI, on trigonometry, influenced by the history of the subject, develops the half-angle formulas at the outset, then the laws of sines and of cosines; it treats of functions of the sum of two angles in only the easiest case with no suggestion that the sum of two acute angles may exceed 90°; and the chapter concludes with sections devoted to Alexandrian astronomy and arithmetic. The tangent of 90° is 1 divided by 0, where 0 is "a quantity so small that we cannot measure it" (p. 238). This phrase crops up again in a later chapter on the calculus in connection with Δx and Δy. The treatment of algebra in Chapter VII is closer to present methods than are the earlier chapters. In Chapter VIII the geometry and trigonometry of the sphere are developed in order to introduce the reader to the astronomical triangle and its application to problems in navigation.

The treatment of analytic geometry in Chapter IX is highly anecdotal. Those hardy readers who have persevered to this point will have learned not to interrupt a good story with questions which might disconcert the narrator. The excitement of "gunpowder plotting," i.e., tracing the path of a cannon-ball in vacuo, carries the reader through the graphing of the parabola $y=3x/2-x^2/4$ before he has met $y=x^2$. A similar ardor carries him through a treatment of the ellipse much longer and more involved than the usual treatment in ordinary textbooks. He is led to accept the derivation of the polar equation of the ellipse, though it is not used later. In Chapter X logarithms are developed by first considering Napier's method of converting the problem of multiplying the cosines of two angles into the simpler problem of adding the cosines of two related angles—though Napier did not describe it in just these terms. Then the binomial expansion and fractional exponents indicate how tables can be constructed and used; and further refinements are added by exhibiting the exponential series, which "leads us to the pronoun e." We are going fast now, and dash headlong through fifteen breath-taking pages to consummate the union of sine, cosine, the pronoun e, and the gerund i. The reviewer would welcome a brief respite at this point, but right ahead in Chapter XI the differential and integral calculus is set forth succinctly in only 64 pages; and then The Arithmetic of Human Welfare, i.e., statistics, is given in Chapter XII, the concluding chapter, in 62 pages. The first example in differentiation, i.e., the calculation of "dee-wy-by-dee-eks," is related to the cannon-ball equation $y=3x/2-x^2/4$ mentioned above. If one has happened to differentiate just the right functions one can solve the second order differential equation associated with simple harmonic motion before one has been formally introduced to integration. Mr. Hogben's disciples do this. Then they integrate functions to find areas and volumes and take up the logarithmic series. The statistics of the final chapter makes almost no use of the calculus but is probably put last to emphasize the human and social value of mathematics.

It must be clear that this book is crammed with interesting mathematics.

To revise certain statements and developments to suit the professional mathematician would be relatively easy; but to make this book suit the million would be much more difficult. It demands more of the ordinary reader than he is used to giving: most people do not think easily in three dimensions or in terms of astronomy. There are about 30 exercises at the end of each chapter, but these are hardly enough to keep the reader abreast of the author. If the million are bewildered by the ordinary text in mathematics, they will find this book doubly bewildering. They will be fascinated by the clever style, but baffled too. So much good mathematics need not go to waste, however. No matter what the million get from it, teachers and members of mathematics clubs will find it highly entertaining and instructive.

RALPH BEATLEY

Spherical Trigonometry. By Pramatha Nath Mitra. Calcutta, University of Calcutta Press, 1935. 22+163 pages.

In a short historical introduction (eight pages) the names and dates of forty-eight mathematicians are given with a very brief statement, or none at all, about the contribution of each to mathematics, particularly with reference to spherical trigonometry or astronomy. Historical references in footnotes are abundant.

The first two chapters are devoted to solid geometry. All theorems relating to the sphere that underlie the proofs of the formulas of spherical trigonometry are proved in detail. A few exercises are given.

Chapters III and IV cover the development of the usual formulas for the solution of spherical triangles. The formulas for the oblique triangle are developed first. The formulas for the right-angled triangle are then developed independently (Chapter IV).

The remaining three chapters deal with properties of spherical triangles. L'Huilier's theorem, the theorem of Menelaus (for a spherical triangle), and similar theorems are proved and used to prove other relations among the parts of a triangle. The properties of the incircle, circumcircle, and excircles of a spherical triangle are studied and many interesting theorems developed.

Only seven exercises for numerical computation are given. The remaining 164 exercises deal with proofs of minor theorems in solid geometry and various identical relations among the trigonometric functions of the sides and angles of a spherical triangle. Many exercises are taken from previous degree-examinations of British and Indian universities. The book is primarily a treatise on theory.

J. O. HASSLER

MATHEMATICS CLUBS

EDITED BY F. W. OWENS AND HELEN B. OWENS, State College, Pa.

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to F. W. Owens, 462 East Foster Ave., State College, Pa.

INTER-CLUB ACTIVITIES

A letter from a lively club says: "The aim of our club has been extended to include the furthering of good fellowship and mutual sharing of ideas and ideals with mathematics clubs of other colleges." This aim is well illustrated by many joint meetings, exchange of speakers, and club letters. The Harvard Mathematical Club and the Yale Undergraduate Mathematics Club annually exchange speakers. Wellesley, Tufts, and Boston University clubs held a joint meeting last year. The Mathematics Club of the State College of Washington annually entertains the Yakima Junior College Mathematics Club. Kappa Mu Epsilon chapters exchange annual reports and letters full of detail and personal notes. The Intercollegiate Mathematics Association brings students of five colleges in Milwaukee into closer relations. The mathematics clubs of Rutgers University and of New Jersey College for Women extend invitations to each other for meetings which feature special guest lecturers. So the list grows. A study of the Directory of Mathematics Clubs which appeared in this MONTHLY, vol. 43, 1936, pp. 420–431 and the Supplementary List in this MONTHLY, vol. 44, 1937, pp. 476–477 may give your club a suggestion as to a neighbor with whom to exchange ideas and enthusiasms. Any one of these clubs may be reached through the mathematics department of the school it represents.

ROAD TO SUCCESS

A letter from the director of one of the most successful clubs in the country carries suggestions for those interested in helping clubs.

"The chief reason for the success we have had with our meetings is that both the speakers and other members look upon the mathematics club as an important part of their mathematical experience and the faculty also show their interest by fairly regular attendance. Having the subjects announced in advance on a printed program undoubtedly helps also."

CLUB REPORTS

1936–37

Mathematics Club of Brown University

Committee on Program: Dr. J. S. Frame, Grace K. Anderson, B. B. Colvin, Lenora Grozen, R. A. Sheldon. Committee on Arrangements: Dr. M. L. Kales, D. G. Clark, Agnes M. Galligan, M. L. Grover, Elsie L. Rawson, Dora P. Roberts. Twice during the year the club enjoyed a social hour after its formal meeting and each member was permitted to invite a friend to hear the guest speaker. Professor W. A. Noyes, of the department of chemistry, spoke on "Why a theoretical chemist should know mathematics" and Professor W. C. Graustein of Harvard University spoke on "The trisectors of the angles of a triangle." At the monthly meetings topics discussed included: Duality in projective geometry; Complete quadrilateral and quadrangle; Theorems of Pascal and Brianchon; Theorems of Fermat and Wilson; Confocal conics; Introduction to space-time in relativity; Some ideas about abstract spaces; Game of Nim; Indeterminate linear and quadratic

equations. This very serious minded and highly successful club reports that, before its formal program is printed for distribution in the autumn, invitations to speak are extended to students who are either taking honors work in mathematics or are doing exceptionally well in their mathematics courses. These undergraduate speakers are coached by faculty members and usually spend fifteen to twenty hours studying their respective subjects and rehearsing their talks. As a result the talks are uniformly worth while and the attendance at meetings is excellent.

The Mathematics Club, University of Nevada

President, Elizabeth Juniper; Vice-President, J. Galvin; Secretary-Treasurer, Margaret Jensen. Topics discussed at the monthly meetings included: The Rhind papyrus; The slide rule; Mathematics for chemistry students; The Maya numerals; Probability; The principle of duality. The club held a social hour after each program and finished the year with a picnic at Lake Tahoe.

Kappa Mu Epsilon, State Teachers College, Emporia, Kansas

President, A. Bryan; Vice-President, D. Anderson; Secretary, Vida G. Walker; Treasurer, M. Horn; Historian, M. Bohn; Corresponding Secretary, A. W. Phillips; Sponsor, Dr. O. J. Peterson. The club held several open meetings with discussions of the rôle of mathematics in other sciences, particularly astronomy, physics, and chemistry. From the records of attendance it is evident that not even picnics command the interest shown in the discussion and demonstration of astronomical instruments. The annual letter appeared in leaflet form during the summer.

Mathematics Club, Women's College of the U. of N. C.

President, Margaret LeRoy; Vice-President, Olga Mallo; Secretary-Treasurer, Annabel Lee; Faculty Adviser, Cornelia Strong. The topics discussed at monthly meetings included: History of the work of United States Coast and Geodetic Survey; Facts of Einstein's theory of relativity; Mathematical books before the advent of printing; Early printed mathematical books, illustrated by slides. A social hour followed each talk and a picnic closed the year's activities.

William S. Hall Mathematics Club, Lafayette College

President, R. C. Wolf; Vice-President, J. F. Young; Secretary, W. F. Plume, Jr.; Librarian, T. C. Bagg. The club conducted its annual problem solving contest with W. F. Ganskopp winning first honors and W. F. Plume holding second place. The chief topic for the year was number theory. The program was varied by talks on paradoxes and the tricks of numerology. At the final meeting of the year Professor Brinkmann of Swarthmore College, the guest speaker, spoke on "Certain Diophantine equations."

Mathematics Club, University of Kansas

President, R. Hemphill; Vice-president, Martha Peterson; Secretary-Treasurer, Dorothy Whitla; Social chairman, Anita Rottler; Faculty Adviser, Professor W. Babcock. With forty-three members this club held meetings twice each month, concluding the year with a picnic. Topics discussed included: Hyperbolic functions; Discontinuities; History of number systems; Simpson's work in mathematics; Time; Squaring the circle. Diversion came with explanation of card tricks and breadth of view with a discussion of the supervision of our territories.

Pi Mu Epsilon, University of Illinois

Director, H. G. Bergmann; Secretary, F. M. Pulliam, Treasurer, Mary Iball. Twenty-eight associate and thirty-one active members helped carry on a successful year's work. In addition to social affairs in connection with pledging and initiation of new members, talks on mathematical subjects were presented at monthly meetings and an effort was made "to further mathematical interest in the community." Topics discussed included: Transfinite cardinal numbers; Logarithmic decrement of a ballistic galvanometer; Legendre polynomials and their applications to physical problems; Number systems.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 299. *Proposed by V. Thébault, Le Mans, France.*

If a certain three-digit number is divided by the product of its digits, the quotient will be its hundreds digit. Find the number and show that the solution is unique.

E 300. *Proposed by Daniel Finkel, New York City.*

When n is a positive integer, the coefficients in the binomial theorem may be divided into three groups as follows:

1) ${}_nC_0$, ${}_nC_3$, ${}_nC_6$, ${}_nC_9$, etc.
2) ${}_nC_1$, ${}_nC_4$, ${}_nC_7$, ${}_nC_{10}$, etc.
3) ${}_nC_2$, ${}_nC_5$, ${}_nC_8$, ${}_nC_{11}$, etc.

Two of these three sets are composed of identical terms. Prove that the sum of the terms in the remaining set differs from the sum of the terms in either of the equal sets by unity.

E 301. *Proposed by D. L. MacKay, Evander Childs H. S., N. Y.*

If in triangle ABC, $B-C=90°$, and S and T are the intersections of the internal and external bisectors of angle A with the side BC, prove:

a) $\sin A=(b^2-c^2)/(b^2+c^2)$,
b) ST is twice the altitude from A,
c) a^2 is the harmonic mean of $(b-c)^2$ and $(b+c)^2$.

E 302. *Proposed by F. A. Alfieri, New York City.*

If A, B, and C are the angles of a plane triangle, prove that

$$\cot A+\cot B+\cot C=\frac{1}{2}\left[\frac{\sin A}{\sin B\cdot\sin C}+\frac{\sin B}{\sin C\cdot\sin A}+\frac{\sin C}{\sin A\cdot\sin B}\right].$$

E 303. *Proposed by J. E. Trevor, Cornell University.*

Given fixed rectangular axes, x and y, and a straight line T perpendicular to the x-axis but capable of displacement parallel to itself. One end of a straight line segment A of constant length a is pivoted to T at the point where T cuts the x-axis. The other end of A makes sliding contact with the y-axis. One end

of a straight line segment B of constant length b is pivoted at the origin. The other end maintains sliding contact with T. The segments A and B, produced if necessary, intersect at P.

Find a rational algebraic equation of the locus of P. Describe the graph of this equation according as a is less than, equal to, or greater than b. Find the point at which the locus cuts the x-axis in the cases where a equals or exceeds b.

SOLUTIONS

E 262 [1937, 104]. *Proposed by Cezar Coşniţă, Roumanian Mathematical Institute.*

Find the locus of the center of a circle which so varies that its radical axes with two fixed circles pass always respectively through two fixed points.

Solution by L. M. Kelly, Northeastern University, Boston.

Let A and B be the centers of the two given circles, X the center of the variable circle, and R_A, R_B, and R_X the radii of the three circles respectively.

Now if P_1 is a point on the radical axis of X and A,

$$(P_1X)^2 - R_X^2 = (P_1A)^2 - R_A^2 = K_1$$

and if P_2 is on the radical axis of B and X,

$$(P_2X)^2 - R_X^2 = (P_2B)^2 - R_B^2 = K_2.$$

Therefore

$$(P_1X)^2 - (P_2X)^2 = K_1 - K_2 = K_3.$$

This is a well known locus, a straight line perpendicular to the line P_1P_2.

Also solved by N. A. Court, J. W. Kitchens, D. L. MacKay, C. E. Springer, E. P. Starke, C. W. Trigg, Simon Vatriquant, and the proposer.

E 263 [1937, 104]. *Proposed by D. L. MacKay, Evander Childs High School, New York.*

In the triangle ABC, the bisector of angle A, the median from vertex B, and the altitude from vertex C, are concurrent. Show that the triangle may be constructed with ruler and compasses if the lengths of sides b and c are given.

Solution by J. W. Kitchens, Student, University of Oklahoma.

Analysis: Let ABC be the required triangle. If the internal bisector AW, the median BB' and the altitude CF are concurrent, we have by Ceva's theorem, $(AF/FB)(BW/WC)(CB'/B'A)=1$, and since $(BW/WC)=c/b$, and $CB'=B'A$, we have $AF/FB=b/c$.

Construction: Draw AB equal to c and locate F so that $AF/FB=b/c$. If C is the intersection of the perpendicular to AB erected at F and the circle centered at A with radius b, then ABC is the required triangle.

Discussion: There is always essentially one and only one solution since

$AF = bc/(b+c)$ is always less than b. Since the point of intersection of the internal bisector and a median is inside the triangle, then F will divide AB internally. There is an analogous solution in case an external bisector, a median and an altitude are concurrent.

Also solved by W. B. Clarke, Daniel Finkel, W. R. Hardman, L. M. Kelly, Seymour Sherman, C. W. Trigg, Simon Vatriquant, and the proposer.

E 264 [1937, 104]. *Proposed by N. A. Court, University of Oklahoma.*

Given four spheres, (A), (B), (C), and (D), with non-coplanar centers. The sphere (AB) is constructed coaxal with [i.e., having same radical plane as] the spheres (A) and (B), and passing through the given point P. The spheres $(AC), \cdots, (CD)$ are similarly constructed. Show that the six spheres thus obtained have their centers in the same plane.

Solution by W. C. Arnold, DePauw University.

The six radical planes of the four original spheres have a common point O and this point O has the same power with respect to all ten of the named spheres. Let the line OP meet the spheres $(AB), \cdots, (CD)$ again in points $P_1, \cdots, P_6$. But $OP \cdot OP_1 = OP \cdot OP_2 = \cdots = OP \cdot OP_6$, and hence the points $P_1, P_2, \cdots, P_6$ are coincident at a point we shall call P'. Since the spheres $(AB), \cdots, (CD)$ have P and P' in common, their centers must be coplanar.

Also solved by L. M. Kelly, C. E. Springer, and the proposer.

E 265 [1937, 104]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

A right triangle has integer sides without any common factor. When each digit is replaced by a code letter, the sides are $SSWTVU$, $PTWTS$, and $RRWWQ$. Solve the code and show that the solution is unique.

Solution by William Douglas, Courtenay, British Columbia.

We note from inspection that $SSWTVU$ is the hypothenuse. Then we denote the hypothenuse by m^2+n^2 and the two sides by $2mn$ and m^2-n^2. Since the hypothenuse is a six-place number and the other two sides are five-place numbers, it follows that $S=1$. Hence we may write

$$m^2 + n^2 = 11WTVU$$
$$m^2 - n^2 = PTWT1$$
$$2mn = RRWWQ.$$

Using this knowledge, we see that $279 < m < 332$. Since m^2-n^2 is a five-place number ending with 1, we may discard as possible values of m all numbers between 280 and 331 whose squares end in 4 or 9. A brief examination of the remaining squares reveals the fact that there is but one solution. It is $SSWTVU = 116549$, $RRWWQ = 88660$, $PTWTS = 75651$.

Also solved by Mary L. Constable, E. P. Starke, C. W. Trigg, and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any sclutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3844. *Proposed by Gertrude S. Ketchum, Urbana, Ill.*

Let

$$p_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

be a polynomial of degree n. Then

$$\sum_{s=0}^{n} (-1)^{n+s}\binom{n}{s} p_n(s) = a_n n!$$

If the degree is less than n, then the above sum is zero. Also if

$$\sum_{s=0}^{n} p_n(s)\psi(n, s) = a_n n!$$

for every polynomial of degree n, then

$$\psi(n, s) = (-1)^{n+s}\binom{n}{s}.$$

3845. *Proposed by R. E. Gaines, University of Richmond.*

A right triangle ABC is inscribed in a conic with the side AB and the hypotenuse CA as normal chords. Prove that the conic must be an ellipse. Determine the position of A for such a triangle.

3846. *Proposed by V. Thébault, Le Mans, France.*

The five points A_1, A_2, A_3, A_4, A_5 on a sphere with the center O determine five tetrahedrons by omitting in turn $A_1, A_2, \cdots$. Let M_i, $i=1, 2, 3, 4, 5$, be the points which divide in the same ratio k the segments which join O with the corresponding Monge points Ω_i. Show that the five straight lines A_iM_i pass through a single point, and determine its position. Generalize for n points on a sphere.

SOLUTIONS

3757 [1935, 572]. *Proposed by V. Thébault, Le Mans, France.*

The sum of the powers of the vertices of a tetrahedron with respect to the sphere having a diameter with end points at the centroid and the Monge point is equal to the sum of the squares of the bimedians of the tetrahedron. (The

bimedians are the segments of straight lines joining the middle points of opposite edges of the tetrahedron.)

I. *Solution by C. W. Trigg, Cumnock College, Los Angeles.*

We first note that the three bimedians of a tetrahedron are concurrent at the centroid and bisect each other; and that the sum of the squares of the edges of a tetrahedron is equal to four times the sum of the squares of its bimedians.

In the tetrahedron $A_1A_2A_3A_4$ let the midpoints of the edges a_{14}, a_{23}, etc. be B_{14}, B_{23}, etc., and the corresponding bimedians be b_{14}, etc. Let X be the midpoint of the segment joining the centroid, G, to any point, M. Then applying the familiar theorem regarding the median of a triangle to triangle $B_{14}MB_{23}$, we have $\overline{MB_{14}}^2+\overline{MB_{23}}^2=2\overline{GM}^2+\frac{1}{2}(b_{14})^2$. Applying the same theorem to the other triangles determined by M and the bimedians, and adding the three equations, yields

$$\sum \overline{MB}_{ij}^2 = 6\,\overline{GM}^2 + \tfrac{1}{2}\sum b_{ij}^2.$$

Similarly, from the six triangles determined by M and the edges, we secure

$$\begin{aligned}3\sum \overline{MA}_i^2 = 2\sum \overline{MB}_{ij}^2 + \tfrac{1}{2}\sum a_{ij}^2 &= 12\,\overline{GM}^2 + \sum b_{ij}^2 + \tfrac{1}{2}\sum a_{ij}^2 \\ &= 12\,\overline{GM}^2 + 3\sum b_{ij}^2.\end{aligned}$$

This is equivalent to the theorem: The sum of the squares of the distances of any point to the vertices of a tetrahedron is equal to the sum of the squares of the bimedians increased by four times the square of the distance of that point from the centroid of the tetrahedron.

If G and M coincide, $\sum\overline{GA}_i^2=\sum b_{ij}^2$.

The power, P_i, of the vertex A_i with reference to the sphere with diameter $\overline{GM}$ and center X is $\overline{XA}_i^2-(\overline{GM}/2)^2$. Now in the triangle GMA_i, $\overline{GA}_i^2+\overline{MA}_i^2=2\overline{XA}_i^2+\frac{1}{2}\overline{GM}^2$. It follows that $P_i=\frac{1}{2}(\overline{GA}_i^2+\overline{MA}_i^2-\overline{GM}^2)$. Furthermore,

$$\begin{aligned}\sum P_i = \tfrac{1}{2}\left(\sum \overline{GA}_i^2+\sum \overline{MA}_i^2 - 4\overline{GM}^2\right) &= \tfrac{1}{2}\left(\sum b_{ij}^2 + 4\overline{GM}^2 + \sum b_{ij}^2 - 4\overline{GM}^2\right) \\ &= \sum b_{ij}^2.\end{aligned}$$

This proves the proposition for any point coupled with the centroid and does not restrict the relationship to the Monge point alone.

II. *Solution by Roy MacKay, University of Michigan.*

Let H, G, O be respectively Monge point, centroid, and circumcenter of the tetrahedron. Then if P is the center of the sphere and M_{ij} the midpoint of the edge A_iA_j, the theorem may be stated

$$\sum_i \left(\overline{PA}_i^2 - \overline{PH}^2\right) = 2\sum_{i,j} \overline{GM}_{ij}^2.$$

Inasmuch as an analogous theorem is true for the n-simplex with proper definition of P and H we give a proof for this more general case.

The Monge point of an n-simplex is defined as the point H dividing OG in the same ratio that the orthocenter divides OG in the case of an orthocentric n-simplex; i.e. $\overline{HG}/\overline{HO}=2/(n+1)$. Let P be a point on HO such that $\overline{HP}/\overline{HO}=1/(n+1)$. The power of A_i with respect to the n-sphere with center P and radius PH is clearly $PA_i^2-PH^2$. From the readily established facts that

$$\sum_{i=1}^{n+1}\overline{A_iG}^2=S/(n+1),$$

and

$$\overline{HO}^2=\{(n+1)^2R^2-S\}/(n-1)^2,$$

where S is the sum of the squares of the edges, and R is the circumradius of the simplex, it is easy to show that the sum of the powers of the vertices with respect to the n-sphere is $S/(n+1)$.

Using the median relationship in the triangle GA_iA_j, we find

$$\overline{GM}_{ij}^2=\tfrac{1}{2}\overline{A_iG}^2+\tfrac{1}{2}\overline{A_jG}^2-\tfrac{1}{4}\overline{A_iA_j}^2.$$

The sum of the $n(n+1)/2$ such equalities readily reduces to $(n-1)S/4(n+1)$, which proves the theorem:

$$\sum(\overline{PA}_i^2-\overline{PH}^2)=\frac{4}{n-1}\sum_{i<j}\overline{GM}_{ij}^2.$$

For $n=3$ this reduces to the desired result.

Solved also by L. M. Kelly and the proposer.

Editorial Note. The Monge point for a simplex may also be defined by the following theorem:

Given a simplex (non-degenerate) in euclidean space of n dimensions; for each pair of its vertices there is a plane perpendicular to the edge joining this pair and passing through the centroid of the remaining $n-1$ vertices. There are $(n+1)n/2$ such planes, and they intersect in a unique finite point M which lies on the straight line through the centroid G of the simplex and its circumcenter C, so that $\overline{GM}=2\overline{CG}/(n-1)$. If the simplex is orthocentric, then M is its orthocenter.

The proof is simple. Let G be chosen as origin of vectors $\boldsymbol{a}_i$ to the vertices A_i; then the Monge plane for A_iA_j and the perpendicular bisecting plane for the same edge have, respectively, the equations

$$[(n-1)\boldsymbol{x}+\boldsymbol{a}_i+\boldsymbol{a}_j]\cdot(\boldsymbol{a}_i-\boldsymbol{a}_j)=0, \tag{1}$$

$$[2\boldsymbol{x}-(\boldsymbol{a}_i+\boldsymbol{a}_j)]\cdot(\boldsymbol{a}_i-\boldsymbol{a}_j)=0, \qquad (i,j=1,2,\cdots,n+1). \tag{2}$$

Since the simplex is non-degenerate the second system of equations has a unique finite vector solution $\boldsymbol{c}$; as shown in the solution of problem 3752 [1937, 403]. Hence the second system has a unique solution and it is necessarily $\boldsymbol{m} = -2\boldsymbol{c}/(n-1)$, where $\boldsymbol{c}$ is the vector to C and $\mathbf{m}$ is the vector to M. If the simplex is orthocentric and $\mathbf{h}$ is the vector to the orthocenter H, then

$$(\boldsymbol{h} - \boldsymbol{a}_k)\cdot(\boldsymbol{a}_i - \boldsymbol{a}_j) = 0, \tag{3}$$

for any three distinct subscripts i, j, k. Let i, j be fixed, then the sum of equations (3) gives

$$[(n-1)\boldsymbol{h} + \boldsymbol{a}_i + \boldsymbol{a}_j]\cdot(\boldsymbol{a}_i - \boldsymbol{a}_j) = 0. \tag{4}$$

Hence $\boldsymbol{h}$ satifies the system (1) and we must have $\boldsymbol{h} = \boldsymbol{m}$. The difference of a pair of equations (3) gives

$$(\boldsymbol{a}_k - \boldsymbol{a}_l)\cdot(\boldsymbol{a}_i - \boldsymbol{a}_j) = 0, \qquad (i, j, k, l = 1, 2, \cdots, n+1). \tag{5}$$

where i, j, k, and l are all distinct. This is a necessary condition, and it will be shown to be sufficient. If i, j, k are fixed, we have by summing (5) and from (1)

$$[(n-1)\boldsymbol{a}_k + \boldsymbol{a}_i + \boldsymbol{a}_j]\cdot(\boldsymbol{a}_i - \boldsymbol{a}_j) = 0,$$

$$[(n-1)\boldsymbol{m} + \boldsymbol{a}_i + \boldsymbol{a}_j]\cdot(\boldsymbol{a}_i - \boldsymbol{a}_j) = 0.$$

Hence

$$(\boldsymbol{m} - \boldsymbol{a}_k)(\boldsymbol{a}_i - \boldsymbol{a}_j) = 0.$$

This says that A_kM is perpendicular to the face opposite A_k, and that the altitudes meet in M.

It was also shown in Kelly's solution that the theorem of the problem for three dimensions is true for any point in place of the Monge point. A similar theorem will be shown to be true for the simplex. Let (S) be any sphere with center S passing through G. If $\boldsymbol{s}$ is the vector of S, then the sum of the powers P_i of the vertices A_i with respect to (S) is

$$\sum P_i = \sum (\boldsymbol{a}_i^2 - 2\boldsymbol{s}\cdot\boldsymbol{a}_i) = \sum_{i=1}^{n+1} \boldsymbol{a}_i^2 .$$

This says that the sum of the powers is equal to the moment of inertia about G of the system of unit masses at the vertices. If M_{ij} is the mid-point of A_iA_j, we have in turn

$$\begin{aligned}
\sum (GM_{ij})^2 &= \tfrac{1}{4}\sum (\boldsymbol{a}_i + \boldsymbol{a}_j)^2 &&= \tfrac{1}{4}[n\sum \boldsymbol{a}_i^2 + 2\sum \boldsymbol{a}_i\cdot\boldsymbol{a}_j],\\
0 &= \tfrac{1}{4}(\sum \boldsymbol{a}_i)^2 &&= \tfrac{1}{4}[\sum \boldsymbol{a}_i^2 + 2\sum \boldsymbol{a}_i\cdot\boldsymbol{a}_j],\\
\sum P_i &= \sum \boldsymbol{a}_i^2 &&= \frac{4}{n-1}\sum (\overline{GM}_{ij})^2.
\end{aligned}$$

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall. University of Chicago, Chicago, Illinois.

Among the delegates appointed by the United States Department of State to the Descartes Congress held in Paris, early in August, were Dean G. D. Birkhoff and Professors Arnold Dresden and L. C. Karpinski. The occasion was the three-hundredth anniversary of the publication of Descartes's "Discourse on Method."

Professor C. H. Richardson represented the Association at the inauguration of Doctor Levering Tyson as president of Muhlenberg College, October 2, 1937; and Professor J. R. Kline represented the Association at the one hundred fiftieth anniversary of the founding of Franklin and Marshall College, October 14–17, 1937.

Professor J. W. Alexander of the Institute for Advanced Study, Professor D. J. Struik of the Massachusetts Institute of Technology, and Professor Louis Weisner of Hunter College, have been added to the staff of contributing editors of Science and Society.

Professor Archibald Henderson of the University of North Carolina delivered two lectures at the East Tennessee Teachers College on mathematics on October 29. On November 16, Professor Henderson spoke before the League for Political Education at the Town Hall in New York on "Relativity and what it means to the average man."

The following mathematicians are in residence at the Institute for Advanced Study this academic year: Dr. P. G. Bergmann; Dr. R. P. Boas, Jr.; Professor Marie M. Johnson of Oberlin College; Professor N. H. McCoy of Smith College; Professor C. C. MacDuffee of the University of Wisconsin; Professor C. B. Morrey of the University of California; Dr. A. P. Morse; Dr. C. J. Nesbitt; Professor Rufus Oldenburger of the Armour Institute of Technology; Dr. J. F. Randolph; Dr. Moses Richardson; Dr. Hyman Serbin; Dr. M. F. Smiley; Dr. A. E. Taylor; Dr. C. B. Tompkins II; Professor H. S. Wall of Northwestern University; Professor Aurel Wintner of the Johns Hopkins University.

The following have been appointed National Research Fellows in mathematics for the year 1937–38: R. P. Boas, Jr., D. M. Dribin, Dorothy Manning, A. E. Taylor, C. B. Tompkins II, H. S. Zuckerman. This list includes renewals.

At the University of Illinois, Dr. D. G. Bourgin, Dr. J. L. Doob, and Dr. P. W. Ketchum have been promoted to assistant professorships, while Assistant Professors H. J. Miles and G. E. Moore have been promoted to associate professorships.

Dr. E. A. Cameron of the University of North Carolina has been promoted to an assistant professorship.

Assistant Professor Leonard Carlitz of Duke University has been promoted to an associate professorship.

Dr. George Comenetz will be at St. John's College, Annapolis, during the coming year.

Dr. Elizabeth M. Cooper has been appointed chairman of the mathematical department of Hunter College High School.

Assistant Professor A. H. Copeland of the University of Michigan has been promoted to an associate professorship.

Associate Professor C. C. Craig of the University of Michigan is on leave of absence for the year 1937–38, and is studying at the Galton Laboratory, University College, London, England.

Dr. D. B. DeLury has been appointed a lecturer in mathematics at the University of Toronto.

Dr. A. H. Diamond of the University of California at Berkeley has been appointed lecturer in mathematics for 1937–38 at the University of California at Los Angeles.

Professor Eleanor C. Doak of Mount Holyoke College has retired.

Dr. H. P. Doole of the University of Nebraska has been promoted to an assistant professorship.

Professor Jesse Douglas of the Massachusetts Institute of Technology is on leave of absence during the scholastic year 1937–38.

Dr. F. W. Dresch has been appointed Florence Noble Travelling Fellow at the University of California.

Assistant Professor W. L. Duren of Tulane University has been promoted to an associate professorship.

Dr. P. S. Dwyer has been appointed to an assistant professorship at the University of Michigan.

E. D. Eaves of the University of Tennessee has been promoted to an assistant professorship.

Dr. R. L. Echols has been appointed to an assistant professorship in physics at the United States Naval Postgraduate School, Annapolis.

Professor W. B. Fite of Columbia University will be absent on leave during the winter session. During his absence Professor J. F. Ritt is acting as executive officer of the department of mathematics.

Assistant Professor D. A. Flanders of New York University has been promoted to an associate professorship.

Associate Professor Philip Franklin of the Massachusetts Institute of Technology has been promoted to a full professorship.

Dr. Kurt Friedrichs has been appointed visiting professor of applied mathematics at New York University.

Assistant Professor L. L. Garner of the University of North Carolina is on leave of absence for the session of 1937–38. His place is being filled by Professor Reinhold Baer from the Institute for Advanced Study.

Associate Professor G. I. Gavett of the University of Washington has been promoted to a professorship.

Associate Professor J. J. Gergen of Duke University has been appointed chairman of the department of mathematics.

O. H. Hamilton of the University of Texas has been appointed to an assistant professorship at Oklahoma Agricultural and Mechanical College.

Rev. B. A. Hausman, has been appointed to a professorship at the University of Detroit.

Ernest Hawkins of the United States Naval Academy, Annapolis, has been promoted to an assistant professorship.

Dr. E. A. Hedberg has been appointed professor of mathematics and head of the department at the University of South Dakota.

R. E. Henry has been appointed assistant professor and chairman of the mathematical department of the University of Newark.

Dr. I. M. Hostetter has been appointed an associate professor at Howard University.

Assistant Professor V. A. Hoyle of the University of North Carolina has been promoted to an associate professorship.

Professor Alfred Hume of the University of Mississippi, who was reported in a recent issue of this MONTHLY, page 487, to have retired with the title of emeritus, has informed the editors that this announcement is incorrect. He retired from the Chancellorship of the University, and is now Chancellor Emeritus, and head of the department of mathematics, on full time.

Assistant Professor A. R. Jerbert of the University of Washington has been promoted to an associate professorship.

Professor Edward Kasner of Columbia University has been appointed to the Adrain Professorship of Mathematics.

Dr. E. C. Kennedy has been appointed to an associate professorship at the Texas College of Arts and Industries.

Professor W. J. Kirkham has been granted a year's leave of absence from Oregon State College to do statistical research in Washington, D. C.

Dr. S. C. Kleene of the University of Wisconsin has been promoted to an assistant professorship.

Professor H. I. Lane of the University of South Dakota has been appointed to a professorship at Hendrix College, Conway, Arkansas.

Professor C. G. Latimer of the University of Kentucky will be a visiting lecturer at the University of Wisconsin during the second semester.

Professor Solomon Lefschetz of Princeton University has been elected a corresponding member of the Reale Accademia di Scienze, Lettere ed Arti di Padova.

Canon Georges Lemaître, professor of mathematics and theoretical physics at the University of Louvain, Dr. Kurt Gödel of the University of Vienna, and Professor Emil Artin of the University of Hamburg, have joined the faculty of the University of Notre Dame.

Associate Professor Olive C. Hazlett, Assistant Professor P. W. Ketchum, and Assistant Professor H. Levy, all of the University of Illinois, are at present on leave of absence.

Assistant Professor Marie Litzinger of Mount Holyoke College has been promoted to an associate professorship and appointed chairman of the department of mathematics.

Dr. L. L. Lowenstein has been appointed to an assistant professorship at Alfred University.

Professor H. B. MacDougal of South Dakota State College has been appointed acting head of the department of mathematics.

Dr. S. W. McCuskey of the Case School of Applied Science has been promoted to an assistant professorship in mathematics and astronomy.

Dr. L. E. Mehlenbacher has been appointed professor and head of the department of mathematics at Arizona State Teachers College at Flagstaff.

Associate Professor W. O. Menge of the University of Michigan has resigned to become Associate Actuary of The Lincoln National Life Insurance Company, Fort Wayne, Indiana.

Professor R. L. Menuet of Tulane University after a year as acting president, has been given a leave of absence for a year. At the end of this time he will return to his former position as professor of mathematics.

Dr. G. C. Munro has been appointed assistant professor of mathematics at Kansas State College.

Dr. A. C. Olshen has been appointed actuary in the Oregon State Insurance Commission.

Dr. L. A. Pipes has been appointed research fellow at the University of Wisconsin.

Assistant Professor H. H. Pixley of Wayne University has been promoted to an associate professorship.

Assistant Professor D. H. Porter of Marion College has been promoted to an associate professorship.

Dr. E. J. Purcell of the University of Arizona has been promoted to an assistant professorship.

Dr. J. F. Randolph, who is spending his second year at the Institute for Advanced Study, has been appointed associate professor of mathematics at the University of Oklahoma; his duties to begin in September 1938.

G. E. Reves has been appointed an assistant professor at The Citadel, the Military College of South Carolina.

Dr. D. P. Richardson of the University of Arkansas has been promoted to an associate professorship.

Assistant Professor J. H. Roberts of Duke University has been granted a leave of absence for the year 1937–38. During this time he will be a visiting lecturer at Princeton University.

Assistant Professor W. E. Roth of the University of Wisconsin at Milwaukee has been made an associate professor.

Associate Professor J. B. Scarborough of the U. S. Naval Academy has been promoted to a professorship.

Assistant Professor W. S. Schlauch of New York University has been promoted to an associate professorship.

Dr. I. J. Schoenberg of Colby College has been promoted to an assistant professorship.

Dr. Wladimir Seidel of the University of Rochester has been promoted to an assistant professorship.

Dr. L. W. Sheridan has been appointed to a professorship at the College of Mount Saint Vincent.

C. E. Smith has been appointed to an assistant professorship in astronomy at the San Diego State College, San Diego, California.

Dr. S. Roscoe Smith of the Carnegie Institute of Technology has been made an assistant professor.

Dr. O. Snodgrass has been appointed professor of mathematics at Yankton College, Yankton, South Dakota.

Associate Professor H. E. Stelson of Kent State University has been promoted to a professorship.

Assistant Professor J. J. Stoker of the Carnegie Institute of Technology has been appointed to an assistant professorship at New York University.

Dr. E. C. Stopher of the University of Iowa has been appointed assistant professor and head of the department of mathematics at Ashland College, Ashland, Ohio.

Professor W. T. Stratton has been appointed head of the department of mathematics at Kansas State College. He succeeds Professor B. L. Remick, who has retired to part-time teaching, after having served as head of the department for thirty-seven years.

Professor J. D. Tamarkin of Brown University is on sabbatical leave for the first semester 1937–38.

Dr. E. W. Titt has been appointed to an assistant professorship at the University of Maryland.

Dr. Stanislaw Ulam is a visiting lecturer at Brown University.

Associate Professor H. S. Vandiver of the University of Texas has been promoted to a professorship.

Professor J. H. Van Vleck of Harvard University is a visiting professor at Princeton University during the first term of the current scholastic year.

At the recent centennial celebration of Mount Holyoke College the degree of doctor of science was conferred on Professor Anna Pell Wheeler of Bryn Mawr College.

Professor W. M. Whyburn, acting chairman of the department of mathematics at the University of California at Los Angeles, has been made chairman of that department.

Associate Professor R. B. Wildermuth of Capital University, Columbus, Ohio, has been promoted to a professorship.

Associate Professor Oscar Zariski of Johns Hopkins University has been promoted to a professorship.

The following appointments to instructorships have been announced:
University of Alabama: Dr. H. C. Ayres
University of Arkansas: Dr. E. G. H. Comfort
Armour Institute of Technology: Dr. D. G. Fulton, Dr. I. E. Perlin
Brooklyn College: Jack Wolfe

Brown University: Dr. Douglas Derry.
University of California: Dr. R. M. Robinson
University of California, Extension division: Dr. P. M. Singer
University of California at Los Angeles: Dr. W. C. Risselman
Carleton College: Dr. C. Grace Shover
Case School of Applied Science: Dr. R. F. Rinehart
University of Cincinnati: Dr. Paul Pepper
Columbia University: Dr. E. R. Lorch
Cornell University: Edwin Galbraith, W. J. Harrington, Karl Johannes, D. S. Miller, Seymour Sherman, G. B. Thomas, W. D. Wray
Drexel Institute: W. B. Campbell
Florida Southern College: A. R. Truquette
Hunter College: Dr. L. A. Knowler
University of Illinois: Dr. H. E. Vaughan, Dr. Ruth G. Mason, Josephine H. Chanler
Johns Hopkins University: D. L. Netzorg; junior instructor, Sidney Kaplan
Kansas State College: part-time instructor, Harold Wierenga
Lafayette College: F. A. Ficken
Lehigh University: H. W. Alexander
Marietta College: Dr. Theodore Bennett
Massachusetts Institute of Technology: part-time instructors, W. R. Hydeman, S. W. Stewart
University of Michigan: Dr. T. N. E. Greville, Dr. M. L. Kales, E. D. Rainville, Dr. R. M. Thrall
University of Missouri: Dr. R. J. Michel
Mount Holyoke College: Miss Dorothy L. Bernstein
University of Nebraska: J. D. Novak
College of the City of New York: Dr. G. G. Harvey (Physics)
New York University: C. H. Helliwell (School of Commerce), Dr. J. C. C. McKinsey (Washington Square College)
University of Notre Dame: Dr. A. N. Milgram
Ohio State University: Dr. C. E. Rhodes, Dr. J. W. T. Suckau
Oregon State Agricultural College: Dr. O. G. Harrold, Dr. Henry Scheffe
Purdue University: Dr. A. H. Bailey, J. F. Carlson
Queens College, Flushing, N. Y.: Dr. Banesh Hoffman
Randolph-Macon Woman's College: Dr. Virginia Modesitt, Miss E. R. Schneckenberger
University of Rochester: Dr. A. S. Galbraith, Dr. S. E. Warshawski
University of Southern California: Dr. F. A. Butter, Jr.
Agricultural and Mechanical College of Texas: Dr. H. A. Luther
University of Wisconsin: Churchill Eisenhart, Bernard Friedman, Dr. D. H. Hyers, S. B. Jackson, R. B. Kershner, R. W. Wagner, Dr. Margarete C. Wolf
Yale University: Dr. Marshall Hall

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, E. J. MOULTON, Northwestern University, Evanston, Ill.

BOOKS FOR REVIEW should be addressed to REVIEW EDITOR, American Mathematical Monthly, 531 West 116th Street, New York, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of addresss to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-first Summer Meeting, Pennsylvania State College, Sept. 6–7, 1937.
Twenty-second Annual Meeting, Indianapolis, Ind., December 30–31, 1937.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1937 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Waynesburg, Pa., May 1; Pittsburgh, October 23.
ILLINOIS, DeKalb, May 14–15.
INDIANA, Greencastle, April 30–May 1.
IOWA, Dubuque, April 16–17.
KANSAS, Wichita, April 3.
KENTUCKY, Louisville, May 1.
LOUISIANA-MISSISSIPPI, Hammond, La., March 5–6.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Lynchburg, Va., May 8; Baltimore, Md., December 4.
MICHIGAN, Ann Arbor, March 20.
MINNESOTA, St. Paul, May 15.
MISSOURI.
NEBRASKA, Lincoln, May 7.
OHIO, Columbus, April 1.
OKLAHOMA, Tulsa, February 5.
PHILADELPHIA, Haverford, Nov. 27.
ROCKY MOUNTAIN, Greeley, Colo., April 16–17.
SOUTHEASTERN, Nashville, Tenn., April 16–17.
SOUTHERN CALIFORNIA, Los Angeles, March 6.
SOUTHWESTERN, State College, N.M., April 2–3.
TEXAS, Houston, April 23–24.
WISCONSIN, Milwaukee, May 8.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

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The Chauvenet Prize

The ASSOCIATION in 1925, with the aid of its president, Professor J. L. Coolidge, established a prize of one hundred dollars for a noteworthy expository paper published in English during successive periods of five years by a member of the ASSOCIATION. In 1928 Professor W. B. Ford, then president of the ASSOCIATION, gave five hundred dollars, the income from which was to supplement the original fund so that the prize could be awarded every three years. In 1936 an anonymous gift of one hundred dollars was added to this fund.

The purpose of the prize is to stimulate expository contributions in mathematical journals on the part of the younger American scholars. The award does not apply to books. It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE should tend to stimulate such production.

The first award, covering the five years preceding 1925, was made to GILBERT AMES BLISS for his paper on "Algebraic Functions and their Divisors" published in the *Annals of Mathematics*. The second award, covering the four years preceding 1929, was made to THEOPHIL HENRY HILDEBRANDT for his paper on "The Borel Theorem and its Generalizations" published in the *Bulletin* of the American Mathematical Society in 1926. The third award, covering the three years preceding 1932, was made to GEORGE H. HARDY for his paper on "An Introduction to the Theory of Numbers" published in the *Bulletin* of the American Mathematical Society in 1929. The fourth award, covering the three years preceding 1935, was made to DUNHAM JACKSON for three related papers: "The Convergence of Fourier Series" published in the *American Mathematical Monthly* in 1934, "Series of Orthogonal Polynomials" and "Orthogonal Trigonometric Sums" published in the *Annals of Mathematics* in 1933.

THE

MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

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FOR THE ACADEMIC YEAR

1937–1938

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PUBLISHED BY THE ASSOCIATION
1937

For a list of all officers and trustees for preceding years, see pages 66–67

† Deceased May 21, 1937.

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COLERAINE. Fattu, Tangjerd.
COLLEGEVILLE. Danzl, Winkelmann.
DULUTH. Cothran, Cowan, Strane.
EVELETH. Pollard.
GILBERT. Schey.
MINNEAPOLIS. Brink, Brooke, Bussey, Carlson, Dalaker, Daoust, Dimsdale, Edwards, Gibbens, Hart, Hartig, Jackson, Kirchner, Ness, Priester, Quaid, Scammon, Saunders, Scherberg, Shawhan, Shuman, Shumway, Teeter, Thorp, Underhill, Wegner, Wilder.
MOORHEAD. Andersen, Rasmusen.
NORTHFIELD. Carlson, Gingrich, Shover.
ROCHESTER. Oberg.
ST. JOSEPH. Claudette.
ST. PAUL. Blackall, Bush, Lewis, O'Toole, Rysgaard, Taylor, Thielman.
ST. PETER. Rundstrom.
VIRGINIA. Hancock.

MISSISSIPPI. (15)

CLEVELAND. Hickey.
CLINTON. Hitt.
HATTIESBURG. Dearman.
JACKSON. Babbitt, McCoy, Mitchell.
LONG BEACH. Willey.
RAYMOND. McDonald.
STATE COLLEGE. Cox, Ollivier, C. D. Smith.
UNIVERSITY. Bickerstaff, Hume, Quarles.
WESSON. Felder.

MISSOURI. (40)

CANTON. Ingold.
CLAYTON. Haertter, Rosskopf.
COLUMBIA. Blumenthal, Callaway, Wahlin, Westfall.
FAYETTE. Fleet.
FULTON. Sweazey.
JEFFERSON CITY. Jason, Talbot.
KANSAS CITY. Cutting, Pierson, Sigley.
KIRKSVILLE. Cosby, Jamison.
KIRKWOOD. Harris.
ROBERTSON. Rice.
ROLLA. Hinsch, Miles.
ST. CHARLES. Karr.
ST. LOUIS. Buell, Callaghan, Case, Dunkel, Gove, King, Middlemiss, Nagle, Osborn, Pennell, Rider, Roever, Siroky, E. Stephens, J. Y. Stephens, Szegö.
SPRINGFIELD. Finkel, H'Doubler.
WEBSTER GROVES. Clarke.

MONTANA. (6)

BOZEMAN. Frick, Hurst.
HELENA. Canning, Topel.
MISSOULA. Carey, Merrill.

NEBRASKA. (27)

BEAVER CROSSING. Thompson.
CHADRON. Sanders.
GILEAD. Erwin.
HASTINGS. McDill, Westhafer.
LINCOLN. Basoco, Brenke, Camp, Candy, Daum, Gaba, Howie, Mundhjeld, Novak, Pierce, Runge, Specht.
OMAHA. Bettinger, Dwyer, Earl, Fitzpatrick, Gunn, Marrin.
PERU. Hill.
WAYNE. Boyce, Hove.
YORK. Feemster.

NEVADA. (1)

RENO. Wood.

NEW HAMPSHIRE. (17)

CONCORD. Conwell.
DURHAM. Slobin, Wilbur.
EXETER. Funkhouser, Pennell, Sweet.
HANOVER. Brown, Forsyth, Mathewson, Morgan, Perkins, Robinson, Silverman, Wilder.
MANCHESTER. O'Leary.
PLYMOUTH. G. M. Smith.
STRATHAM. Wiggin.

NEW JERSEY. (56)

BELLEPLAIN. Durell.
DOVER. Cavalli.
EAST ORANGE. Nordgaard.
ENGLEWOOD. Echols.
HIGHTSTOWN. Harrison, Litterick.
HOBOKEN. Hazeltine, Murray.
JERSEY CITY. J. P. Smith.
LAWRENCEVILLE. Kimball, Mikesh.
MONTCLAIR. Davis, Fehr, Mallory, Stabler, Turner.
MORRISTOWN. Emmons.
NEWARK. Conkling, Klein, Mosesson, Strock.
NEW BRUNSWICK. Bunyan, Meder, Morris, Nelson, Starke, Walter, Wilson.
ORANGE. Strock.
PRINCETON. Adams, Alexander, Boas, Clifford, Eisenhart, Flood, Gillespie, L. W. Johnson, M. M. Johnson, Ketchum, Knebelman, Lefschetz, MacDuffee, Morse, Oldenburger, Olmsted, Randolph, Thomas, Tompkins, Veblen, von Neumann, Wedderburn.
SOUTH ORANGE. Stanwick.
TRENTON. Shuster.
UPPER MONTCLAIR. Campbell, Hildebrandt.
WESTFIELD. Marshall.
WORTENDYKE. F. E. Smith.

NEW MEXICO. (18)

ALBUQUERQUE. Anderman, Barnhart, Bauer, Byram, Haskins, Larsen, Newsom.
HACHITA. Roberts.
LAS VEGAS. Roberts, Rodgers.
PORTALES. MacKay.
ROSWELL. Harp.
SILVER CITY. Mickelson.
SOCORRO. Reece.
STATE COLLEGE. Branson, Fuller, Hazlewood, Rees.

NEW YORK. (243)

ALBANY. Beaver, Birchenough, Do Bell, Frankel, Lester, Stokes.
ALFRED. Lowenstein, Polan, Seidlin, Titsworth, Whitford.
ANNANDALE-ON-HUDSON. Phalen.
AURORA. Hollcroft, Rusk.
BALDWIN. Grove, Skelding.
BRONX. Kirby.
BROOKFIELD. Whitford.
BROOKLYN. Antonina, Berry, Bowden, Boyer, Charosh, Cowles, Fleisher, Griffin, Hertzler, R. A. Johnson, Karnow, Kennison, Koch, A. W. Landers, M. K. Landers, Lavoie, Lieber, Locke, Lorell, MacNeish, Milkman, Moore, Penn, Ruderman, Schuyler, Simpson, F. E. Smith, Tabatchnik, Thompson, Walter, Welkowitz, Whitford, Woodbridge.
BUFFALO. Gehman, Harrington, Montague, Ott, Podmele, Pound, Smokowski.
CLINTON. Brown, Carruth, Ferry, Fitch, Patterson.
CROWN POINT. Henderson.
ELMIRA. Suffa, Wright.
FLUSHING. Cope, Lehmann, Raudenbush.
FOREST HILLS. Walker.
GENEVA. Durfee, Hubbs.
HAMILTON. Aude, A. W. Smith, Wardwell.
HOUGHTON. Davison.
ITHACA. Agnew, Barbour, Carver, Curtiss, Flexner, Hurwitz, B. W. Jones, Karapetoff, Snyder, Trevor, Walker, Wray.
KENMORE. Brockett.
NEW YORK. Adams, Alfieri, Allen, Allison, Anderson, Archibald, Berger, Bergstresser, Berkeley, Bernard, Bernstein, Berry, Boehm, Bradley, A. B. Brown, Burgess, Mrs. J. H. Bushey, J. H. Bushey, G. A. Campbell, Chebotar, Cooley, Cooper, Courant, Doermann, Eisele, Fagerstrom, Farnum, Feld, Fisanick, Fiske, Fite, Flanders, Foster, Fry, Gentzler, Gilder, Gill, A. M. Ginsburg, J. Ginsburg, Graham, Gray, Grove, Hall, Hamilton, Harvey, Hawkes, Henderson, Hill, Hubert, Hurwitz, Hussey, Jablonower, Joffe, John, P. C. Jones, Kasner, Katsh, Koopman, Kutman, M. K. Landers, Larkin, Lawton, Lehmann, Linehan, MacColl, McKenna, Mead, Miller, Mirick, Molina, Mullins, Nehrbas, Oehler, Payne, Pedersen, Penney, Peters, Putnam, Quilty, Reddick, Rees, Reeve, Richmond, Ritt, Robinson, Roos, Roth, Schelkunoff, Schlauch, Shaw, Sheridan, Shewhart, Siceloff, Simons, Skelding, D. E. Smith, R. F. Smith, Snoke, Tanzola, Tilley, Turner, Upton, Wahlert, Walker, Weaver, Wechsler, Weisner, Whelan, Whitford, Wirth, Wood, Wright, Yanosik.
NIAGARA FALLS. O'Connor.
ONEONTA. Sanford, Schoonmaker.
PARISH. Church.
PELHAM. Milos.
POTSDAM. Waltz.
POUGHKEEPSIE. Hopper, Weiss, Wells.
ROCHESTER. Betz, Chesna, Eastham, Gale, Harding, Long, Mestler, Seidel, Watkeys.
ROME. Wardwell.
ST. BONAVENTURE. Nickol, Wheeler.
SCARSDALE. Lawton, MacNeish.
SCHENECTADY. Fox, Morse, Poritsky, Snyder, Ulrich.
SYRACUSE. Campbell, Carroll, Church, Decker, Harwood, Taylor.
TROY. Allen, McGiffert, Nash, Street.
WEST POINT. Echols, Jones.
YONKERS. Hubert, Yanosik.

NORTH CAROLINA. (38)

BOONE. Wright.
BUIES CREEK. R. E. Smith.
CHAPEL HILL. Browne, Cameron, Henderson, Hill, Kattsoff, Lasley, Linker, Mackie.
CHARLOTTE. O. M. Jones, Woodson.
DAVIDSON. McGavock, Mebane, Ward.
DURHAM. Dressel, Elliott, Gergen, Hickson, Patterson, Rankin, Thomas.
GREENSBORO. Barton, Pegram, Strong.
GREENVILLE. Graham, ReBarker.
HICKORY. Fritz.
MARS HILL. Robinson.

RALEIGH. Bullock, Cell, Eason, Levine, Winton.
RED SPRINGS. Prince.
SALISBURY. Dearborn.
WINGATE. Hendricks, Ledford.

NORTH DAKOTA. (7)

DICKINSON. Swanson.
FARGO. Householder, I. W. Smith.
GRAND FORKS. Mason, Staley.
JAMESTOWN. Jackson.
VALLEY CITY. Meyer.

OHIO. (115)

ADA. Whitted.
AKRON. Bender.
ALLIANCE. Hildner.
ATHENS. Marquis, Reed, Starcher.
BEREA. Baur, Dustheimer.
BLUFFTON. Hirschler.
BOWLING GREEN. Mathias, Overman.
CANAL WINCHESTER. Bareis.
CHILLICOTHE. Mathias.
CINCINNATI. Barnett, Brand, Butler, Hancock, Justice, Kennedy, Kersten, Lubin, Merriman, Moore, E. S. Smith, Yowell.
CLEVELAND. Boyce, O. E. Brown, Burington, Focke, Johnson, Jonah, Justin, Morris, Musselman, Nassau, Patterson, Rinehart, Sauté, Simon, Thomas, Tolar, Torrance.
CLEVELAND HEIGHTS. Joliat.
COLUMBUS. Bamforth, Beatty, Blumberg, Hartung, Horn, M. E. Jones, Kuhn, LaPaz, Manson, Morris, Radó, Rasor, Rhodes, Rickard, Singer, Toops, Weaver, Westhafer, Wildermuth, Wylie.
DAYTON. McGee.
DEFIANCE. MacCullough.
DELAWARE. Crane, Rowland.
FINDLAY. Roots.
GAMBIER. Allen, Bumer.
GRANVILLE. Ladner, Wiley.
HIRAM. Clarke.
KENT. Brooks, Harshbarger, Manchester, Rogers, Stelson.
MARIETTA. Bennett, Sandt.
MOUNT ST. JOSEPH. Corona.
NEW CONCORD. Gilbert.
NEW LEXINGTON. Hoops.
NORTH CANTON. Schug.
NORWOOD. Wishard.
OBERLIN. Cairns, Carr, Johnson, Sinclair, Smyth, Yeaton.
OXFORD. Anderson, Pollard, Spenceley, Tappan, Wolfe.
PAINESVILLE. Lewis.
SEVEN MILE. Baird.
SOUTH EUCLID. Garvin.
SPRINGFIELD. Tripp.
TIFFIN. Pierce.
TOLEDO. Brandeberry, Dancer, Koley, Mercedes, Welker, Winslow, Yeager.
WESTERVILLE. Glover.
WILMINGTON. Spinks.
WOOSTER. Crandell, Knight, Williamson, Yanney.
YELLOW SPRINGS. Astrachan, Burr.
YOUNGSTOWN. Foard.

OKLAHOMA. (34)

ADA. Heimann, Winn.
ALVA. Hall.
BARTLESVILLE. Hagler.
CHECOTAH. Harrison.
DURANT. Dragoo.
EDMOND. Johnson.
ENID. Butchart.
HOLDENVILLE. Wedel.
NORMAN. Brixey, Court, Duval, Gentry, Hassler, LaFon, McFarland, Reaves, Springer, Townes.
OKLAHOMA CITY. Whitney.
PERRY. Dolezal.
SHAWNEE. Doerfler, Dwight, Short.
STILLWATER. Allen, Barnett, Flanders, Garretson, Gundersen, H. W. Smith, Zant.
TULSA. Veatch, West.
WEATHERFORD. McCormick.

OREGON. (12)

ALBANY. Keeler.
CORVALLIS. Beaty, Kirkham, Milne, Williams.
EUGENE. DeCou, Moursund.
GRANT'S PASS. Feinler.
McMINNVILLE. Ramsey.
PORTLAND. Griffin, Hadley, Merriss.
SALEM. Luther.

PANAMA. (2)

BALBOA. McNair.
PANAMA CITY. Linares.

PENNSYLVANIA. (146)

ALLENTOWN. Deck, Kunkel.
ANNVILLE. Black, Wagner.
BEAVER FALLS. Cleland.
BETHLEHEM. Ashbaugh, Cairns, Cutler, Fort, Latshaw, Lehmer, Rau, Raynor, Reynolds, Shook, Smail, Van Arnam.
BRYN ATHYN. Allen.
BRYN MAWR. Hedlund, Lehr, Peterson, Wheeler.
BUTLER. Robb.
CAMP HILL. Foberg.
CARLISLE. Ayres, Landis.
COLLEGEVILLE. Clawson, Manning.
DENVER. Marburger.
EASTON. Benner, Beverley, Cawley, Hatch, W. M. Smith.
ELLWOOD CITY. Johnston.
ERIE. Benedicta, Kraus, Oergel, Sullivan, Wells.
GETTYSBURG. Wilson.
GREENSBURG. McNeil.
GROVE CITY. Grimes, Renwick.
HARRISBURG. Whited.
HAVERFORD. Oakley, Wilson.
HUNTINGDON. Hess, Shively, Stayer.
KUTZTOWN. Knedler.
LANCASTER. Charles, Long, Marburger, Murray, Worthington.
LATROBE. Seubert.
LEWISBURG. Gold, Lindemann, Richardson.
LOCKHAVEN. S. J. Smith.
MEADVILLE. Beisel.
NEW KENSINGTON. Sturm.

NEW WILMINGTON. Black.
PHILADELPHIA. Campbell, Caris, Constable, Davis, Eggert, Evans, Graves, Kaltenborn, Kline, Latshaw, Linton, McDonough, Mitchell, Robertson, Rosengarten, Safford, Shohat, Spencer, Tartler, Tyler.
PITTSBURGH. Aberle, Baird, Briant, Bryson, Buker, Calkins, Cowley, Dines, Foraker, Hicks, Hoover, Johnson, Karpov, Leifer, Moskovitz, Neelley, Olds, Riggs, Rosenbach, Saibel, S. R. Smith, Starr, Taber, Taylor, Wagner, Whitman.
READING. Speicher.
SCRANTON. Bertrand, Mary Daniel.
SHARON. Manning.
SHIPPENSBURG. Kieffer.
SLIPPERY ROCK. Lady.
STATE COLLEGE. Cohen, Curry, Dunlap, Frink, Gordon, Gravatt, Graves, Hagen, Jeffries, Moody, F. W. Owens, H. B. Owens, Sheffer, West.
SWARTHMORE. Brinkmann, Dresden, Kovalenko, Marriott.
SWISSVALE. Foraker, Zimmerman.
UPPER DARBY. McDonough.
WASHINGTON. Atchison, Bert, Dorwart, Rasel, Shaub, Thomas.
WAYNESBURG. Moston.
WILKES-BARRE. Miller.
YORK. Baker.

PHILIPPINE ISLANDS. (1)

MANILA. Hizon.

RHODE ISLAND. (16)

NEWPORT. Chase.
PROVIDENCE. Adams. Adkins, Archibald, Bennett, Carlen, Currier, Frame, Gilman, Manning, Richardson, Seidel, Smiley, Tamarkin, Watt, Wehausen.

SOUTH CAROLINA. (18)

CHARLESTON. Dye, Hair, Holt, Myers, Reves, Saunders.
CLINTON. Spencer.
COLUMBIA. Coker, Coleman, Jackson, Peele, Weber, Williams.
DUE WEST. Leslie.
HARTSVILLE. Reaves.
NEWBERRY. Gaver.
ROCK HILL. Stokes.
SPARTANBURG. Peck.

SOUTH DAKOTA. (9)

BROOKINGS. MacDougal, Walder, Wente.
HURON. Titt.
MITCHELL. Knox.
RAPID CITY. Petrie.
SPEARFISH. Hesseltine.
SPRINGFIELD. Hoopes.
VERMILLION. Ekman.

TENNESSEE. (24)

COOKEVILLE. Hutchinson.
JACKSON. Cordrey.
JEFFERSON CITY. Sloan.
JOHNSON CITY. Carson, Cloyd.
KNOXVILLE. Blincoe, Bond, Eaves, Gillis, Purviance, Sisk.
MARYVILLE. Knapp.
MEMPHIS. Locke.
NASHVILLE. Blair, Hyden, Karnes, McPherson, N. P. Miser, W. L. Miser, Morrel, Peterson, Van Horn, Wren.
TOWNSEND. Keller.

TEXAS. (64)

ABILENE. Burnam, Mullings, Tate.
ALPINE. Gilley.
ARLINGTON. Lynch.
AUSTIN. Batchelder, Craig, Decherd, Dodd, Ettlinger, Lubben, Moore, Vandiver.
BROWNSVILLE. De la Garza.
CANYON. Murray.
COLLEGE STATION. Binney, Blumberg, Edmonson, Ross.
COMMERCE. Box.
CORPUS CHRISTI. Woodard.
DALLAS. Mouzon, Thomas.
DENTON. Barksdale, Brown, Hanson, White.
EDINBURGH. Searcy.
EL PASO. Turrittin.
FORT WORTH. Howard, Sherer.
GALVESTON. Underwood.
GEORGETOWN. Wapple.
HEREFORD. Rice.
HOUSTON, Beckenbach, Blau, Bray, Dean, Lovett, W. A. Rees, Slotnick.
HUNTSVILLE. Querry.
KINGSVILLE. Kennedy.
LUBBOCK. May, Michie, Miller, Sparks, Thompson, Underwood, Woodward.
NACOGDOCHES. Ferguson.
PRAIRIE VIEW. Randall, Turner.
SAN ANTONIO. Hurry, McNelly, Schnepp, Wunder.
SAN BENITO. Rogers.
STEPHENVILLE. Cromwell, McSweeny, Redden.
WAXAHACHIE. Newton.
WICHITA FALLS. Adams.

UTAH. (7)

LOGAN. Bird.
ST. GEORGE. Everett.
SALT LAKE CITY. Gibson, Hayes, Horsfall, Pehrson, Stafford.

VERMONT. (12)

BURLINGTON. Bullard, Butterfield, Millington, Swift, Thomas.
MIDDLEBURY. Bowker, Hazeltine, Perkins, Wiley.
NORTHFIELD. Dix.
PUTNEY. Alliot.
VERGENNES. Patterson.

VIRGINIA. (46)

ASHLAND. Simpson.
AYLOR. Aylor.
BLACKSBURG. Hatcher, O'Shaughnessy, Rasche, Williams.
BLUEFIELD. Berry, Wright.
CHARLOTTESVILLE. Luck, McShane.
EMORY. Miller.
FARMVILLE. Taliaferro.
HAMPTON. Atkinson, Perkins.
LANGLEY FIELD. Pinkerton.

LEXINGTON. Byrne, Knox, Paxton, Purdie, L. W. Smith.
LYNCHBURG. Garland, Larew, Modesitt, Wiggin.
MILLER SCHOOL. Watson.
MONTEREY. Colaw.
PORTSMOUTH. Downing.
RICHMOND. Drew, Gaines, Harris, Whaley, Wheeler.
SALEM. Carpenter.
SOUTH BOSTON. Patten.
STAUNTON. Taylor.
SWEET BRIAR. Cole, Morenus.
UNIVERSITY. Linfield, Oglesby, Puckett, Sparrow, Whyburn.
WILLIAMSBURG. Calkins, Gregory, Russell, Stetson.

WASHINGTON. (17)

PULLMAN. Butler, Hacker, Isaacs.
SEATTLE. Ballantine, Beegle, Cramlet, Haller, Jerbert, McFarlan, Moritz, Mullemeister, Neikirk, Winger.
SPOKANE. Carlson.
TACOMA. Martin.
WALLA WALLA. Bratton.
YAKIMA. Whitney.

WEST VIRGINIA. (13)

BELLE. Bennett.
ELKINS. Vest.
HUNTINGTON. Hackney.
MONTGOMERY. W. F. Smith.
MORGANTOWN. Colwell, Davis, Eiesland, Reynolds, Turner, Vehse.
SMITHERS. Bell.
WEST LIBERTY. Kiplinger.
WHEELING. Bagby.

WISCONSIN. (42)

BELOIT. Bigelow, Conwell, Huffer.
GREEN BAY. Kalcik.
LACROSSE. Adkins.
MADISON. Allen, Cook, Evans, Ingraham, Langer, MacDuffee, March, Sokolnikoff, Trump, Van Vleck.
MILWAUKEE. Bardell, Beckwith, Ericson, Fitzpatrick, Knight, Luteyn, Marden, Mary Felice, Nordhaus, Norris, Parkinson, Pettit, Quarles, Roth, Vass, Wilczewski.
OSHKOSH. Beenken, Price.
RIVER FALLS. Eide.
SHEBOYGAN. Battig.
SUPERIOR. Flogstad, C. W. Smith.
WAUKESHA. Batha, Dancey, Hopkins.
WEST ALLIS. Wolf.
WEST DE PERE. De Cleene.
WISCONSIN RAPIDS. McMillan.

WYOMING. (5)

LARAMIE. Barr, Bellamy, Neubauer, Rechard, Stewart.

FOREIGN MEMBERS
(Other than Canada.)

ARGENTINA. (1)

BUENOS AIRES. Baidaff, Barral-Souto.

BRITISH HONDURAS. (1)

COROZAL. Zimmerman.

CEYLON. (1)

VADDUKODDAI. Lockwood.

CHILE. (1)

SANTIAGO. Salas-Edwards.

CHINA. (5)

AN-CHING. Chang.
CANTON. MacDonald, Woo.
PEKING. Shen-Fu.
SHASI. Reilly.

FRANCE. (2)

LE MANS. Thébault.
PARIS. Fréchet.

GREAT BRITAIN. (8)

BELFAST. Todd.
CAMBRIDGE. Hardy.
DUBLIN. Rowe.
LONDON. Craig, Dalal, Spies, Todd.
NOTTINGHAM. Piaggio.
OXFORD. Frecheville.

INDIA. (7)

ALLAHABAD CITY. Mitra.
BANGALORE. Iyengar.
CALCUTTA. Chatterjee.
DACCA. Vijayaraghavan.
MADRAS. Durairajan.
POONA. Banerji.
SURAT. Shah.

ITALY. (4)

BOLOGNA. Bortolotti.
NAPLES. Crudeli.
ROME. Enriques, Laboccetta.

JAPAN. (1)

TOKYO. Kobayashi.

NEW ZEALAND. (1)

DUNDEDIN. Martyn.

POLAND. (1)

WARSAW. Dickstein.

PORTUGAL. (1)

LISBON. da Cunha.

ROUMANIA. (1)

BUCHAREST. Claudian.

SIAM. (1)

BANGKOK. Hadlock.

SOUTH AFRICA. (2)

BLOEMFONTEIN. Arndt.
JOHANNESBURG. Dalton.

SOUTH AUSTRALIA. (1)

ADELAIDE. Wilton.

STRAITS SETTLEMENTS. (1)

SINGAPORE. Oppenheim.

SWITZERLAND. (3)

FRIBOURG. Bays.
GENEVA. Fehr.
NEUCHÂTEL. DuPasquier.

SYRIA. (1)

BEIRUT. Jurdak.

TURKEY. (1)

ISTANBUL. Mourad.

UKRAINE. (1)

KIEFF. Kryloff.

BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED)

Article I—Name, Purpose and Corporate Seal

1. This organization shall be known as

The Mathematical Association of America (Incorporated)

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field, by holding meetings in any part of the United States or Canada for the presentation and discussion of mathematical papers, by the publication of mathematical papers, journals, books, monographs and reports, by conducting investigations for the purpose of improving the teaching of mathematics, by accumulating a mathematical library and by coöperating with other organizations whenever this may be desirable for attaining these or similar objects.

3. The Corporate Seal of the Association shall have inscribed thereon the name of the Association and the words "Corporate Seal—Illinois."

Article II—Membership

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Any institution in which the Calculus is regularly taught shall be eligible for election to institutional membership in the Association. Such an institution shall have the privilege of sending a voting delegate to the meetings of the Association.

3. Election to membership shall be by vote of the Board upon written application from the individual or institution seeking admission, endorsed in the case of individuals by two members of the Association.

4. Those who were admitted to membership in The Mathematical Association of America (unincorporated) prior to October 1, 1920, and were in good standing as such on that date, were thereby admitted to membership in this Association (Incorporated).

Article III—Board of Trustees and Officers

1. The Officers of the Association shall be a President, two (2) Vice-Presidents, a Secretary-Treasurer, a Librarian and three (3) members of a Committee on Official Journal.

2. The control and management of the affairs and funds of the Association shall be vested in a Board of twenty (20) Trustees (hereinafter called the "Board"), who shall be members of the Association. This Board shall consist of the officers of the Association and twelve (12) additional members.

3. The President shall be elected by the Association's members biennially for a term of two years and shall be ineligible for reëlection. The Vice-Presidents shall be elected by the Association's members annually for a term of one year, and four members of the Board shall be elected by the Association's members annually for a term of three years. They shall be eligible for reëlection, but not for more than two (2) consecutive terms. The Secretary-Treasurer, the Librarian, and the Committee on Official Journal, consisting of the Editor-in-Chief, the Manager and one other member, shall be appointed by the Board. All Officers and other Trustees shall hold over until their respective successors are elected or appointed and qualify.

4. The Board shall transact the official business of the Association and shall report its actions at the annual business meeting of the Association and in the official journal. A statement regarding any proposed action of the Board which makes or alters a question of policy shall be published in the official journal, or notice of such proposed action shall be mailed to each member, before final action has been taken, so that members of the Association may make known to the Board their individual views.

5. The Board shall have authority to fill vacancies *ad interim* in any office, including vacancies in the Board and in the Committee on Official Journal, and to make any other appointments necessary for the transaction of the business of the Association.

6. At all meetings of the Board of Trustees a quorum shall consist of not less than five (5) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Board, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than announcement of such meeting. Informal action based on a mail ballot by the members of the Board, if ratified at a properly convened meeting of the Board, shall be as valid and effective as if originally authorized at such meeting.

7. Approximately two months before the date of the annual meeting all members shall be given an opportunity to nominate by mail a candidate for each office to be filled by the members for the ensuing year. Approximately one month before the annual meeting the Board shall announce two candidates for each office to be filled by the members, one being the person who received the highest vote in the nominations and the other being selected from among the several nominees next in order. The election shall be by mail or in person and shall close on the day of the annual meeting.

8. The President shall be the Executive Officer of the Association, shall preside at all meetings of the Board of Trustees and at the annual meeting of the Association. He shall have the usual duties pertaining to his office and such other duties as may from time to time be assigned him by the Board of Trustees.

9. The Vice-Presidents shall, in the absence of the President, have and exercise the powers of the President, their order being determined alphabetically. The Board of Trustees may assign to the Vice-Presidents such duties as may from time to time be determined.

10. The Secretary-Treasurer shall have the usual duties pertaining to the Office of Secretary and of Treasurer, including the custody of the records of the Association and of its Corporate Seal, the keeping of minutes of the meetings of the Board of Trustees and of the annual meeting and special meetings of members, and giving of due notice of all regular and special meetings of the Association and of the Board of Trustees and the supervision and safekeeping of the funds of the Association. The Secretary-Treasurer shall also have the duty of seeing that whenever Trustees are elected, including the election of Trustees to fill vacancies, a Certificate, under the Seal of the Association, giving the names of those elected and the term of their office, shall be recorded in the Office of the Recorder of Deeds for Cook County, Illinois. Such Certificates shall be signed by the Secretary-Treasurer and verified by oath of the President.

11. The Committee on Official Journal shall have supervision of the official journal subject to the control of the Board of Trustees.

12. The Librarian shall have general charge of the library of the Association and shall direct its affairs, including the exchange of the publications of the Association, subject to the control of the Board.

Article IV—Meetings

1. A meeting of the Association shall be held annually, at such time and place as the Board may direct. Special meetings of the Association may be called from time to time by the Board, or while the Board is not in session by the President of the Association, to be held at such time and place as may appear from the call.

2. The outgoing Board shall hold a meeting immediately preceding the annual meeting of the Association next succeeding their election, and the members of the new Board shall hold a meeting and organize, by completing the Board, immediately succeeding the annual meeting of the Association at which the new members thereof were elected. Further meetings of the Board may be held from time to time at the call of the President or of any three (3) members of the Board.

3. Notice of any meeting of members of the Association shall be given by the Secretary-Treasurer at least thirty (30) days prior to the date set for each meeting. Notice of all meetings of the Board other than the regular meetings provided in Section 2 shall be given to each member of the Board at least fifteen (15) days prior to the date set therefor.

4. Any member of the Association or of the Board may waive notice with the same effect as if due notice had been given him.

5. At all meetings of the Association a quorum shall consist of not less than twenty-five (25) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Association, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than the announcement at such meeting.

6. Members may take part and vote in person or by proxy at all meetings of the Association.

Article V—Sections

1. Any group of not less than ten (10) members of this Association may petition the Board for authority to organize a Section of the Association for the purpose of holding local meetings. The Board shall have power to specify the conditions under which such authority shall be granted.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such Sections.

Article VI—Offical Publications

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Board shall have full control of the publication and sale of the official journal and of all other official publications.

3. The official journal shall be under the general management of the Committee on Official Journal. There shall also be appointed by the Board a body of Associate Editors who shall give assistance in connection with the official journal and under the direction of the Committee on Official Journal.

4. The Board shall from time to time, as the need arises, make special provision for the management of any other official publications.

5. The Board shall fix the price of the official journal and of any other official publications of the Association, but in no case shall the journal be sold to non-members for less than the annual dues of individual members.

Article VII—Dues

1. Individual members of the Association shall pay an initiation fee of Two Dollars ($2) at the time of election.

2. The annual dues of each individual member shall be Four Dollars ($4), including a subscription to the official journal.

3. The annual dues of each institutional member shall be Seven Dollars ($7), including two (2) subscriptions to the official journal.

4. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list after due notice.

5. New members entering the Association after April 1 of any year shall have their dues pro-rated for the balance of the year, except when they desire to receive the full current volume of the official journal.

6. The life membership fee shall be the present value, according to McClintock's Male Annuitant Table based upon four (4) per cent interest, of an annuity due of Four Dollars ($4) a year at the attained age of the member; an annual valuation of the life membership fund shall be made under the McClintock Male Four (4) Per Cent Table; and the reserve thus computed shall be held as a liability.

Article VIII—Amendments to the Articles of Association and By-Laws

1. Changes in the Articles of Association or amendments to the By-Laws may be made at any annual meeting of the Association, or at any adjourned session thereof, or at any special meeting of the Association called for such purpose, by a two-thirds ($\frac{2}{3}$) vote of those present and entitled to vote; *provided* that due notice concerning such amendment shall have been printed in the official journal, or mailed to each member, at least one (1) month before the date of such meeting.

2. No changes in the Articles of Association shall have legal effect until a certificate thereof, verified by oath of the President and under Seal of the Association, attested by the Secretary-Treasurer, shall be filed in the office of the Secretary of State of the State of Illinois and recorded in the office of the Recorder of Deeds for Cook County, Illinois.

PERIODS OF SERVICE OF THE OFFICERS OF THE ASSOCIATION

Presidents

E. R. Hedrick	1916	J. L. Coolidge	1925
Florian Cajori	1917	Dunham Jackson	1926
E. V. Huntington	1918	W. B. Ford	1927–1928
H. E. Slaught	1919	J. W. Young	1929–1930
D. E. Smith	1920	E. T. Bell	1931–1932
G. A. Miller	1921	Arnold Dresden	1933–1934
R. C. Archibald	1922	D. R. Curtiss	1935–1936
R. D. Carmichael	1923	A. J. Kempner	1937–
H. L. Rietz	1924		

Vice-Presidents

E. V. Huntington	1916
G. A. Miller	1916
D. N. Lehmer	1917, 1918
Oswald Veblen	1917
J. W. Young	1918, 1926
R. G. D. Richardson	1919
H. L. Rietz	1919
Helen A. Merrill	1920
E. J. Wilczynski	1920
R. C. Archibald	1921
R. D. Carmichael	1921, 1922
B. F. Finkel	1922
A. B. Chace	1923
L. P. Eisenhart	1923
J. L. Coolidge	1924
Dunham Jackson	1924, 1925
A. A. Bennett	1925, 1933, 1934
W. B. Ford	1926
A. J. Kempner	1927, 1928, 1935
Clara E. Smith	1927
F. D. Murnaghan	1928
E. T. Bell	1929, 1930
W. C. Graustein	1929, 1930
Arnold Dresden	1931
C. N. Moore	1931
W. H. Bussey	1932
G. C. Evans	1932
E. B. Stouffer	1933
E. P. Lane	1934
L. L. Dines	1935
N. A. Court	1936
T. C. Fry	1936
T. H. Hildebrandt	1937
E. J. Moulton	1937

Secretary-Treasurer

(Appointed by the Trustees after 1918)

(W. D. Cairns .... 1916–)

Committee on Offical Journal

(Appointed by the Trustees)

H. E. Slaught	1916–1937
R. D. Carmichael	1916–1918
W. H. Bussey	1916–1918, 1926–1931
R. C. Archibald	1919–1921
W. A. Hurwitz	1919–1921
A. A. Bennett	1922
H. P. Manning	1921–1922
W. B. Ford	1923–1925
J. L. Coolidge	1923
A. J. Kempner	1924–
W. B. Carver	1932–1936, 1937–
E. J. Moulton	1937–

Elected Members of the Board

D. N. Lehmer	1916–1918, 1922–1924, 1930–1932
R. E. Moritz	1916–1918
K. D. Swartzel	1916
Oswald Veblen	1916, 1920–1922, 1926–1931
R. C. Archibald	1916–1917, 1923–1930
Florian Cajori	1916, 1918–1923, 1926–1930
M. B. Porter	1916–1917
J. W. Young	1916–1917, 1920–1922
B. F. Finkel	1916–1921, 1930–1935
E. H. Moore	1916–1921, 1923–1928
Alexander Ziwet	1916–1918
E. R. Hedrick	1917–1922, 1924–1929, 1932–
J. N. Van der Vries	1916–1918
Helen A. Merrill	1917–1919
D. E. Smith	1917–1919, 1921–1926, 1937–
Elizabeth B. Cowley	1918–1920
G. A. Miller	1918–1920, 1922–1924
E. J. Wilczynski	1918–1919, 1922–1926
L. P. Eisenhart	1919–1922, 1925–1930
E. V. Huntington	1917, 1919–1927, 1933–1935
E. L. Dodd	1920
R. D. Carmichael	1920, 1924–1929
A. A. Bennett	1921, 1930–1932
H. L. Rietz	1921–1923, 1925–1930, 1934–1936
C. F. Gummer	1921–1925
Dunham Jackson	1923–1929
Clara E. Smith	1923–1925
A. B. Chace	1924–1925
J. L. Coolidge	1926–1931
E. T. Bell	1927–1928
E. P. Lane	1928–1933
W. B. Ford	1929–1934
E. R. Smith	1929
W. L. Hart	1930–1935
Lao G. Simons	1930–1931
L. L. Dines	1931–1933
T. C. Fry	1931–1933
J. W. Glover	1931–1933
H. E. Buchanan	1932–
W. R. Longley	1932–1934, 1936–
E. J. Moulton	1933–1936
R. W. Brink	1934–
D. R. Curtiss	1934, 1937–
J. L. Walsh	1934–1936
Arnold Dresden	1935–
J. O. Hassler	1935–1936
F. D. Murnaghan	1935–
G. C. Evans	1936–
Mary Emily Sinclair	1936–
J. M. Thomas	1937–
Marie J. Weiss	1937–

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THE TARRY-ESCOTT PROBLEM

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1. *Introduction.* The problem to be discussed in this paper can be stated either in terms of elementary number theory or in terms of elementary algebra. In the former, we are *to find two sets of integers, equal in number* such that the integers in each set have the same sum, the same sum of squares, etc., up to and including the same sum of k^{th} powers*, i.e., we are to find solutions in integers of the system of equations

$$\sum_{i=1}^{s} a_i^j = \sum_{i=1}^{s} b_i^j, \qquad (j = 1, 2, \cdots, k). \tag{1}$$

In this form the problem has sometimes been called the problem of Equal Sums of Like Powers.

The algebraic formulation requires us *to determine two equations of the same degree s, each having integral roots, and such that the first $k+1$ coefficients of the one equation are equal to the first $k+1$ coefficients of the other.* The roots of these equations are the a's and b's of (1) and the equivalence of the two formulations is at once evident from a consideration of the elementary symmetric functions of these roots.

The problem certainly dates back as far as 1750–51 when Goldbach and Euler noted certain simple solutions of (1). In 1800 Delambre used simple cases of the equations. L. E. Dickson devotes chapter 24 of volume II of his *History of the Theory of Numbers* to brief reviews of 63 papers on the problem, including the above, and recently interest in it has been revived by the appearance of papers to which later reference will be made.

Because of the contributions made in the study of this problem by G. Tarry and E. B. Escott, it will, at the suggestion of Professor Dickson, be referred to in this paper as the Tarry-Escott problem.

In spite of the simplicity of its statement, the Tarry-Escott problem does not seem to be widely known; and while the complete solution is yet to be found, the partial results that are known are of such an interesting nature that they ought to appeal to a wide range of readers. Hence the present paper will try to present in expository form the essential material bearing on the problem, some of which is believed to be new. This material will be divided into topics as follows: Existence of solutions and general theorems; Theorems concerning special cases; General solutions of special cases; Applications in various fields.

2. *The existence of solutions and general theorems.* For convenience let a solution of the system of equations (1) be designated by the notation

$$a_1, a_2, \cdots, a_s \overset{k}{=} b_1, b_2, \cdots, b_s, \tag{2}$$

* The cases in which the sets are unequal in number of elements are automatically included, since zeros are allowed as elements.

or, in cases in which it is clear that the parentheses enclose sets of numbers, by the notation

$$(a_i) \overset{k}{=} (b_i). \tag{3}$$

We shall find it convenient also to speak of a set of integers $(a_1, \cdots, a_s; b_1, \cdots, b_s)$ satisfying (1) as *a set of degree k*. A solution of (1) in which the a's merely form a permutation of the b's will be called *trivial*, and in what follows, unless the contrary is stated, we will be concerned only with non-trivial solutions.

Although general solutions have been found for only a few values of k and s, the literature on the problem is replete with particular solutions in both numerical and parametric forms. For $k=2$, we cite the solutions

$$2, 3, 7 \overset{2}{=} 1, 5, 6;$$
$$a+c, b+c, 2a+2b+c \overset{2}{=} c, 2a+b+c, a+2b+c;$$
$$AD, AG+BD, BG \overset{2}{=} AG, AD+BG, BD;$$

while for $k=3$, examples of solutions of (1) are

$$0, 5, 5, 10 \overset{3}{=} 1, 2, 8, 9;$$
$$1, 4, 12, 13, 20 \overset{3}{=} 2, 3, 10, 16, 19;$$
$$a, b, 3a+3b, 2a+4b \overset{3}{=} 2a+b, a+3b, 3a+4b, 0;$$
$$AB, CD, CD+AD+BC, AB+AD+BC \overset{3}{=} AD, BC, AB+CD+AD, AB+CD+BC.$$

Fortunately we are not dependent upon illustrations for the proof of the existence of solutions of (1) for large values of k as will be shown by certain general theorems. But first we state the following theorem due to M. Frolov [1].

THEOREM 1. *If*

$$a_1, \cdots, a_s \overset{k}{=} b_1, \cdots, b_s,$$

then

$$Ma_1 + K, \cdots, Ma_s + K \overset{k}{=} Mb_1 + K, \cdots, Mb_s + K,$$

where M and K are arbitrary integers. That is, if $(a_1, \cdots, a_s; b_1, \cdots, b_s)$ is a solution of (1), then $(Ma_1+K, \cdots, Ma_s+K; Mb_1+K, \cdots, Mb_s+K)$ is also a solution of (1).

Although this theorem is very easily established and possibly appears trivial, it is very useful in that it enables us to operate on the symbolic form

$$a_1, \cdots, a_s \overset{k}{=} b_1, \cdots, b_s$$

according to rules of elementary algebra, i.e., we can multiply all the members of both sides by the same constant, we can add the same constant to all the members of both sides, and evidently also we can cancel any element common to both sides.

If one solution comes from another through the application of this theorem, the two are called *equivalent.* When we speak of distinct solutions we refer to solutions which are not equivalent.

From Theorem 1 it follows that for each solution there is an equivalent one such that the sum of the a's is zero, and the sum of the b's is zero. This equivalent solution has been called the *standard form* by Escott.
Thus in

$$2, 3, 7 \overset{2}{=} 1, 5, 6$$

if we subtract 4 from each term, we have

$$-2, -1, 3 \overset{2}{=} -3, 1, 2$$

in standard form; or in

$$1, 5, 8, 12 \overset{3}{=} 2, 3, 10, 11$$

if we multiply by 2 in order to make the sum divisible by 4, and then subtract one fourth of this new sum, or 13, from each member, we have

$$-11, -3, 3, 11 \overset{3}{=} -9, -7, 7, 9,$$

or

$$\pm 3, \pm 11 \overset{3}{=} \pm 7, \pm 9$$

in standard form.

In 1910, Escott [2] published the following:

THEOREM 2. *If $F(x)$ and $G(x)$ are two polynomials with integral coefficients having their first r terms alike, then $F(x)G(x+d)$ and $G(x)F(x+d)$, where d is any integer, are two polynomials having their first $r+1$ terms alike.*

As Escott remarks, this amounts to differencing the two polynomials, and it forms the basis of his extensive calculations on the problem.

In 1912, Tarry [3] stated essentially the same result in the following form:

THEOREM 3. *If $(a_1, \cdots, a_s; b_1, \cdots, b_s)$ is a set of degree k, then $(a_1, \cdots, a_s, b_1+h, \cdots, b_s+h; b_1, \cdots, b_s, a_1+h, \cdots, a_s+h)$, where h is an arbitrary integer, is a set of degree $k+1$.*

Obviously the h in the above theorem may be taken so large that no equality of the form $a_i+h=a_j$ or $b_i+h=b_j$ is possible, and the set of degree $k+1$ resulting from the theorem is non-trivial if the set of degree k assumed is non-trivial. Hence starting from any particular solution we can build up a set of any desired degree. However, when $a_i+h \neq a_j$ and $b_i+h \neq b_j$, each time we apply the theorem we double the number of terms while merely increasing k by one. Therefore in order to keep down the number of terms, it is advisable to choose for h that difference among the a_i-a_j and b_i-b_j which occurs most frequently. To illus-

trate the power of this theorem, we exhibit the following sequence which contains certain interesting features.

$$
(4)\quad \begin{cases}
 & 1, 9 \overset{1}{=} 4, 6; \\
h = 2 & 1, 8, 9 \overset{2}{=} 3, 4, 11; \\
h = 1 & 1, 5, 8, 12 \overset{3}{=} 2, 3, 10, 11; \\
h = 7 & 1, 5, 9, 17, 18 \overset{4}{=} 2, 3, 11, 15, 19; \\
h = 8 & 1, 5, 10, 18, 23, 27 \overset{5}{=} 2, 3, 13, 15, 25, 26; \\
h = 13 & 1, 5, 10, 16, 27, 28, 38, 39 \overset{6}{=} 2, 3, 13, 14, 25, 31, 36, 40; \\
h = 11 & 1, 5, 10, 24, 28, 42, 47, 51 \overset{7}{=} 2, 3, 12, 21, 31, 40, 49, 50.
\end{cases}
$$

A number of writers, especially recently, have been interested in finding the least value of s for which (1) will have solutions. It is easy to show that $s \geqq k+1$. Since the sets $(a_i; b_i)$ have the same sums of powers from the first to the kth, they have the same symmetric functions up to the degree k, and hence the sets constitute the roots of two equations of degree s having their first $k+1$ terms alike. If s were less than $k+1$ these equations would have all terms alike, and therefore have the same roots, possible only for trivial solutions. The resulting theorem, first established by L. Bastien [4], may be stated as follows:

THEOREM 4. *If equations (1) have a non-trivial solution, then* $s \geqq k+1$.

Solutions of the system (1) with $s=k+1$ have been called *ideal* solutions by J. Chernick, whose paper on the subject appears elsewhere in this issue of the MONTHLY.

It will be noted that all sets of the sequence (4) are ideal except the one of degree 6. The ideal set of degree 7 of (4) was first obtained by Tarry [5] by the procedure used above. It has since been found by Escott, Crussol, and Chernick by other methods. It contains the smallest integers of any ideal set of degree 7 on record. The ideal set of degree 6 with smallest integers that has been reported is (1, 19, 28, 59, 65, 90, 102; 2, 14, 39, 45, 76, 85, 103), given by Escott [2]. So far, no ideal set of degree $k>7$ has been found.

Seeking to find such solutions if they exist for all values of k, recent writers, including Alfred Moessner and Werner Schulz [6] and E. Maitland Wright [7] have defined the function $\rho(k)$ as the least number ρ such that

$$a_1, \cdots, a_\rho \overset{k}{=} b_1, \cdots, b_\rho$$

has non-trivial solutions. From Theorem 4 it is clear that $\rho(k) \geqq k+1$, and examples given in this paper show that $\rho(k)=k+1$ for $k=1, 2, \cdots, 7$. The latest published results for $k=8, 9, \cdots, 24$ are given in the table

k	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
ρ	10	14	14	18	24	32	33	50	58	70	80	102	140	280	504	648	864

Escott [8] has noted that Theorems 2, 3 can be reversed. He calls the resulting operation "integration." We have

THEOREM 5. *If* $(a_1, \cdots, a_s; b_1, \cdots, b_s)$ *is a set of degree* k, *and if* $a_i \equiv b_i \pmod h$, $(i=1, 2, \cdots, s)$, *with* $a_i < b_i$, $(i=1, 2, \cdots, r)$, *and* $b_i < a_i$, $(i=r+1, \cdots, s)$, *then* $(a_1, a_1+h, a_1+2h, \cdots, b_1-h, a_2, a_2+h, \cdots, b_2-h, \cdots, a_r, a_r+h, \cdots, b_r-h; b_{r+1}, b_{r+1}+h, \cdots, a_{r+1}-h, \cdots, b_s, b_s+h, \cdots, a_s-h)$ *forms a set of degree* $k-1$.

For example, "integrating" the set (1, 5, 9, 17, 18; 2, 3, 11, 15, 19) of degree 4 gives the set (1, 5, 8, 12; 2, 3, 10, 11) of degree 3, if we take $h=7$.

By repeated application of Theorem 2, Escott [2] has obtained:

THEOREM 6. *If we take as the a's all possible sums of odd numbers of the integers* $x_1, x_2, \cdots, x_n$ *and as the b's, the number zero and all possible sums of even numbers of those integers the set* $(a_i; b_i)$ *forms a set of degree* $n-1$.

Another theorem of some importance is the following:

THEOREM 7. *For* $n>1$, *the first* $2^n(2q+1)$ *positive integers can be separated into two groups, equal in number, such that they form a set of degree* $n-1$ *if* $q=0$, *or a set of degree* n, *if* $q>0$.

This was established by Tarry [9] for the case $q>0$ and by Barbette [10] for the case $q=0$, the latter having stated his theorem for $2^n(2q+1)$ numbers in arithmetic progression, an obvious corollary in view of Theorem 1.

Much use has been made in the search for solutions of (1) of sets possessing some kind of symmetry. For example, when the number of terms s is even, the assumptions

$$a_1 = -a_s,\ a_2 = -a_{s-1}, \cdots, b_1 = -b_s,\ b_2 = -b_{s-1}, \cdots$$

will reduce the conditions for a set of degree k to

$$\sum_{i=1}^{s/2} a_i^j = \sum_{i=1}^{s/2} b_i^j, \qquad (j = 2, 4, \cdots, k).$$

Thus for the system

$$(x+a)^k+(x-a)^k+(y+b)^k+(y-b)^k=(z+a)^k+(z-a)^k+(t+b)^k+(t-b)^k, \qquad (k=2, 4, 6),$$

Crussol [11] found three particular solutions, one of which is $x=23$, $y=41$, $z=29$, $t=37$, $a=19$, $b=9$. This gives

$$2^k + 16^k + 21^k + 25^k = 5^k + 14^k + 23^k + 24^k, \qquad (k = 2, 4, 6),$$

and leads at once to

$$-25, -21, -16, -2, 2, 16, 21, 25 \overset{7}{=} -24, -23, -14, -5, 5, 14, 23, 24,$$

(since the sums on each side for odd powers will obviously be zero), which is

the standard form of Tarry's ideal set of degree 7 which we mentioned above.

That not all solutions of (1) with s even are of this type follows from the example

$$-91, -17, -5, 113 \overset{3}{=} -77, -55, 23, 109.$$

Likewise, if the number of terms is odd, the assumptions

$$a_1 = -b_1, \cdots, a_s = -b_s$$

will reduce the conditions to

$$\sum_{i=1}^{s} a_i^j = \sum_{i=1}^{s} b_i^j = 0, \qquad (j = 1, 3, \cdots),$$

for a symmetrical type of solution.

3. *Theorems concerning special cases.* For $k=2$ Frolov [1] has given us the following theorem:

THEOREM 8. *If* $(a_1, \cdots, a_s; b_1, \cdots, b_s)$ *is a set of degree* 2 *and if* $r_1, \cdots, r_s$ *are any* s *integers satisfying*

$$\sum_{i=1}^{s} r_i(a_i - b_i) = 0,$$

then $(a_1+r_1, \cdots, a_s+r_s; b_1+r_1, \cdots, b_s+r_s)$ *is a set of degree* 2.

We shall refer to two sets of degree 2 as *projective* if one arises from the other by application of Theorem 8. The following five theorems are believed to have been first established by O. E. Brown:

THEOREM 9. *Every set* $(a_1, \cdots, a_s; b_1, \cdots, b_s)$ *of degree 2 is projective with a trivial set* $(0, c_1, \cdots, c_{s-1}; c_1, \cdots, c_{s-1}, 0)$ *of degree* 2.

For we have

$$\begin{aligned} 0 &= (\sum a_i - \sum b_i)^2 \\ &= \sum a_i^2 + \sum b_i^2 + 2\sum a_ia_j + 2\sum b_ib_j - 2\sum a_ib_h \\ &= 2\sum a_i^2 + 2\sum a_ia_j + 2\sum b_ib_j - 2\sum a_ib_h, \quad (i \neq j;\, i, j, h = 1, \cdots, s). \end{aligned}$$

Dividing by 2 and rearranging, we have

$$\begin{aligned} 0 = (-a_1)(b_1 - a_1) &+ (b_1 - a_1 - a_2)(b_2 - a_2) \\ &+ (b_2 + b_1 - a_1 - a_2 - a_3)(b_3 - a_3) + \cdots \\ &+ (b_{s-1} + b_{s-2} + \cdots + b_1 - a_1 - a_2 - \cdots - a_s)(b_s - a_s). \end{aligned}$$

Hence, Theorem 8 is satisfied with

$$r_1 = -a_1,$$

$$r_2 = -a_1 - a_2 + b_1.$$

$$r_3 = -a_1 - a_2 - a_3 + b_2 + b_1,$$

$$\cdots\cdots\cdots$$

$$r_{s-2} = -a_1 - \cdots - a_{s-2} + b_{s-3} + \cdots + b_1 = -b_s - b_{s-1} - b_{s-2} + a_{s-1} + a_s,$$

$$r_{s-1} = -a_1 - \cdots - a_{s-1} + b_{s-2} + \cdots + b_1 = -b_s - b_{s-1} + a_s,$$

$$r_s = -a_1 - \cdots - a_s + b_{s-1} + \cdots + b_1 = -b_s.$$

Hence (A_i, B_i) is a set of degree 2, where

$$A_1 = a_1 + r_1 = 0,$$

$$A_2 = a_2 + r_2 = b_1 - a_1 = b_1 + r_1 = B_1,$$

$$A_3 = a_3 + r_3 = b_1 + b_2 - a_2 - a_1 = b_2 + r_2 = B_2,$$

$$\cdots\cdots\cdots$$

$$A_{s-2} = a_{s-2} + r_{s-2} = a_s + a_{s-1} + a_{s-2} - b_{s-2} - b_{s-1} - b_s = b_{s-3} + r_{s-3} = B_{s-3},$$

$$A_{s-1} = a_{s-1} + r_{s-1} = a_s + a_{s-1} - b_{s-1} - b_s = b_{s-2} + r_{s-2} = B_{s-2},$$

$$A_s = a_s + r_s = a_s - b_s = b_{s-1} + r_{s-1} = B_{s-1},$$

$$0 = b_s + r_s = B_s,$$

where the set $(A_1, \cdots, A_s; B_1, \cdots, B_s)$ is obviously trivial.

THEOREM 10. *If a set $(a_1, \cdots, a_s; b_1, \cdots, b_s)$ of degree 2 is projective with a set $(x_1, \cdots, x_s; y_1, \cdots, y_s)$ of degree 2, then $(a_1+x_1, \cdots, a_s+x_s; b_1+y_1, \cdots, b_s+y_s)$ is a set of degree 2.*

PROOF. By hypothesis, $x_i = a_i + r_i$, $y_i = b_i + r_i$ for $i = 1, \cdots, s$, where

$$\begin{aligned} 0 = \sum r_i(a_i - b_i) &= \sum a_i r_i - \sum b_i r_i \\ &= \sum a_i(x_i - a_i) - \sum b_i(y_i - b_i) \\ &= \sum a_i x_i - \sum b_i y_i, \end{aligned}$$

from which the theorem follows.

The transformations thus far introduced by which we may obtain solutions from a given solution of (1) have been symmetrical in the a_i and also in the b_i. We now show a transformation which affects h of the b's and $h+2$ of the a's and leaves the others unaltered and which preserves the relation that $(a_i; b_i)$ be a set of degree 2, in view of

THEOREM 11. *If $(a_1, \cdots, a_s; b_1, \cdots, b_s)$ is a set of degree 2, and if, by definition,*

$$(T_h) \begin{cases} A_i = S - a_i \quad (i = 1, \cdots, h+2), & B_i = S - b_i, \quad (i = 1, \cdots, h) \\ A_i = a_i \quad (i = h+3, \cdots, s), & B_i = b_i, \quad (i = h+1, \cdots, s) \\ S = a_1 + a_2 + \cdots + a_{h+2} - b_1 - b_2 - \cdots - b_h, & h \leqq s-2, \end{cases}$$

then $(A_1, \cdots, A_s; B_1, \cdots, B_s)$ is a set of degree 2.

THEOREM 12. *Any transformation T_h is its own inverse.*

These last two theorems are very easily established and the proof will be omitted. Their usefulness comes from the fact that by T_h, trivial sets of degree 2 may readily be transformed into non-trivial sets. For example, if we apply T_1 to the set (5*, 3*, 2*, 1; 1*, 2, 3, 5), where the elements to be affected are starred, we have $S=5+3+2-1=9$, and the set becomes (4, 6, 7, 1; 8, 2, 3, 5), a non-trivial set.

THEOREM 13. *By the transformations T_h and trivial transformations any existing non-trivial set of degree 2 may be transformed into a trivial set.*

This theorem follows if we can show that by the use of trivial transformations we may always satisfy the conditions

$$\left\{\begin{array}{l} b_1 > a_1 \geqq a_2 \geqq \cdots \geqq a_s, \\ b_1 > a_1 + \cdots + a_h > 0, \\ b_1 < a_1 + \cdots + a_h + a_{h+1} + a_{h+2}. \end{array}\right. \tag{5}$$

For we have

$$b_1 < a_1 + a_2 + \cdots + a_h + a_{h+1} + a_{h+2} < b_1 + a_{h+1} + a_{h+2}.$$

Let $b_2=b_3=\cdots=b_h=0$ and subtract b_1 from these inequalities to obtain

$$0 < S < a_{h+1} + a_{h+2};$$

hence we have

$$0 < hS < h(a_{h+1} + a_{h+2}) \leqq 2ha_{h+1} \leqq 2(a_1 + \cdots + a_h) < 2b_1,$$

or

$$hS - 2b_1 < 0.$$

Now

$$\sum_1^s B_i^2 = hS^2 - 2S\sum_1^h b_i + \sum_1^s b_i^2$$
$$= hS^2 - 2Sb_1 + \sum_1^s b_i^2,$$

whence

$$\sum_1^s B_i^2 - \sum_1^s b_i^2 = hS^2 - 2Sb_1 = S(hS - 2b_1) < 0.$$

Thus, the norm of the B_i is less than the norm of the b_i. If the A_i, B_i do not form a trivial set we may cancel out any common terms and repeat the foregoing process, ultimately arriving at a trivial set.

To show that conditions (5) are always attainable we shall first choose the notation so that

$$\begin{array}{l} b_1 > a_1 \geqq a_2 \geqq \cdots \geqq a_p > 0 > a_{p+1} \geqq \cdots \geqq a_s, \\ b_1 \geqq b_2 \geqq \cdots \geqq b_q > 0 > b_{q+1} \geqq \cdots \geqq b_s, \end{array} \tag{6}$$

and consider the three cases: (I) $p=s$; (II) $p<s$, $a_s>b_s$; (III) $p<s$, $a_s<b_s$.

Case I. If $p=s$ all the a's are positive and b_1 is greater than any of the a_i. Let h be as large as possible to have (5_2) hold. Then

$$b_1 \leqq a_1 + a_2 + \cdots + a_h + a_{h+1}, \tag{7}$$

and the inequality (5_3),

$$b_1 < a_1 + a_2 + \cdots + a_h + a_{h+1} + a_{h+2}$$

is possible unless there are not enough of the a's to exceed b_1 in sum, that is, unless $a_1+a_2+\cdots+a_s \leqq b_1$. Now, since all a's are supposed positive this would lead to

$$a_1^2 + \cdots + a_s^2 \leqq (a_1 + \cdots + a_s)^2 \leqq b_1^2 \leqq b_1^2 + \cdots + b_s^2.$$

But, by hypothesis, the sum of the squares of the a's is equal to the sum of the squares of the b's so that the last line of relations can only hold with all signs read as "equals." This implies that only one of the a's and one of the b's is not zero, contrary to the hypothesis that the solution is non-trivial.

Case II. In this case $p<s$ and $a_s>b_s$, and we shall realize (5) at once if

$$a_1 + \cdots + a_p > b_1. \tag{8}$$

If

$$a_1 + \cdots + a_p \leqq b_1 \tag{9}$$

we shall realize (5) by changing all signs and changing the order of the terms if

$$b_s > a_s + a_{s-1} + \cdots + a_{p+1}. \tag{10}$$

If the relations

$$b_s \leqq a_s + a_{s-1} + \cdots + a_{p+1} \tag{11}$$

and (9) both held we should have

$$\begin{aligned}(a_1^2 + \cdots + a_p^2) &+ (a_{p+1}^2 + \cdots + a_s^2)\\ &\leqq (a_1 + \cdots + a_p)^2 + (a_{p+1} + \cdots + a_s)^2\\ &\leqq b_1^2 + b_s^2 \leqq b_1^2 + \cdots + b_s^2.\end{aligned}$$

Since the last member of this set of relations is equal to the first, it can hold only with all equality signs effective, which implies that the solution is trivial.

Case III. Here $p<s$ and $a_s<b_s$, and we may still obtain (5) if (8) holds. In the contrary case (9) holds and we may realize (5) by changing all signs, interchanging the a's with the b's and rearranging terms if we have

$$a_s > b_s + b_{s-1} + \cdots + b_{q+1}. \tag{12}$$

If

$$a_s \leqq b_s + b_{s-1} + \cdots + b_{q+1}, \tag{13}$$

we may conclude from (9), (13), and the obvious relation

$$a_{p+1} + a_{p+2} + \cdots + a_{s-1} \leqq 0 \leqq b_2 + b_3 + \cdots + b_q,$$

that

$$a_1 + a_2 + \cdots + a_s \leqq b_1 + b_2 + \cdots + b_s,$$

a relation in which the equality sign is known to hold. Hence the equality signs hold in both (9) and (13), and we have also $p=s-1$ and $q=1$. We may then, as a trivial transformation, make all terms positive and return the problem to Case I.

4. *General solutions.* General parametric solutions of (1) have been found for only a few special cases of k and s. L. E. Dickson [**12**] has proved that all integral sets $(x, y, z; u, v, w)$ of degree 2 are obtainable by adding a constant to each term of the set $(AD, AG+BD, BG; AG, AD+BG, BD)$, and the general set $(x, y, z, w; r, s, t, u)$ of degree 2 is found by adding a constant to each term of the set $(u+gGb, v, gQ+Gc, gG(a+b); u+gG(a+b), v+gGb, gQ, Gc)$, where a is prime to b, g is prime to c, $u=scQ+bT$, $v=rcQ-aT$, and $as+br=1$.

In view of Theorem 9, the problem of finding the general set $(a_1, \cdots, a_s; b_1, \cdots, b_s)$ of degree 2 reduces to that of finding an expression for the general solution of the equation

$$0 = r_1(c_1) + r_2(c_2 - c_1) + r_3(c_3 - c_2) + \cdots + r_{s-1}(c_{s-1} - c_{s-2}) + r_s(-c_s),$$

where the c's are arbitrary integers. Attack on the problem by this method with $s=3$ and $s=4$ will yield the solutions given by Dickson by other methods.

We note, in passing, that any solution of (1) with $k=2$ may be built up from trivial solutions by the use of the transformations T_h and trivial transformations, in view of Theorems 12 and 13.

Methods for finding all integral solutions of (1) with $k=3$ and $s=4$ have also been given by Dickson [**12**].

5. *Applications.* 1) *Calculation of logarithms.* Solutions of the Tarry-Escott problem at once furnish the means for forming rapidly converging series convenient for the computation of logarithms. Starting with the familiar series

$$(14)\qquad \log_{10}\frac{M}{N} = 2\log_{10} e\left[\frac{M-N}{M+N} + \frac{1}{3}\left(\frac{M-N}{M+N}\right)^3 + \frac{1}{5}\left(\frac{M-N}{M+N}\right)^5 + \cdots\right],$$

the first step is to replace M and N by polynomials in x whose zeros are all integers such that $(M-N)/(M+N)$ becomes a fraction whose numerator is constant, i.e., we need just such a pair of polynomials as we have been discussing.

In finding logarithms of integers, we need only to calculate the logarithms of prime integers; and furthermore if the M and N of (14) are taken as numbers whose difference is as small as convenient* (preferably unity), we can find the

* A table of such numbers for calculating logarithms of primes less than 100 was given by Huygens (Opera Varia, vol. 2, pp. 456–8) and has been improved and extended for primes less than 200 by Escott [**2**; p. 164].

logarithm of the largest prime factor in M or N in terms of the logarithms of the smaller factors and the infinite series (14). The following example is taken from Escott's work [**2**].

Consider the following two polynomials of degree three which differ only in their constant terms:

$$4M = (x+2)(x-1)^2 = x^3 - 3x + 2,$$
$$4N = (x-2)(x+1)^2 = x^3 - 3x - 2,$$

and take $x=28898$. Then

$$M = 5^2 11^2 17^2 37^2 71^2 = 2456245^2$$
$$N = 2^3 3^5 7\cdot 13^4 19^2 43 = 2456245^2 - 1,$$

and we find

$$\begin{aligned} 2\log 71 = {} & 3\log 2 + 5\log 3 - 2\log 5 + \log 7 - 2\log 11 \\ & + 4\log 13 - 2\log 17 + 2\log 19 - 2\log 37 + \log 43 \\ & + 2\log e\left[\frac{1}{12{,}066{,}279{,}000{,}049} + \cdots\right]. \end{aligned}$$

In computing log 71 by this relation, since the number given by the first term of the series is a decimal commencing with 14 zeros, it will not be necessary to use the series if logarithms are required to not more then 13 places, and only the first term of the series need be used if not more than 41 places are desired.

The above method is particularly useful in calculating to a large number of decimal places the logarithm of a number beyond the limits of existing tables [**13**]. Escott shows how it is possible to find, using the first term of the series obtained from two sixth degree polynomials, the logarithm of 43867 (a prime factor of the ninth Bernouillian number) correct to 72 places of decimals.

2) *Computation of* π. Escott [**14**] has pointed out a connection of the problem with the derivation of the following type of formulas for the computation of π:

$$\tan^{-1}\frac{a}{x+\alpha} + \tan^{-1}\frac{b}{x+\beta} + \cdots \equiv \tan^{-1}\frac{p}{X},$$

where X is a real polynomial whose degree equals the number of fractions in the left hand member.

Since

$$\tan^{-1}\frac{a}{y} = \frac{1}{2i}\log\frac{y+ai}{y-ai},$$

the above expression becomes

$$\frac{x+\alpha+ai}{x+\alpha-ai}\cdot\frac{x+\beta+bi}{x+\beta-bi}\cdot\cdots = \frac{X+pi}{X-pi}.$$

Thus the polynomials

$$(x+ai)(x+bi)(x-ai-bi) = x^3+(a^2+ab+b^2)x+ab(a+b)i$$
$$(x-ai)(x-bi)(x+ai+bi) = x^3+(a^2+ab+b^2)x-ab(a+b)i$$

give

$$\tan^{-1}\frac{a}{x}+\tan^{-1}\frac{b}{x}-\tan^{-1}\frac{a+b}{x}\equiv\tan^{-1}\frac{ab(a+b)}{x^3+(a^2+ab+b^2)x}.$$

3) *Certain reducible polynomials.* H. L. Dorwart and Oystein Ore [15] have shown that polynomials of the form

$$f(x) = a(x-x_1)(x-x_2)\cdots(x-x_n)\pm p$$

where $x_i\neq x_j$ and p is a prime, for $n>6$, are irreducible if n is odd, and if n is even, they may have only two factors of degree $n/2$. Furthermore, Dorwart [16] has shown that the conditions for reducibility of these polynomials are essentially equivalent to finding ideal solutions of (1), with $a_i\neq a_j$ and $b_i\neq b_j$. For example

$$1, 5, 8, 12 \stackrel{3}{=} 2, 3, 10, 11$$

gives rise to the identity

$$(x-1)(x-2)(x-3)(x-5)(x-8)(x-10)(x-11)(x-12)+179 =$$
$$[(x-1)(x-5)(x-8)(x-12)+179][(x-2)(x-3)(x-10)(x-11)-179].$$

4) *Polynomials of degree n taking $\pm N$ $2n$ times.* The maximum number of times that a polynomial $g(x)$ of degree n can take $\pm$ some particular value N is evidently $2n$, since the equations $g(x)=+N$ and $g(x)=-N$ cannot have more than $2n$ roots. Dorwart [16] has also shown that ideal sets of degree k are necessary conditions for such polynomials, e.g.

$$(x-1)(x-5)(x-8)(x-12)+90=(x-2)(x-3)(x-10)(x-11)-90.$$

5) *Multiple unit roots of certain equations.* Consider the equation

$$F(x) = x^{12}-x^{11}-x^{10}+x^8+x^5-x^3-x^2+x=0.$$

The conditions for 1 to be a root of multiplicity 4 of this equation are

$$F(1)=0,\qquad F'(1)=0,\qquad F''(1)=0,\qquad F'''(1)=0.$$

The first condition is evidently satisfied. The second one is

$$12-11-10+8+5-3-2+1=0$$

or

$$12+8+5+1=11+10+3+2.$$

If instead of forming $F''(1)$, we first multiply $F'(x)$ by x before differentiating

and then substitute 1 for x (this will merely introduce a zero root and will not affect the unit roots), we have for the third condition

$$12^2 + 8^2 + 5^2 + 1^2 = 11^2 + 10^2 + 3^2 + 2^2.$$

In like manner, the fourth condition becomes

$$12^3 + 8^3 + 5^3 + 1^3 = 11^3 + 10^3 + 3^3 + 2^3.$$

These conditions are satisfied because we have already seen that (12, 8, 5, 1; 11, 10, 3, 2) is an ideal set of degree 3.

In addition to pointing out the above, Escott [**8**] has also noted that his Theorem 5 can be used to form equations of the above type. Thus from the sequence (4), we have the following equations:

$$(x^6-x)(x^3-1) = x^9-x^6-x^4+x = 0,$$

$$(x^6-x)(x^3-1)(x^2-1) = x^{11}-x^9-x^8+x^4+x^3-x = 0,$$

$$(x^6-x)(x^3-1)(x^2-1)(x-1) = x^{12}-x^{11}-x^{10}+x^8+x^5-x^3-x^2+x = 0,$$

$$(x^6-x)(x^3-1)(x^2-1)(x-1)(x^7-1) = x^{19}-x^{18}-x^{17}+x^{15}+x^{11}-x^9-x^5+x^3+x^2-x = 0,$$

and so on, in which it will be noted that, except for the first one, the factors are of the form x^h-1.

6) *Connection with the Waring problem.* An interesting consequence of work by Moessner and Schulz [**7**] is the identity

$$2^8+12^8+18^8+19^8+29^8+35^8+39^8+x^8 = 3^8+9^8+20^8+21^8+26^8+37^8+38^8+x^8.$$

There are similar identities for the exponents 9 and 10. Since these relations show that there are infinitely many positive integers N which can be represented in at least two ways as the sum of k positive kth powers for $k=8, 9, 10$, they give some information concerning Hypothesis K of Hardy and Littlewood [**17**] in the Waring problem—viz., in their notation, there are infinitely many numbers N for which $r_{kk}(N) \geqq 2$, $k=8, 9, 10$.

7) *Functional equations.* The variety of problems to which solutions of the Tarry-Escott problem apply is illustrated by a paper by Joseph Grant [**18**] on doubly homogeneous functional equations.

Note by Editor. The reader's attention may be called to several recent papers on the Tarry-Escott problem in the *Quarterly Journal of Mathematics* (Oxford), two by E. M. Wright (vol. 7, 1936, pp. 43–5; vol. 8, 1937, pp. 48–50) and one by Burchnall and Chaundy (vol. 8, 1937, pp. 119–130). E. J. M.

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IDEAL SOLUTIONS OF THE TARRY-ESCOTT PROBLEM*

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1. *Introduction.* We may assume that the reader is acquainted with the foregoing article by Dorwart and Brown on the Tarry-Escott problem and shall plunge *in medias res.*

In the present paper we shall consider integral, non-trivial solutions of the system

$$a_1, \cdots, a_s \overset{k}{=} b_1, \cdots, b_s \tag{1}$$

with $s=k+1$. Since $s \geqq k+1$, such solutions are denoted as *ideal.* We shall obtain parametric solutions for all values of $k \leqq 7$. For $k>5$ few numerical examples were hitherto known.†

Two well-known theorems‡ will be employed so frequently that they are here stated as lemmas.

Lemma I. The system (1) implies every system

$$Ma_1 + K, \cdots, Ma_s + K \overset{k}{=} Mb_1 + K, \cdots, Mb_s + K. \tag{2}$$

Lemma II. The system (1) implies the system

$$a_1, \cdots, a_s, b_1 + h, \cdots, b_s + h \overset{k+1}{=} b_1, \cdots, b_s, a_1 + h, \cdots, a_s + h.$$

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* Received by the editors on March 26, 1937.

† Since this paper was written, correspondence with E. B. Escott has revealed that he has obtained similar (unpublished) results. These include cases for $k=4$, 5 of a type not herein considered, several numerical solutions for $k=6$, and a solution for $k=7$ which is essentially that given in section 8 of this paper. The reader may also compare the work of Burchnall and Chaundy: A type of magic square in Tarry's Problem, *Quarterly Journal of Mathematics*, vol. 8, 1937, pp. 119–130; note their use of the phrase "of the $k+1$st order" where we say "of the kth degree."

‡ Numbered 1 and 3 in the paper by Dorwart and Brown.

By virtue of Lemma I we shall speak of all systems (2) as *equivalent*. Then by the results of this paper we may state:

THEOREM 1. *There exists an infinite number of non-equivalent, ideal solutions of the system* (1) *for every value of* $k \leqq 7$.

Consider a solution of (1) with $\sum a_i = 0$, $(a_i, b_i) = 1$. Such solutions may be termed *reduced*. By the use of Lemma I every solution of (1) is equivalent to a reduced form. But no two distinct reduced forms are equivalent; else there exists a second such set in x_i, y_i, where $x_1, \cdots, x_s = Ma_1 + K, \cdots, Ma_s + K$ in some order with M and K rational, and similarly for the y's in terms of b's. Now $\sum a_i = \sum x_i = 0$, and $\sum b_i = \sum y_i = 0$. Hence $K = 0$, $M = \pm 1$ only.

2. *Theorem 1 for the second degree.* The general solution

$$AD + k,\ AG + BD + k,\ BG + k \overset{2}{=} AD + BG + k,\ AG + k,\ BD + k \tag{3}$$

is due to L. E. Dickson. However it suffices to invoke the special system

$$a,\ b,\ -a-b \overset{2}{=} a+b,\ -a,\ -b. \tag{3a}$$

The latter is reduced if a is prime to b. If there exists only a finite number of reduced solutions of (3a) there must exist among them a greatest finite value of a. Call this a_1. Since a is arbitrary it may always be taken prime to some b and $> a_1$. Thus there are an infinitude of reduced systems (3) and Theorem 1 follows immediately.

A similar argument holds for all the degrees herein treated.

3. *A parametric solution of the third degree.* We shall solve generally the system

$$a_1,\ a_2,\ -a_1,\ -a_2 \overset{3}{=} b_1,\ b_2,\ -b_1,\ -b_2. \tag{4}$$

To satisfy (4) we merely require

$$a_1^2 + a_2^2 = b_1^2 + b_2^2,$$

while the general solution of the latter equation is given by

$$\begin{cases} a_1 = p_1p_2 + p_3p_4, & b_1 = p_1p_2 - p_3p_4, \\ a_2 = p_1p_3 - p_2p_4, & b_2 = p_1p_3 + p_2p_4. \end{cases}$$

If we now apply Lemma I to (4) with K and M arbitrary integers we obtain the six-parameter solution

$$\begin{aligned} &Ma_1 + K,\ Ma_2 + K,\ -Ma_1 + K,\ -Ma_2 + K \\ &\qquad \overset{3}{=} Mb_1 + K,\ Mb_2 + K,\ -Mb_1 + K,\ -Mb_2 + K. \end{aligned} \tag{5}$$

However not all solutions are included in (5). We shall find another general expression by first considering rational solutions of the third degree.

4. *All rational solutions of the third degree.* If we take $k=3$ in (1), the application of Lemma I with $M=1$, $K=-\sum a_i/4$ yields

$$u_1,\ u_2,\ u_3,\ u_4 \overset{3}{=} v_1,\ v_2,\ v_3,\ v_4, \tag{6}$$

where $\sum u_i = \sum v_i = 0$. We can now justify the transformation

$$\begin{cases} u_1 = -x_1 + x_2 + x_3, & u_3 = x_1 + x_2 - x_3, \\ u_2 = x_1 - x_2 + x_3, & u_4 = -x_1 - x_2 - x_3. \end{cases}$$

In the identical manner express the v_i in terms of y_i. Equations (6) will then reduce to

$$x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2, \qquad x_1x_2x_3 = y_1y_2y_3. \tag{7}$$

Write

$$x_1 = my_1, \qquad x_2 = ny_2, \qquad y_3 = mnx_3,$$

where m and n are fractions in their lowest terms. If $m = \pm 1$ we obtain only trivial results, while $m=0$ returns the problem to (4), the general rational solution of which may be written

$$\begin{cases} a_1 = ru + y, & b_1 = ru - y, \\ a_2 = u - ry, & b_2 = u + ry. \end{cases}$$

Aside from this exceptional case, it is apparent from the symmetry of (7) that we may take $|m|$ and $|n| > 1$ without the loss of generality. The given transformation then resolves (7) into

$$(m^2 - 1)y_1^2 + (n^2 - 1)y_2^2 = (m^2n^2 - 1)x_3^2. \tag{8}$$

The general solution of (8) is known* if a single solution is given and the coefficients of the squared terms are integers. Hence write $m = m_1/m_2$ where the m_i are relatively prime and treat n similarly. Equation (8) becomes

$$(m_1^2 - m_2^2)n_2^2y_1^2 + (n_1^2 - n_2^2)m_2^2y_2^2 = (m_1^2n_1^2 - m_2^2n_2^2)x_3^2. \tag{9}$$

A rational solution of (9) is $(n, 1, 1)$. By inspection the coefficient of $x_3^2 > 0$, and the discriminant of (9) is negative. Hence the general solution of (8) is given by

$$y_1 = Mr, \qquad y_2 = Ms, \qquad x_3 = Mt,$$

$$\begin{cases} r = -(m_1^2 - m_2^2)n_1n_2U^2 - 2(n_1^2 - n_2^2)m_2^2UV + \dfrac{(n_1^2 - n_2^2)m_2^2n_1V^2}{n_2}, \\ s = (m_1^2 - m_2^2)n_2^2U^2 - 2(m_1^2 - m_2^2)n_1n_2UV - (n_1^2 - n_2^2)m_2^2V^2, \\ t = (m_1^2 - m_2^2)n_2^2U^2 + (n_1^2 - n_2^2)m_2^2V^2, \end{cases} \tag{10}$$

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* See L. E. Dickson, Introduction to the Theory of Numbers, University of Chicago Press, 1929, p. 44.

where M is rational and U, V are relatively prime integers. We can now use (10) to obtain integral solutions of (6) differing from (5) by taking M integral and $n_2=m_2=1$. Then

$$\begin{cases} a_1 = M(-mr+ns+t)+K, & b_1 = M(-r+s+mnt)+K, \\ a_2 = M(mr-ns+t)+K, & b_2 = M(r-s+mnt)+K, \\ a_3 = M(mr+ns-t)+K, & b_3 = M(r+s-mnt)+K, \\ a_4 = M(-mr-ns-t)+K, & b_4 = M(-r-s-mnt)+K, \end{cases}$$

where m and n are integers $\neq 0, \pm 1$ and

$$\begin{cases} r = -(m^2-1)nU^2 - 2(n^2-1)UV + (n^2-1)nV^2, \\ s = (m^2-1)U^2 - 2(m^2-1)nUV - (n^2-1)V^2, \\ t = (m^2-1)U^2 + (n^2-1)V^2. \end{cases}$$

5. *Solutions of the fourth degree.* (A). Conjecture a solution of the third degree of the form

$$a,\ a+2k,\ -a,\ -a-2k \overset{3}{=} b,\ k,\ -b,\ -k. \tag{11}$$

By the successive application of Lemma II with $h=2k$ and then Lemma I with $M=1/2$, $K=-k/2$, we find

$$a_1,\ a_2,\ \cdots,\ a_5 \overset{4}{=} -a_1,\ -a_2,\ \cdots,\ -a_5, \tag{12}$$

where $(a_i) = \left(\dfrac{-a-3k}{2},\ \dfrac{-b+k}{2},\ \dfrac{a-k}{2},\ \dfrac{b+k}{2},\ k\right)$. Now (11) reduces to

$$k^2 + 2(a+k)^2 = b^2,$$

while a solution of the latter relation is given by

$$b = m^2 + 2n^2, \qquad k = m^2 - 2n^2, \qquad a = -m^2 + 2mn + 2n^2.$$

Hence $(a_i) = (m^2,\ -2n^2,\ -m^2-mn+2n^2,\ -m^2+mn+2n^2,\ m^2-2n^2)$ in (12) will yield a two-parameter solution of the fourth degree.

(B). Similarly if we assume the existence of

$$-3k,\ -k,\ k,\ 3k \overset{3}{=} u,\ v,\ -u,\ -v, \tag{13}$$

we are again led to a solution of (12) with $(a_i) = (-2m^2-2n^2,\ -m^2+mn+2n^2,\ -3mn+n^2,\ m^2+3mn,\ 2m^2-mn-n^2)$. Further, inspection reveals that (13) ordinarily will yield solutions other than those derived in (5A) and conversely.

6. *Solutions of the fifth degree.* (A). Apply successively Lemma II to (13) with $h=m^2-4mn-n^2$ and then Lemma I with $M=2$ and $K=-m^2+4mn+n$. We get

$$a_1,\ a_2,\ a_3,\ -a_1,\ -a_2,\ -a_3 \overset{5}{=} b_1,\ b_2,\ b_3,\ -b_1,\ -b_2,\ -b_3, \tag{14}$$

with

$$\begin{cases}(a_i) = (-5m^2+4mn-3n^2,\ -3m^2+6mn+5n^2,\ -m^2-10mn-n^2),\\ (b_i) = (-5m^2+6mn+3n^2,\ -3m^2-4mn-5n^2,\ -m^2+10mn-n^2).\end{cases}$$

(B). The system (14) is equivalent to

$$a_1^2,\ a_2^2,\ a_3^2 \overset{2}{=} b_1^2,\ b_2^2,\ b_3^2. \tag{15}$$

We use the known general solution (3) with $k=0$ and write

$$\begin{cases}AD = a_1^2, & AG+BD = a_2^2, & BG = a_3^2,\\ AG = b_1^2, & AD+BG = b_2^2, & BD = b_3^2.\end{cases}$$

Then (15) becomes

$$b_1^2 + b_3^2 = a_2^2, \qquad a_1^2 + a_3^2 = b_2^2, \qquad a_1a_3 = b_1b_3. \tag{16}$$

Solving the first two equations of (16) by

$$\begin{cases}b_1 = m^2-n^2, & b_3 = 2mn, & a_2 = m^2+n^2,\\ a_1 = u^2-v^2, & a_3 = 2uv, & b_2 = u^2+v^2,\end{cases}$$

we are confronted with

$$mn(m^2-n^2) = uv(u^2-v^2).$$

A simple solution of the latter is found when $m=u$,

$$\begin{cases}m = q^2+qr+r^2, & n = q^2-r^2,\\ u = q^2+qr+r^2, & v = 2qr+r^2.\end{cases}$$

This yields a solution of (14) with

$$\begin{aligned}(a_i) = (&q^4+2q^3r-q^2r^2-2qr^3,\ 2q^4+2q^3r+q^2r^2+2qr^3+2r^4,\\ &4q^3r+6q^2r^2+6qr^3+2r^4),\\ (b_i) = (&2q^3r+5q^2r^2+2qr^3,\ q^4+2q^3r+7q^2r^2+6qr^3+2r^4,\\ &2q^4+2q^3r-2qr^3-2r^4).\end{aligned}$$

(C). If we apply Lemma I to (3), with $M=3$, $K=-(A+B)(G+D)-3k$, there results

$$a_1,\ a_2,\ a_3 \overset{2}{=} b_1,\ b_2,\ b_3, \tag{17}$$

where $\sum a_i=\sum b_i=0$, and hence $\sum a_i^4=\sum b_i^4$. We thus have at once a solution of (14) with

$$\begin{cases}(a_i) = (3AD+N,\ 3BG+N,\ 3AG+3BD+N),\\ (b_i) = (3AG+N,\ 3BD+N,\ 3AD+3BG+N),\\ N = -(A+B)(G+D).\end{cases}$$

7. *Solutions of the sixth degree.* (A). Assume a solution of (14) where

$$(a_i) = (a, a+2k, b), \qquad (b_i) = (k, 3k, b+2r). \tag{18}$$

This is equivalent to the system

$$\begin{cases} \text{(a)} & a^2 + (a+2k)^2 + b^2 = k^2 + (3k)^2 + (b+2r)^2, \\ \text{(b)} & a^4 + (a+2k)^4 + b^4 = k^4 + (3k)^4 + (b+2r)^4. \end{cases} \tag{19}$$

In consequence of (19a) we may write

$$a^2 + 2ak - 3k^2 = 2Mr, \; b = M - r, \; a + k = 2Q.$$

Then (19b) reduces to

$$M^2 - Mr + r^2 = 7k^2,$$

a solution of which is

$$\begin{cases} k = m^2 + mn + n^2, & M = -2m^2 + 2mn + 3n^2, \\ r = m^2 + 6mn + 2n^2; \end{cases} \tag{20}$$

whence (19a) becomes

$$2Q^2 = n(n+2m)(2n+3m)(4n-m).$$

If we now let

$$\begin{cases} Q = 2u_1u_2u_3u_4, & n = 2u_1^2, \quad n + 2m = 4u_2^2, \\ 2n + 3m = u_3^2, & 4n - m = u_4^2, \end{cases}$$

we obtain

$$9u_1^2 - 2u_2^2 = u_4^2, \qquad u_1^2 + 6u_2^2 = u_3^2. \tag{21}$$

The obvious solution of (21) of $(u_1, u_2) = (1, 2)$ leads to a non-trivial solution of (18). The use of Lemma II with $h = 2k$ then results in a solution equivalent to

$$x_1, x_2, \cdots, x_7 \overset{6}{=} -x_1, -x_2, \cdots, -x_7, \tag{22}$$

with $(x_i) = (-134, -75, -66, 8, 47, 87, 133)$.

Consider a set of positive integers (p_1, q_1, d_1) which satisfy the equation

$$p^4 + 220p^2q^2 + 4q^4 = 9d^2, \; (p, q, d) = 1. \tag{23}$$

We then obtain a second such set (p_2, q_2, d_2) by

$$\pm d_2 = 3d_1^4 - 7(4p_1q_1)^4, \qquad \pm p_2 = (p_1^4 - 4q_1^4)/3, \qquad q_2 = 2p_1q_1d_1. \tag{24}$$

Thus the solution $(1, 1, 5)$ yields $(p_2, q_2, d_2) = (1, 10, 83)$. But from any solution of (23) we may proceed to solve (21) by

$$\begin{cases} u_1 = (p^2 + 2q^2)/3, & u_4 = p^2 - 2q^2, \\ u_2 = 2pq, & u_3 = d. \end{cases}$$

We thus can find an infinite number of solutions of (22) by the recursion formula (24) which are readily shown to be distinct and reduced. Hence Theorem 1 follows for $k=6$ without the use of a parametric formula for this degree.

(B). As an alternative one may replace (18) by

$$(a_i) = (a, a+2k, b), \qquad (b_i) = (k, a+2r, a+2r+2k).$$

A treatment similar to (7A) will then yield new solutions of (22). One thus obtained is $(x_i) = (-11894, -6586, -5200, -121, 5893, 6001, 11907)$.

(C). To determine a parametric solution we return to a brief consideration of (6). From a rational solution of (7) we derive not only (6) but also $\sum_1^4 u_i^5 = -80x_1x_2x_3(x_1^2+x_2^2+x_3^2) = \sum_1^4 v^5$, and hence of the non-ideal

$$u_1, \cdots, u_4, -v_1, \cdots, -v_4 \overset{6}{=} v_1, \cdots, v_4, -u_1, \cdots, -u_4.$$

For ideal solutions set $u_1=0$, or $x_1=x_2+x_3$ and write $x_2=my_1$. Equations (7) may then be written as

$$(25) \qquad \begin{cases} \text{(a)} & (2m^2-1)y_1^2 = y_2^2 - 2y_2y_3/m + y_3^2, \\ \text{(b)} & x_3^2 + my_1x_3 - y_2y_3/m = 0. \end{cases}$$

Solve (25a) in terms of relatively prime integers r, s where

$$\begin{cases} y_1 = r^2 - 2rs/m + s^2, \\ y_2 = (m+1)r^2 - 2mrs - (m-1)s^2, \\ y_3 = mr^2 + 2(m-1)rs - \dfrac{(m^2+2m-2)s^2}{m}. \end{cases}$$

Now (25b) is satisfied if and only if

$$m^2y_1^2 + 4y_2y_3/m = u^2.$$

Using the values of the y's in (25a) we get

$$(26) \qquad a^2r^4 + br^3s + cr^2s^2 + drs^3 + es^4 = u^2,$$

where

$$\begin{cases} a = m+2, \quad b = -4(m^2+2)/m, \quad c = 2(m^4-12m^3+6m^2+4)/m^2, \\ d = -4(m^2-8m+6)/m, \quad e = (m^4+4m^3+4m^2-16m+8)/m^2. \end{cases}$$

A solution of (26) is

$$\begin{cases} u = ar^2 + prs + qs^2, & q = \dfrac{c}{2a} - \dfrac{b^2}{8a^3}, \\ p = b/2a, & r/s = (q^2-e)/(d-2pq). \end{cases}$$

Retracing our steps we find

$$\begin{cases} y_1 = k(4m^8+32m^7+96m^6+144m^5+145m^4+140m^3+113m^2+48m+7), \\ y_2 = k(4m^9+28m^8+60m^7+12m^6-113m^5-159m^4-105m^3-41m^2-9m-1), \\ y_3 = k(4m^9+40m^8+156m^7+296m^6+251m^5-14m^4-183m^3-120m^2-25m), \\ u = k(4m^9+40m^8+160m^7+304m^6+209m^5-154m^4-343m^3-222m^2-69m-10), \\ k=m^2-1, \qquad x_1=(u+my_1)/2, \qquad x_2=my_1, \qquad x_3=(u-my_1)/2. \end{cases}$$

Since m is rational we can now set $m = m_1/m_2$ and by the use of Lemma I obtain an integral two-parameter solution of the sixth degree.

8. *Solution of the seventh degree.* For simplicity we will inquire only into the integral possibilities of the system

$$a_1, \cdots, a_4, -a_1, \cdots, -a_4 \overset{7}{=} b_1, \cdots, b_4, -b_1, \cdots, -b_4.$$

This is, of course, equivalent to

$$a_1^k + a_2^k + a_3^k + a_4^k = b_1^k + b_2^k + b_3^k + b_4^k, \qquad (k = 2, 4, 6). \tag{27}$$

If we postulate an integral solution of

$$3w, 2w, w, -w, -2w, -3w \overset{5}{=} u, v, u+v, -u, -v, -u-v, \tag{28}$$

the application of Lemma II with $h=w$ results in the non-ideal

$$w, -3w, u+w, -u+w, v+w, -v+w, u+v+w, -u-v+w \overset{6}{=} 0, 4w, u, v, u+v, -u, -v, -u-v.$$

The use of Lemma II with $h=u, v$, or $u+v$ will now yield a solution of (27). Taking $h=u$ we find

$$u-7w, u-2v+w, 3u+w, 3u+2v+w \overset{k}{=} u+7w, u-2v-w, 3u-w, 3u+2v-w, \qquad (k = 2, 4, 6).$$

Now (28) reduces to

$$u^2 + uv + v^2 = 7w^2,$$

a parametric solution of which is afforded by (20). Hence we possess at once an integral, two-parameter solution of the 7th degree. A few new numerical examples for $k=2$, 4, or 6 are

$$7, 24, 25, 34 \overset{k}{=} 14, 15, 31, 32,$$

$$7, 31, 36, 50 \overset{k}{=} 18, 20, 41, 49,$$

$$9, 47, 49, 67 \overset{k}{=} 23, 31, 61, 63.$$

This method lends itself readily to small integer solutions. For $w=7$, 13, 19, 31 alone there are 21 distinct reduced solutions, all in integers $\leqq 313$.

THE MATHEMATICAL ASSOCIATION AND MATHEMATICS IN THE SECONDARY SCHOOL SYSTEM*

By AUBREY KEMPNER, University of Colorado

Please be patient with me if I speak seriously. The least the Association may expect from its President is that he give earnest thought to the general policies of the Association.

The Association was founded to look after the interests of collegiate mathematics. This purpose it has performed admirably. One chief source of strength in this work has been the close and happy affiliation with the Mathematical Society. About seventy percent of our membership is overlapping. Our liaison with research interests is thereby guaranteed and represents one of our most treasured assets.

However, tendencies of the last generation should have made it frightfully clear to all of us that the whole of mathematics, from the grades through high school, college, graduate school and beyond, form an indivisible unit. Whatever harms mathematics at one level, harms it at all levels; whatever benefits it at one level, benefits it at all levels.

I vividly remember my student days in Göttingen, so far in the past. One day Earle Hedrick asked me—as it seemed to me, from a blue sky—whether I had ever thought of going to America after I had my degree. He started writing letters, with the result that I was offered an instructorship at the University of Illinois. I was informed that I should teach Freshman and Sophomore mathematics—and, may I add, for quite a number of years I did no other teaching; not exactly an ideal training for research. Told that algebra would be among my courses, I remembered my first year algebra work under Frobenius in Berlin, and had visions of intensive review of group theory and Galois theory. I wrote to the department at Illinois, asking for textbooks for the algebra and other Freshman courses. This was my first introduction to American mathematical texts.

In my bewilderment and dismay, I turned to Hedrick. I still recall his drawling: "I'll give you a demonstration Freshman lecture to-morrow!" He did, and ended with: "And then the student will have 1 over 2 over 3 over 4 on the board, and will wonder what it is all about!" A quarter of a century is a long time! I have learned a few things, and forgotten many; but my feeling of amazement that that should be called university mathematics, has never left me, and, I firmly hope, never will.

And yet, in these five and twenty years, the mathematical level has still been constantly falling. The third half year of algebra is doomed in many parts of the country, and its absence from the high school curriculum has helped drag down our university standards. But this is, of course, only an indication of the

* From an after-dinner speech at the joint meeting of the Association and the Society at Pennsylvania State College, State College, Pennsylvania, September 9, 1937.

general trend. More than a few of our large state universities are at present forced to admit students without any high school mathematics; in some of them it is possible for a student to obtain his bachelor's degree or even his doctor's degree without ever having had any mathematics beyond work in the grades. Some of my friends, even mathematicians whose opinion I value, contemplate this situation without alarm, even with mild approval. I cannot help thinking that they are making a terrible mistake.

I am considering not only the cultural values involved—in the long run the most frightening aspect—but also the danger of having in a democracy the future rest upon the shoulders of men and women who have no introduction to a language which so superbly dominates our modern world.

In the whole vast secondary educational system, mathematics is fighting with its back to the wall. This is so generally recognized that it is nearly an insult to repeat it to an audience such as this. But the percentage of those among us who feel a deep personal responsibility for the situation is, I fear, very small.

A generation ago, it was entirely fitting and proper that mathematicians of this country should subordinate everything else to building up research; and research is, and will remain, the crowning glory of our mathematical structure. But now the future of mathematical research in this country is secure. For this, we have the testimony of leading European mathematicians.

Can we say as much of the foundations of mathematics, which have their very roots in our secondary schools?

It is a good sign that we are beginning to realize two facts:

First: We are dealing with a situation for which we mathematicians carry a greater share of blame than we have been ready to admit.

Second: It can be remedied only by well-directed concerted action of the mathematics teachers of the country, acting as a whole, grade school and high school teachers, principals, college and university teachers, and graduate faculties.

It is exactly in this respect that the Association can, in my opinion, at present perform its most valuable service.

The creation, a few years ago, of a joint commission of the National Council of Teachers of Mathematics and our Association on "The place of mathematics in the secondary schools," is a step in the right direction. The preliminary work done by this committee promises much of value.

The 1935 report on training of teachers is another. This report touches on the important question of the best preparation for high school teachers going on to their master's degree. (It is certain that we cannot indefinitely side-step the problem of clearly defining the purpose of a master's degree: whether it represents a goal in itself, or a stepping stone to the advanced degree, or merely a year's work in graduate mathematics; or a combination, or a selection, of these aims. One sometimes feels that a three-valued logic is required to deal adequately with this trilemma.) This report also has something to say about the pressing question of a doctor's degree in mathematics with less emphasis on re-

search, and more on preparation for inspired teaching by a comprehensive integration of knowledge and proper historical perspective.

The Association should interest itself in the problem of revising the absurd minimum requirement of the North Central Association for accrediting high schools: fifteen semester hours of mathematics, of which six may be counted on high school mathematics. This means that a teacher who has taken a short course in Freshman algebra, trigonometry, and analytic geometry is considered sufficiently prepared to lead students to the door of the university—if they take mathematics at all.

Do not tell me that this was intended as a minimum requirement that would in practice always be exceeded! In some parts of the country this may possibly be true. But I speak of my own knowledge when I say that in our Western Mountain States this minimum requirement is by many principals and superintendents considered an irksome maximum requirement, which they evade and whittle down wherever possible.

The Association should have an open mind concerning such fundamental questions as differentiation of mathematics courses in high school according to ability and purpose.

It seems to me that, up to a few years ago, a quite unreasonable amount of distrust and suspicion existed between university people and high school people. Common pressure in a common cause is beginning to drive us together. We look at each other, talk to each other, listen to each other, and discover that we really speak the same language.

The National Council of Teachers of Mathematics has approached the Association concerning the possibility of a joint session, and the Association sincerely hopes that such a meeting may be arranged.

When I asked Dean Stouffer to suggest means of strengthening our Rocky Mountain Section he told me that in Kansas they had found a common session of the Association and high school teachers the most valuable part of their regular program. Since then our Professor Hutchinson at Boulder has organized for our section such joint meetings, and we are much encouraged by the first results.

I notice that other sections are moving in the same direction. If you look at recent reports of sectional meetings in the MONTHLY, you will find in nearly every one of them several papers dealing with high school and pedagogical questions.

These are, if you like, all only straws in the wind! But it is a wind which is gathering strength and blowing steadily from one direction. To derive its full benefit, the Association must set its sails so as to secure the fullest cooperation of all progressive elements in the secondary school systems.

To accomplish this, determined efforts should be made to encourage them to join the Association. I said that about seventy percent of the members of the Association and of the Society overlap. But how is it, when we glance at the enrollment in the Association of high school teachers, principals, and superintendents? We find that of about two thousand members, roughly a hundred and twenty, that is, about five to six percent, belong to these groups. If we

could have five hundred, instead of one hundred, of the best high school teachers and officials actively interested in our organization, we might begin to feel that we speak for the mathematical interests of the country as a whole.

Nor should we have to fear a lowering of our standards! The best high school teachers are very superior people, and would represent a decided strengthening of the Association.

In no other way can the Association hope to acquire a position of leadership in all mathematical educational questions—a position it should aspire to—because high school teachers will, with perfect justification, never recognize the authority in their own field of an organization in which they have only negligible representation.

In closing, permit me to say that, however critical and pessimistic I may sound, I am really, at the bottom of my heart, profoundly optimistic. It would be quite unfair to attempt to compare our situation with those prevailing in European countries. Our difficulties are unavoidable. They are the inescapable consequences of the most ambitious undertaking ever started—the educating of a population of far over a hundred million to a general level unheard of in the history of mankind.

A METHOD FOR THE SOLUTION OF POLYNOMIAL EQUATIONS

By F. H. STEEN, Georgia School of Technology

1. *Introduction.* This paper presents a method of finding a fractional approximation to a real root of a rational integral equation with numerical coefficients. The method usually requires less computation than the customary processes for "extraction" of square root and cube root, and less than "Horner's Method" of solving equations, and is better adapted to the use of computing machines. It is of particular value when a high degree of accuracy is required in the approximation.

2. *General method.* Let $P_n(x)=a_0+a_1x+\cdots+a_nx^n$ be a polynomial with real, numerical coefficients. Form the product

$$\begin{vmatrix} 1 & x & x^2 & x^3 & \cdots & x^r \\ a_2 & a_1 & a_0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & a_0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ a_{r+1} & a_r & a_{r-1} & a_{r-2} & \cdots & a_1 \end{vmatrix} \cdot P_n(x),$$

where r is a positive integer. If $P_n(x)$ in its expanded form is multiplied into each of the elements of the first row of the determinant and the result expanded in powers of x, the product takes on the form

$$a_0D_{r-1,r} + D_{r,r+1}x + (-1)^r \sum_{k=2}^{n} E_{r+1,r+k}x^{r+k},$$

where

$$D_{p,s} = \begin{vmatrix} a_1 & a_0 & 0 & 0 & \cdots 0 \\ a_2 & a_1 & a_0 & 0 & \cdots 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_p & a_{p-1} & a_{p-2} & a_{p-3} & \cdots a_0 \\ a_s & a_{s-1} & a_{s-2} & a_{s-3} & \cdots a_{s-p} \end{vmatrix},$$

$$E_{p,s} = \begin{vmatrix} a_2 & a_1 & a_0 & 0 & \cdots 0 \\ a_3 & a_2 & a_1 & a_0 & \cdots 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_p & a_{p-1} & a_{p-2} & a_{p-3} & \cdots a_1 \\ a_s & a_{s-1} & a_{s-2} & a_{s-3} & \cdots a_{s-p+1} \end{vmatrix}$$

If $P_n(x)$ vanishes at x_0, the product also vanishes there. Then

$$x_0 = -a_0(D_{r-1,r}/D_{r,r+1}) + R/D_{r,r+1},$$

where

$$R = (-1)^{r+1} \sum_{k=2}^{n} E_{r+1,r+k}x_0^{r+k}.*$$

If x_0 is near zero, x_0^{r+k} will be small.

The determinants $D_{1,2}, D_{2,3}, D_{3,4}, \cdots$ can be evaluated successively by a simple procedure. Suppose $D_{1,2}, D_{2,3}, \cdots, D_{p-1,p}$ have been computed. If the determinant $D_{p,p+1}$ is expanded by minors of the last row, the result is

$$D_{p,p+1} = a_{p+1}(-a_0)^p + a_pD_{0,1}(-a_0)^{p-1} + a_{p-1}D_{1,2}(-a_0)^{p-2} + \cdots + a_1D_{p-1,p},$$

where $D_{0,1}=a_1$. Since the right member is simply the value of the polynomial

$$a_{p+1}y^p + a_pD_{0,1}y^{p-1} + a_{p-1}D_{1,2}y^{p-2} + \cdots + a_1D_{p-1,p}$$

for $y=-a_0$, it follows that $D_{p,p+1}$ can be obtained by dividing this polynomial synthetically by $y+a_0$. The coefficients $a_{p+1}, a_pD_{0,1}, \cdots, a_1D_{p-1,p}$ involve only

* The method described in this paper and also the particular development of the theory are believed by the author to be new, but it has been pointed out by the referee that the approximation term here denoted by $-a_0(D_{r-1,r}/D_{r,r+1})$ with a somewhat different form for the remainder, is derived and discussed in K. A. Poukka, Über die Berechnung der Wurzeln einer algebraischen oder transcendenten Gleichung, #5 in the Festschrift: Commentationes in Honorem Ernesti Leonardi Lindelöf, 1929. This approximation term was obtained earlier by E. T. Whittaker, being the sum of the first $(r+1)$ terms of his series expansion for the root of an equation. See Whittaker and Robinson, The Calculus of Observations, Blackie and Son, London and Glasgow, Second Edition, 1929, p. 120. (From Proceedings of the Edinburgh Mathematical Society, vol. 36, 1918, p. 103.)

known quantities. To obtain these coefficients most easily, the numbers $a_1, a_2, a_3, \cdots, a_n, 0, \cdots$ should be written in a row, and then $D_{0,1}$, $D_{1,2}$, $D_{2,3}, \cdots$ one by one, as they are computed, in a second row directly below the first. For example, suppose $D_{1,2}$ and $D_{2,3}$ have been computed by synthetic division and entered into the proper cells ($D_{0,1}=a_1$ can be entered immediately):

a_1	a_2	a_3	a_4	a_5	$\cdots$	a_n	0	0	$\cdots$
$D_{0,1}$	$D_{1,2}$	$D_{2,3}$							

Since

$$D_{3,4} = a_4(-a_0)^3 + a_3D_{0,1}(-a_0)^2 + a_2D_{1,2}(-a_0) + a_1D_{2,3},$$

the coefficients in the synthetic division for computing $D_{3,4}$ are a_4, $a_3D_{0,1}$, $a_2D_{1,2}$, and $a_1D_{2,3}$.

The simplicity of the method is best demonstrated by examples.

3. *Quadratic equations.* When $n=2$, we have $a_k=0$ for $k>2$, and hence

$$R = (-1)^{r+1}E_{r+1,r+2}x_0^{r+2} = (-1)^{r+1}a_2^{r+1}x_0^{r+2},$$

which reduces to $(-1)^{r+1}x_0^{r+2}$ if the leading coefficient is unity.

For an example, let it be required to find the square root of 39, i.e., a solution of $x^2-39=0$, approximately. The root is about 6 and diminishing the roots by 6 we obtain $x^2+12x-3=0$. The root of this equation is about 1/4. Multiplying the roots by 4 and diminishing by 1 we obtain $x^2+50x+1=0$.*

We now apply the method, carrying it as far as $r=3$, using the third and fourth cells of the second row for the approximation to x_0.

We first enter $a_1=50$, $a_2=1$, $a_3=0, \cdots$ in successive cells in a horizontal row and $D_{0,1}=a_1=50$ in the cell under $a_1=50$ as shown in the chart below. To obtain $D_{1,2}$ we copy the numbers 1 and $(50)(50)=2500$ in this order and apply synthetic division, using $-a_0=-1$ as "divisor":

$$\begin{array}{rr|l} 1 & 2500 & \underline{-1} \\ & -1 & \\ \hline 1 & 2499 = D_{1,2} & \end{array}$$

The value (2499) of $D_{1,2}$ is entered in the cell to the right of $D_{0,1}=50$ in the second row. To obtain $D_{2,3}$ we copy the numbers 0, (1)(50), and (50)(2499) $=124,950$, and again apply synthetic division with -1 as divisor.† The result, $D_{2,3}=124,900$ is entered beside $D_{1,2}=2499$. The coefficients in the next synthetic

* The first transformation would have been sufficient, but the additional ones lead to a more accurate result and do not appreciably complicate the equation.

† Since the zero has no effect on the result it can be omitted.

division are 0, (0)(50), (1)(2499), and (50)(124,900) = 6,245,000, and the process gives $D_{3,4} = 6,242,501$. At this stage the chart has the appearance:

50	1	0	0
50	**2499**	**124,900**	**6,242,501**

If the numbers already obtained do not give sufficiently accurate results the method can be continued. We now have

$$x_0 = -\frac{a_0 D_{2,3}}{D_{3,4}} + \frac{R}{D_{3,4}} = -\frac{124,900}{6,242,501} + \frac{(-1)^4}{6,242,501} x_0^5.$$

Since x_0 is approximately $-.02$, the last term is less than 10^{-15} and hence $-124,900/6,242,501$ gives x_0 to at least fourteen significant figures. We now substitute the value of x_0, computed by means of this fraction to five significant figures, into the remainder term. Using a five place table of logarithms we easily obtain the first five figures of R. Then

$$6 + [1 - (124,900/6,242,501) + R]/4$$

will give $(39)^{1/2}$ to twenty-two significant figures.* Twenty-seven figures can be obtained by using a ten place table of logarithms with x_0 to ten places.

If a less accurate approximation is sufficient we simply use the first transformed equation, $x^2 + 12x - 3 = 0$, and apply synthetic division twice. The resulting chart is:

12	1	0	0
12	**147**	**1800**	

and this gives

$$x_0 = \frac{(3)(147)}{1800} - \frac{x_0^4}{1800} = \frac{49}{200} - \frac{x_0^4}{1800}.$$

Since the last term is less than 10^{-5}, we have $(39)^{1/2} = 6 + 49/200 = 6.24500$ to six significant figures. Computing $R = (.24500)^4/1800$ by means of five place logarithms we obtain $R = (2.0018)(10^{-6})$. Subtraction of this number from .24500 gives $(39)^{1/2} = 6.244,997,998,2$ correct to ten significant figures.†

4. *Cubic equations*. When dealing with a cubic equation, R consists of the two determinants $E_{r+1,r+2}$ and $E_{r+1,r+3}$. If these are written out (with $a_4 = a_5 = \cdots = 0$) it will be seen that they are of the same type as $D_{r,r+1}$ and

* Even though long division "by hand" be required to convert the first term of x_0 to a decimal, the method is much less laborious than the usual square root method. The elementary school device of writing down the first nine multiples of the denominator before beginning the division will be found useful.

† The last digit is in doubt because interpolation was used in computing the logarithms. The actual value is 4.

$D_{r-1,r}$ respectively, with the roles of a_0, a_1, a_2, a_3 simply reversed. Consequently these determinants can also be obtained by the synthetic division process with $-a_3$ taking the place of $-a_0$ as "divisor," a_2 the place of a_1, and so on.

For an example we use Wallis's equation, $x^3-2x-5=0$ which has a root near 2. Diminishing the roots by 2 we obtain the reduced equation $x^3+6x^2+10x-1=0$. Upon applying the synthetic division process three times ($r=3$) with $-a_0=1$ as "divisor" we obtain:

10	6	1	0
10	**106**	**1121**	**11,856**

$$x_0 = \frac{1121}{11,856} + \frac{R}{11,856} = \frac{59}{624} + \frac{R}{11,856}, \quad R = (-1)^4(E_{4,5}x_0^5 + E_{4,6}x_0^6).$$

A repetition of the process with the roles of a_0, a_1, a_2, a_3 simply reversed (hence using $-a_3=-1$ as "divisor") gives $E_{4,5}$ and $E_{4,6}$ in the fourth and third cells respectively:

6	10	-1	0
6	**26**	**95**	**304**

Thus $E_{4,5}=304$ and $E_{4,6}=95$ and therefore

$$x_0 = \frac{59}{624} + \frac{304}{11,856}x_0^5 + \frac{95}{11,856}x_0^6 = \frac{59}{624} + (304 + 95x_0)\frac{x_0^5}{11,856}\cdot$$

Since x_0 is less than $1/10$, the remainder term is less than $(3)(10^{-7})$ and hence the root of the original equation is $2+59/624=2.094551$ to seven significant figures. The value of x_0 correct to five places can now be substituted in R to give

$$R = [304 + (95)(.094551)]\frac{(.094551)^5}{11,856} = (1.9948)(10^{-7})$$

the last result being obtained by logarithms. When this number is added to $(59/624)=.09455128205$, we get, 2.09455148153, which gives the first eleven significant figures of the required root.

5. *Equations of higher degree.* For equations of degree higher than three, one procedure after finding $D_{r-1,r}$ and $D_{r,r+1}$ is to evaluate the first determinant of R by ordinary expansion methods. This will give the value of R approximately since the remaining terms are of higher order in x_0. Or, the $(r+1)$th order determinants of R can be expressed as second order determinants by a simple method which will now be described.

Writing $E_{r+1,r+k}$, ($k=2, 3, \cdots, n$), as a linear combination of $E_{r+1,0}$, $E_{r+1,1}, \cdots, E_{r+1,r}$ with undetermined constants of combination H, we have

$$E_{r+1,r+k} = H^{(k)}_{r+1,0}E_{r+1,0} + H^{(k)}_{r+1,1}E_{r+1,1} + \cdots + H^{(k)}_{r+1,r}E_{r+1,r},$$

which is equivalent to the $(r+1)$ equations in the $(r+1)$ unknowns $H^{(k)}_{r+1,\nu}$

$$a_{r+k-\nu} = H^{(k)}_{r+1,\nu}a_0 + H^{(k)}_{r+1,\nu+1}a_1 + \cdots + H^{(k)}_{r+1,\nu+r}a_r, \qquad (\nu = 0, 1, \cdots, r),$$

where $H^{(k)}_{r+1,\mu}=0$ for $\mu=r+1, r+2, \cdots$. These give

$$H^{(k)}_{r+1,0} = \frac{(-1)^r}{a_0^{r+1}} D_{r,r+k}; \qquad H^{(k)}_{r+1,1} = \frac{(-1)^{r+1}}{a_0^r} D_{r-1,r+k-1}.$$

Since $E_{r+1,0}=(-1)^rD_{r-1,r}a_0$, $E_{r+1,1}=(-1)^rD_{r,r+1}$, and $E_{r+1,2}=E_{r+1,3}=\cdots=E_{r+1,r}=0$, we get

$$E_{r+1,r+k} = \frac{(-1)^r}{a_0^{r+1}} D_{r,r+k}(-1)^rD_{r-1,r}a_0 + \frac{(-1)^{r+1}}{a_0^r} D_{r-1,r+k-1}(-1)^rD_{r,r+1}$$

$$\text{(A)} \qquad = \frac{1}{a_0{}^r}\begin{vmatrix} D_{r-1,r} & D_{r,r+1} \\ D_{r-1,r+k-1} & D_{r,r+k}\end{vmatrix}, \qquad (k = 2, 3, \cdots, n).$$

Expansion of $D_{r,r+k}$ by minors of its last row gives

$$D_{r,r+k} = a_{r+k}(-a_0)^r + a_{r+k-1}D_{0,1}(-a_0)^{r-1} + \cdots + a_kD_{r-1,r}.$$

By comparing this result with the expansion

$$D_{r+k-1,r+k} = a_{r+k}(-a_0)^{r+k-1} + a_{r+k-1}D_{0,1}(-a_0)^{r+k-2} + \cdots + a_1D_{r+k-2,r+k-1},$$

it is seen that $D_{r,r+k}$ is simply the $(r+1)$th coefficient in the "quotient" obtained in the synthetic division process which gives $D_{r+k-1,r+k}$ and hence no additional process is required for its computation.

The coefficients referred to should be written in cells diagonally downward and to the left, beginning as before with the last coefficient. Suppose $D_{1,2}$ and $D_{2,3}$ have been computed. The following outline shows the next synthetic division and the arrangement of the results. The first two rows of the chart are the same as in those discussed above.

$$\begin{array}{cccc|l} a_4 & a_3D_{0,1} & a_2D_{1,2} & a_1D_{2,3} & -a_0 \\ & \underline{\quad} & \underline{\quad} & \underline{\quad} & \\ \hline \mathbf{D}_{0,4} & \mathbf{D}_{1,4} & \mathbf{D}_{2,4} & \mathbf{D}_{3,4} & \end{array}$$

a_1	a_2	a_3	a_4	a_5
$D_{0,1}$	$D_{1,2}$	$D_{2,3}$	$\mathbf{D}_{3,4}$	
$D_{0,2}$	$D_{1,3}$	$\mathbf{D}_{2,4}$		
$D_{0,3}$	$\mathbf{D}_{1,4}$			
$\mathbf{D}_{0,4}$				

All of the elements of the determinants of R given by (A) corresponding to the root approximation $-a_0(D_{r-1,r}/D_{r,r+1})$ are in the two adjacent columns which contain $D_{r-1,r}$ and $D_{r,r+1}$ and consequently the determinants (A) can be copied directly from the chart.* For example, suppose we are using $-a_0(D_{1,2}/D_{2,3})$ as a root approximation. Then the determinants of (A) are

$$\begin{vmatrix} D_{12} & D_{23} \\ D_{13} & D_{24} \end{vmatrix}, \qquad \begin{vmatrix} D_{12} & D_{23} \\ D_{14} & D_{25} \end{vmatrix}, \qquad \cdots$$

the elements of which are obtained from the a_2 and a_3 columns.

As an example, let it be required to find the positive fourth root of 18, i.e. a solution of $x^4-18=0$, approximately. Diminishing the roots by 2 we obtain $x^4+8x^3+24x^2+32x-2=0$. An outline of the method, carried to $r=2$, follows.

$$\begin{array}{rr|l} 24 & 1024 & \underline{2} \\ & 48 & \\ \hline 24 & 1072 & \end{array} \qquad \begin{array}{rrr|l} 8 & 768 & 34,304 & \underline{2} \\ & 16 & 1,568 & \\ \hline 8 & 784 & 35,872 & \end{array} \qquad \begin{array}{rrrr|l} 1 & 256 & 25,728 & \text{——} & \underline{2} \\ & 2 & 516 & \text{——} & \\ \hline 1 & 258 & 26,244 & \text{——} & \end{array}$$

$$\begin{array}{rrrr|l} 32 & 8576 & \text{——} & \text{——} & \underline{2} \\ & 64 & \text{——} & \text{——} & \\ \hline 32 & 8640 & \text{——} & \text{——} & \end{array}$$

32	24	8	1	0
32	1072	35,872		
24	784	26,244		
8	258	8640		
1	32	1072		

At this stage the chart gives

$$\begin{aligned} x_0 &= 2\,\frac{1072}{35,872} + \frac{\begin{vmatrix} 1072 & 35,872 \\ 784 & 26,244 \end{vmatrix}}{(-2)^2(35,872)}\,x_0 \\ &\quad + \frac{\begin{vmatrix} 1072 & 35,872 \\ 258 & 8640 \end{vmatrix}}{(-2)^2(35,872)}\,x_0^5 + \frac{\begin{vmatrix} 1072 & 35,872 \\ 32 & 1072 \end{vmatrix}}{(-2)^2(35,872)}\,x_0^6 \\ &= \frac{67}{1121} - \frac{155}{2242}\,x_0^4 - \frac{111}{2242}\,x_0^5 - \frac{10}{1121}\,x_0^6 \\ &= .059,768,064,23 \cdots + R. \end{aligned}$$

* To complete a particular column it is, of course, unnecessary to carry out all of the synthetic divisions completely.

Since $x_0=.06$ to one significant figure, $(155+111x_0+20x_0^2)$ is approximately 160. Hence R is approximately $(160/2200)x_0^4=.000,001, \cdots$ and therefore $(18)^{1/4}=2.059,767$ to seven figures. Using x_0 to five places, $(155+111x_0+20x_0^2)=161.71$, and the value of R computed by logarithms is $(9.2036)(10^{-7})$. This leads to $(18)^{1/4}=2.059,767,143,87$ with eleven significant figures.

6. *Conclusion.* In finding the approximate value of a root of an equation the best procedure is to diminish all the roots so that the root under consideration is between zero and 1/2. Next, if the root is near 1/2, 1/3, or 1/4, multiply all the roots, by 2, 3, or 4 respectively and diminish by 1.* It is then a relatively simple matter, using the method described, to obtain a rough or a good approximation to the root of the given polynomial equation.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions, and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

RAPID METHOD FOR EXTRACTING A SQUARE ROOT

By E. B. ESCOTT, Oak Park, Illinois

The following method for extracting square root is probably the most rapid method yet discovered. As it gives the square root in the form of an infinite product, it is especially well adapted for use with a computing machine. In computing a table of square roots by the method of differences it is important to have an independent method of computing an occasional value and this method is very good for that purpose.

From the identity

$$\frac{(x+1)^2(x-2)}{(x-1)^2(x+2)}=\frac{x^3-3x-2}{x^3-3x+2} \tag{1}$$

I get

$$\frac{x+2}{x-2}\bigg/\frac{x^3-3x+2}{x^3-3x-2}=\left(\frac{x+1}{x-1}\right)^2. \tag{2}$$

Replacing x by x_n and putting

$$x_{n+1} = x_n^3 - 3x_n$$

* This process gives numbers with fewer digits than does a simple repetition of root diminishing and at the same time hastens the convergence of $-a_0(D_{r-1,r}/D_{r,r+1})$ to the root. If the location of the root is not known well enough, first apply the method to this equation to get a rough approximation. This involves very little calculation since the value of R is not required.

the identity (2) gives, by putting $n=1, 2, 3, \cdots$, the series of identities

$$\frac{x_1+2}{x_1-2}\Big/\frac{x_2+2}{x_2-2}=\left(\frac{x_1+1}{x_1-1}\right)^2,$$
$$\frac{x_2+2}{x_2-2}\Big/\frac{x_3+2}{x_3-2}=\left(\frac{x_2+1}{x_2-1}\right)^2,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\frac{x_n+2}{x_n-2}\Big/\frac{x_{n+1}+2}{x_{n+1}-2}=\left(\frac{x_n+1}{x_n-1}\right)^2.$$

Multiplying the terms on each side and extracting the square root, we obtain

$$\text{(3)}\qquad \sqrt{\frac{x_1+2}{x_1-2}\Big/\frac{x_{n+1}+2}{x_{n+1}-2}}=\left(\frac{x_1+1}{x_1-1}\right)\left(\frac{x_2+1}{x_2-1}\right)\cdots\left(\frac{x_n+1}{x_n-1}\right).$$

As n becomes infinite the values of $(x_{n+1}+2)/(x_{n+1}-2)$ and $(x_n+1)/(x_n-1)$ approach the value 1 if $x_1>2$, so that the limiting form of (3) is

$$\text{(4)}\qquad \sqrt{\frac{x_1+2}{x_1-2}}=\frac{x_1+1}{x_1-1}\cdot\frac{x_2+1}{x_2-1}\cdot\frac{x_3+1}{x_3-1}\cdots$$
$$=\left(1+\frac{2}{x_1-1}\right)\left(1+\frac{2}{x_2-1}\right)\left(1+\frac{2}{x_3-1}\right)\cdots,$$

where $x_{n+1}=x_n(x_n^2-3)$.

Example 1. Let $x_1=3$; then $x_2=3(3^2-3)=18$; $x_3=18(18^2-3)=5778$; $x_4=5778\ (5778^2-3)=192{,}900{,}153{,}618$; $\cdots$. We obtain

$$\sqrt{5}=2(1+2/17)(1+2/5777)(1+2/192{,}900{,}153{,}617)\cdots$$
$$=2.23606\ 79774\ 99789\ 69640\ 99012\ 08267\ 97,$$

true to at least 30 decimals.

Example 2. Let $x_1=7$; then $x_2=322$; $x_3=33{,}385{,}282$; and $x_4=$ a number of 24 digits. From an inversion of (4) we now obtain

$$\sqrt{5}=3(1-1/4)(1-2/323)(1-2/33{,}385{,}283)\cdots.$$

Example 3. Let $x_1=198$; then $x_2=7{,}761{,}798$; $\cdots$. Hence

$$\sqrt{2}=7/5(1+2/197)(1+2/7{,}761{,}797)\cdots$$
$$=1.41421\ 35623\ 730.$$

Example 4. Let $x_1=10$; then $x_2=970$; $x_3=912{,}670{,}090$; $\cdots$. Hence

$$\sqrt{6} = 2(1 + 2/9)(1 + 2/969)(1 + 2/912,670,089) \cdots$$
$$= 2.44948\,97427\,831780,$$

true to 16 decimals.

Slide-rule Solutions of Quadratic and Cubic Equations

By T. J. Higgins, Purdue University

It is not generally known that the real roots of quadratic and cubic equations can be determined from one or two settings of an ordinary slide-rule. I believe that there is no American publication that contains this information. Yet an understanding of such solutions would be of real value to those who must frequently solve such equations, and to whom quickness, accuracy, and release from laborious computation are prime considerations. In particular, students of engineering, the natural sciences, and those engaged in statistical and actuarial studies find these solutions especially useful. Not only are they independent modes of obtaining roots, but they may be used to check the solutions obtained by the use of Horner's Method, Newton's Method, the usual formulas, or other conventional means.

1. *Quadratic equations.* Any quadratic equation can be put in the form

$$x^2 + px + q = 0. \tag{1}$$

Designating the roots of (1) by x_1 and x_2, we have

$$x_1 + x_2 = -p, \qquad x_1x_2 = q. \tag{2}$$

Consider the following slide-rule settings:

(a) If q is positive the C scale is located such that the sum of the reading on the D scale below the C index and the reading on C above q on D equals $|p|$.

(b) If q is negative C is located such that the difference of the reading on D below the C index and the reading on C above $|q|$ on D equals $|p|$.

It follows at once from (2) that these two readings are the desired roots x_1 and x_2. If q is positive the sign of each is opposite that of p. If q is negative they are of opposite sign, the numerically greater being opposite in sign to p.

A more convenient setting is to place the index of the inverted C scale opposite $|q|$ on D and then locate the slide such that:

(a) When q is positive the sum of the readings under the hairline equals $|p|$.

(b) When q is negative the difference of the readings equals $|p|$. As an illustration consider the slide-rule solution of the equation $x^2+6x+6=0$. Here q is positive. Consequently, we set the index on CI to 6 on D and run the slide to 1.27 on CI; beneath, on D, is 4.73. The sum of these two is 6. Hence, p being positive, -1.27 and -4.73, or, more accurately, -1.268 and -4.732 are the roots.

2. *Cubic equations.* Any cubic equation can be put in the form

$$X^3 + a_1X^2 + a_2X + a_3 = 0. \tag{3}$$

RECENT PUBLICATIONS

EDITED BY W. R. LONGLEY, Yale University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Procedures and Metaphysics. A study in the philosophy of mathematical-physical science in the sixteenth and seventeenth centuries. By E. W. Strong. Berkeley, University of California Press, 1937. 7+301 pages. $2.50.

Algebra for Today. By W. Betz. Boston, Ginn and Company, 1937. 12+565 pages. $1.36.

Useful Mathematics. A high school course in fundamentals. By F. M. Dunn, E. H. Allen, J. S. Goldthwaite, and M. A. Potter. Boston, Ginn and Company, 1937. 10+422 pages. $1.32.

Origins of Clerk Maxwell's electric ideas as described in familiar letters to William Thomson. Edited by Sir Joseph Larmor. Cambridge, The University Press; New York, The Macmillan Company, 1937. 56 pages. $1.00.

A First Year of College Mathematics. By R. W. Brink. New York, Appleton-Century Company, 1937. 16+667 pages. $3.50.

Higher Algebra. By S. Barnard and J. M. Child. London, The Macmillan Company, 1936. 14+585 pages. $6.00.

Mechanics. By W. F. Osgood. New York, The Macmillan Company, 1937. 15+495 pages. $5.00.

Plane Trigonometry. By C. N. Mills, E. I. Atkin, and E. B. Flagg. Chicago, Scott, Foresman and Company, 1937. 12+170 pages. $1.60.

Mathematics of Finance. By D. H. Mackenzie. New York and London, McGraw-Hill Company, 1937. 9+214 pages. $3.75.

Approximate Computation. By Aaron Bakst. The twelfth year-book of the National Council of Teachers of Mathematics. New York, Bureau of Publications, Teachers College, Columbia University, 1937. 16+287 pages. $1.75.

Mathematical Snack Bar. By N. Alliston. Cambridge, Heffer, 1936. 7+155 pages. $3.00.

Introduction to the Theory of Groups of Finite Order. By R. D. Carmichael. Boston, Ginn and Company, 1937. 14+447 pages. $5.00.

Projective Geometry. By B. C. Patterson. New York, John Wiley and Sons; London, Chapman and Hall, 1937. 13+276 pages. $3.50.

A Brief Course in Advanced Algebra. By H. E. Buchanan and L. C. Emmons. Under the editorship of J. W. Young. Revised edition. Boston, Houghton Mifflin Company, 1937. 8+185 pages. $1.40.

The Psychology of Teaching Arithmetic. By H. G. Wheat. New York, D. C. Heath and Company, 1937. 10+591 pages. $2.80.

Leçons d'algèbre et de géométrie à l'usage des étudiants des facultés des sciences. By R. Garnier. Volume 3. Élimination. Éléments de géométrie réglée. Transformation de Lie. Applications à la géométrie conforme. D'après la rédaction de B. Guendjian. Paris, Gauthier-Villars, 1937. 6+280 pages. 80 fr.

REVIEWS

Lectures on College Algebra. By S. B. Dandekar. Indore, Vinayak, 1936. 12+402 pages.

At once one sees that this book can not be reviewed in the ordinary way, having been printed in a country where English is not the common language, where the material differs from that presented in our college courses in algebra, and student point of view and reaction are of types not to be found in our American colleges.

Since nine of the fourteen chapters carry headings belonging within our three units of mathematics for college entrance, the title, College Algebra, seems misleading to the American reader, but a knowledge of the "intermediate mathematics" of the universities of India, or a scrutiny of these nine chapters, shows the greater demand made on the student there in the material through progressions, as is also the case with English texts. For instance, the chapter on progressions includes harmonic series, as well as devices for summing various series, while throughout the chapters on elementary algebra a very healthy emphasis is placed on theory. Also the English habit is followed of listing long sets of problems from university degree examinations, very valuable for the Indian student, prepared as he is by one set of mathematicians and examined by another.

In the last four chapters the material is similar to that included in an American college algebra, though again in permutations and combinations, binomial theorem, series, indeterminate coefficients, partial fractions, and determinants, greater emphasis is given theory and challenging problems than is found in the commonly used American texts.

The book, throughout, has characteristics appropriate to the situation in India where a student entering a university is not as far advanced mathematically as is the student admitted to our universities and colleges, but must, in a short time, acquire those topics in a more demanding form than our texts give. The author seems ever mindful of the demands of the Indian "intermediate" examinations. Because of these facts the book is not one that could be used

for college work in this country, but it could very well furnish us examples and challenging applications of the principles presented in the last four chapters.

A marked unevenness of quality is shown. Sometimes no better reason is given for an obscure operation than "an intelligent step." One finds the statement that "higher equations" than the quartic "have no special name," and consistently the treatment of limits is inadequate. Yet often the student is conducted with unusual skill and with expert transition from familiar to new material.

The paper is good and the print far better than in many Indian texts whose printing is so blurred as to be scarcely legible. The faults of printing, notation, and expression, though glaring to an American reader, will not disturb the Indian student accustomed to an English spoken or written by peoples of over forty different major languages, each producing some effect on the idiomatic use of the adopted tongue. Instead of being overcritical of the English idiom of the author, one should, rather, admire his courage in writing in a language not his own, for those whose only common medium is English. Because of the virtues and in spite of the many faults and deficiencies of the book, one can but applaud the enterprise that prompts Indians to write their own texts as a part of the program to Indianize their curriculum.

MARY E. WELLS

Textbook of Trigonometry. By W. H. H. Cowles and J. E. Thompson. New York, D. Van Nostrand Company, 1936. 10+373 pages. $2.50.

This book includes, in addition to the topics common in current texts on trigonometry, sections on polar coördinates and polar graphs, a fuller treatment than usual of complex numbers, de Moivre's theorem, and series expansions. There are supplementary chapters on Coördinates and Graphs, including the graphical solution of simultaneous equations; Logarithms; The Slide Rule. Also —an unusual feature—in the sections on solving triangles the solution by slide rule is given in each case. Cologarithms are not treated in the chapter on logarithms nor are they used elsewhere. There is no spherical trigonometry in the book. (There is a section called Applications to Geodesy and Astronomy, but no spherical trigonometry is used.) The sets of problems are extensive and well chosen. Graphs are emphasized throughout.

The general plan of the book is not unusual. The trigonometric functions are defined at once for the general angle in terms of coördinates. The solution of the right triangle is put early, before the relations between the squares of the functions in fact, then some analytical chapters, followed by the solution of oblique triangles, and then the chapters on Polar Coördinates and Complex Numbers, Series, and the supplementary chapters named above.

Various small treatments are slightly unusual: for instance, a geometric deduction of the functions of 15° and 75°, (the student is told to remember the decimal values of these as well as those of 30°, 45°, and 60°), the trigonometric solution of the algebraic cubic, the "Segment Law" for oblique triangles, (used

as an alternate method for solving the three sides case), and five cases of oblique triangles, the usual Case I being split into two according to the location of the given side with reference to the given angles.

The explanations are very full. On the title page the book is announced as "For Colleges and Engineering Schools." It seems to the reviewer that for such a purpose far too little is left to the reader in many places. Unfortunately, too, at times the clarity of the explanations is marred by inexact phrasing.

The Preface says, "The general question of accuracy, precision, and significant figures is given a prominent place in all numerical work." In fulfillment of this promise a section is devoted to Accuracy, Precision, and Significant Figures. The explanation is not faultless (for example, it would rule out the possibility of zeros immediately to the left of the decimal point being significant when no digits follow it) but a worse fault is the authors' failure to follow in practice their own precepts. Many answers are given with a precision not warranted by the data and even the worked examples are not consistent in this respect. (Page 323 is really bad. By rounding off $1/M$ to 2.3, $\log_e .09685$ is found and left as $\bar{3}.66803$ whose last three digits are wrong.)

Checking the solution of triangles also shows some divergence between precept and practice. A section is devoted to the matter which opens with the sentence, "All triangle solutions should be checked, or verified." Check by the Sine Law, and by the 180° sum for the three sides case are then mentioned, and finally the check by Mollweide's equations is discussed and an example worked. It seems strange that in the worked examples, except in the solution of right triangles by natural functions, no check is actually carried out. Checking by slide rule is regularly mentioned, but of course this is not good enough for 4 or 5 place computations. In the worked example of the solution of the case of two sides and the included angle by the Tangent Law appears this statement, literally true, but woefully misleading, "If the sum of the three angles (one given, two computed) is not equal to 180° the results are incorrect. The solution may also be checked by $\cdots$ the slide rule." If the student should avoid inferring from this that the sum of the angles checks the values of the computed angles it would be strange.

The presswork is excellent throughout.

R. K. MORLEY

First Year College Mathematics. By M. A. Hill, Jr. and J. B. Linker. New York, Henry Holt and Company, 1936. 436 pages + 155 pages of tables. $2.60.

The three parts of this book are devoted to algebra and trigonometry, analytic geometry, and the mathematics of finance. For anyone desiring such a combination of subjects, it is recommended as a good usable textbook. It is designed for the use of students in arts and sciences, commerce, or engineering, in each case with the omission of certain sections. A good feature of the book is its parallel development of algebra and trigonometry. The sections of algebra review on such topics as fractions, factoring, radicals, and exponents, are each

introduced just before the trigonometry sections to which they are relevant. Such algebra review sections might to advantage be included in the ordinary trigonometry textbooks, occupying the space often devoted to long-winded explanations, and "topics which may be omitted at the discretion of the instructor."

The present book includes also more advanced algebraic topics, such as determinants, the binomial theorem, theory of equations, complex numbers, and progressions. Aside from its simultaneous development of algebra and trigonometry, there is nothing very unusual in its treatment of trigonometry, analytic geometry, or the mathematics of finance, which is similar to that of many textbooks on those subjects. No solid analytic geometry or spherical trigonometry is included.

The reviewer found no serious errors, but has the following minor criticisms to make. On page 93 the authors recommend the practice of replacing a factor x^2 inside a square root sign by the factor x outside. On page 45 the notation $\tan^{-1} x$ is said to be unfortunate, yet it is used exclusively thereafter. Also on page 45 is the statement "In general there are two angles less than 360° for which any trigonometric function is true." The principal values of the inverse trigonometric functions are not mentioned. On page 289, an asymptote of a hyperbola is defined as a line which "meets the curve in two infinite points which may be regarded as coincident." This is not the best definition to use for freshmen. It would also have been better to derive the formulas for tangent and normal to a conic once and for all, instead of doing it three times over in different chapters on the circle, parabola, and central conics. As in other books on the subject, in the sections on the mathematics of finance there seems to be too much emphasis on complicated symbols and formulas which only an actuary would find it worth while to remember, and not enough emphasis on the use of algebra in solving problems. In the chapter on bonds, no problems are given on finding the yield of a bond selling at a given price, although problems more complicated than this are given in the section on building and loan associations.

Aside from these few points the reviewer has no fault to find with the book. The typography is excellent, and so is the selection of problems.

ORRIN FRINK, JR.

Mathematical Analysis. By Maximilian Philip. New York, Longmans, Green and Company, 1936. 2+277 pages. $2.75.

This book is designed as a text for freshman mathematics. It seeks to coördinate algebra, trigonometry, analytics, and calculus through the early introduction of function and derivative. It may be worth while, at the outset, to observe that, of the 273 pages, space is alloted to these subjects in the approximate proportion 1:4:4:11, respectively.

The author derives most of the proofs of trigonometric formulas by means of ordinary plane geometry without using the idea of a directed line-segment. Such a derivation, for example, is given for the half- and double-angle formulas, these

preceding the proof of the addition theorem. Although the definitions of the trigonometric functions are given for any angle most of the formulas are derived for acute angles without suggestion as to whether they may be valid for all angles.

Careful proofs are rarely, if ever, given. For example, in the discussion of the binomial expansion for positive integral exponents, the expressions are calculated for $n=1, 2, 3, 4, 5$ and then the general formula is given without any suggestion as to what is needed in order to complete the proof.

If one of the important functions of a freshman course in mathematics is to train the student to think as clearly as possible, the reviewer feels that this book is an undesirable one to place in the hands of a young student. Many of the definitions and statements are open to objection. We list only a few:

"Thus the decimal $.333 \cdots$ is a variable." (p. 17)

"The graph is not continuous at $x=0$, for a small change in x causes a very large change in y." (p. 25)

"The square root of a negative number is called an imaginary number" (p. 28)—this is given as the definition of an imaginary number.

"A quadratic equation can always be changed so that in $ax^2+bx+c=0$, the coefficients a, b, and c shall be integers" (p. 28)—with no assumption concerning a, b, c.

"Any equation of the form $Ax+By+C=0$, where A, B, and C are constants, may be written in the form $y=mx+b$." (p. 70)

"The line PP_x is moving horizontally." (p. 125)

"Δy, the approximate change in y for a small change in $x \cdots$." (p. 130)

"Although the methods of plane geometry do not enable us to calculate an area when the bounding lines are curves other than arcs of circles $\cdots$" (p. 147)—page Archimedes!

There are two features that demand commendation. One is the stress put upon the interpretation of results and the other is a good discussion of linear interpolation.

The only typographical error that came to our attention appears on p. 16: for "$f=(2)$ is" read "$f(2)$ is."

Tables of square and cube roots, four-place common logarithms of numbers and trigonometric functions, five-place natural logarithms, and values of e^x and e^{-x} are included.

C. A. NELSON

A First Course in Statistical Method. By G. I. Gavett. Second Edition. New York, McGraw-Hill Book Company, 1937. 10+400 pages. $3.50.

This edition is changed but slightly. A new chapter, X. Multiple and Partial Correlation, has been inserted between the former chapters IX and X. This adds thirteen pages and closes with a set of eight exercises. All other exercises, except an added, 17th, exercise closing Chapter I, are identical with those of the first edition. There are forty-two pages more in this edition due to an expansion

of tables and subject matter, some in each chapter, aggregating twenty-nine pages.

The author plans to help the students to understand the theory by laboratory work. Each student is required to measure the length and breadth of leaves, these measurements being the basis of a set of exercises on the theory. There are five tables of data in the first chapter. One of these gives the weight, the height, and the blood-pressure of 665 Freshmen of the University of Washington. We appreciate the interest that must thus be aroused and we agree that a first course should be taken leisurely so as to give time for meditation and experimentation. It would be a pleasure to lead students thus prepared through the farther reaches of the subject.

C. C. GROVE

Differential and Integral Calculus, vol. ii. By R. Courant. Translated by E. J. McShane. Blackie and Son, London and Glasgow, 1936. 10+682 pages. \$7.50.

The second and concluding volume of Courant's treatise on the calculus follows the same plan as the first in giving the reader a direct path into the living body of analysis and an intelligent grasp of its major applications, and this without a pedantry which fails to recognize the difference between essentials and non-essentials. In the preface of the first German edition, the author states his aim as follows: "I have tried to give the reader a clear view of the close connection of analysis to its applications and, with all mathematical rigor and precision, to yield to intuition its full prerogative as the original source of mathematical truths." This program has been faithfully followed and systematically carried out. The exposition is always clear and the treatment at least reasonably rigorous. One departure, however, from adequate rigor should be noted. On page 440, the vanishing of the Wronskian of a set of n functions is given as a sufficient condition for their linear dependence. In order that this conclusion hold it is well known that in addition the n Wronskians formed from $n-1$ of the functions must not vanish simultaneously at any point.

The book opens with an introductory chapter on three-dimensional analytic geometry and vector analysis; this is to be studied by the reader as the matters treated therein are needed. The following chapters deal with partial and total derivatives (II); further developments and applications of the same (III); multiple integrals (IV); curve integrals, the integral theorems of Gauss, Stokes, and Green, and their significance (V); differential equations and their applications, including a brief discussion of potential, the wave equation, and Maxwell's equations in empty space (VI); the calculus of variations, principally with regard to Euler's equation and its generalizations (VII); functions of a complex variable, with especial regard to complex integration and its applications (VIII). The book concludes with a supplement on real numbers and the concept of limit, and a 21-page summary of important theorems and formulas applying to both volumes.

A number of theorems are given in both scalar and vector formulation. Although the German edition used the Gibbs cross for the vector product, the English translation uses the German bracket notation. This is perhaps as it should be; for while the rest of the world is gradually conforming to the notation of Gibbs, his homeland is gradually drifting away from it. Indeed several new notations have been devised by recent American authors, so that any approach to notational unanimity in vector analysis seems to be more and more hopeless.

Those familiar with the German edition of this volume should note that the English version is virtually a new book. The chapter in differential equations has been greatly amplified, and the chapters on the calculus of variations and on complex variable are entirely new. Many important additions, such as a treatment of the Fourier integral and of gamma functions, are incorporated in the earlier chapters. Moreover, the book has been further improved as a text by the addition of many sets of exercises and a terminal section of 54 pages giving answers and hints for their solution. These additions serve to make the complete two-volume work a superb course in mathematical analysis, one which should be especially appreciated by students of physics.

LOUIS BRAND

College Algebra. By H. C. Carter. New York, Prentice Hall, 1936. 10+234 pages. $1.50.

The best feature of this text is its arrangement of problems. There are three sets, *A*, *B*, and *C*, following each section of the text. These sets are of equal difficulty, and any one of them is sufficient for the needs of a class. Answers are given to set *A* alone. Often there is added a fourth set, *D*, containing more difficult problems.

An attempt to unify the subject matter about the idea of function is not very successful. It results in little else than a renaming of the chapters, using the word "function" in the title of each, and relegating to a final chapter on "Miscellaneous Topics" all those ideas which refuse to be so named.

Graphical representation is used freely, and the figures are good. Historical footnotes add much to the value and interest of the text. The statements are carefully worded and very concise—almost too concise. The subject matter seems to be more restricted in certain directions than many teachers will desire.

D. F. BARROW

MATHEMATICS CLUBS

EDITED BY F. W. OWENS AND HELEN B. OWENS, State College, Pa.

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to F. W. Owens, 462 East Foster Ave., State College, Pa.

BOOKS FOR CLUBS

During the last year many questions have been received concerning books suitable for the use of clubs. It is impossible to attempt a complete listing. Nevertheless we are beginning a short series of notes on such books. The notes are mostly contributed by men in close contact with mathematics clubs who modestly do the work anonymously. The books will be numbered consecutively for ready reference.

Bibliographies

A. *A list of mathematical books for schools and colleges*, this MONTHLY, vol. 24, 1917, pp. 368–376; see also pp. 58–60.

B. *A List of Books Suitable for School Libraries issued by The Mathematical Association in 1936*, London, 1936, 24 pages.

C. *Report of the committee on assigned collateral reading in mathematics*, this MONTHLY, vol. 35, 1928, pp. 221–228.

D. *A secondary school mathematics club*. By C. W. Newhall. U. S. Bureau of Education, *Bulletin*, 1911, no. 16, pp. 164–168.

E. *Recreational values achieved through mathematics clubs in secondary schools*. By Marie Gugle and others. National Council of Teachers of Mathematics, *Yearbook*, vol. 1, 1926, pp. 194–200.

F. *A List of Books for College Libraries. Approximately 14,000 titles selected on the Recommendation of 200 College Teachers, Librarians and other Advisers*. By Charles B. Shaw, Chicago, American Library Association, 1931; *Mathematics*, pp. 457–476.

G. *Suggestions for Students of Mathematics. Mathematics and Life Activities*. By Brown University. Fifth ed. Providence, R. I., 1928; *Literature List*, p. 8.

General

1. *Men of Mathematics*. By E. T. Bell. New York, Simon and Schuster, 1937. 22+593 pages. An exceedingly attractive work.
2. *Some Great Mathematicians of the Nineteenth Century: Their Lives and Their Work*, 2 vols. By G. Prasad. Calcutta, 1933–34. Vol. 1– Gauss, Cauchy, Abel, Jacobi, Weierstrass, Riemann. Vol. 2– Cayley, Hermite, Brioschi, Kronecker, Cremona, Darboux, Cantor, Mittag-Leffler, Klein, Poincaré.
3. *A Source Book of Mathematics*. By D. E. Smith, ed. New York, McGraw-Hill, 1929. 17+701 pages. An admirably edited work of many collaborators, giving discoveries in the fields of number, algebra, geometry, probability, functions, and quaternions, in the form in which they were originally stated.
4. *A Catalogue of a Special Exhibition of Manuscripts, Books, Portraits and Personal Relics of Nathaniel Bowditch (1773–1838) with a sketch of Nathaniel

Bowditch, by Harold Bowditch, *and an Essay on the scientific achievements of . . . Bowditch, with a Bibliography of his publications*, by R. C. Archibald. Salem, Mass. Peabody Museum, 1937. 40 pages+7 plates.

Club Reports

1936–37

The Mathematics Club, Cooper Union of Technology

President, L. Schumann; Vice-President, C. Pinzka; Secretary-Treasurer, T. Berlin. F. V. Pohle won the Polyphase duplex slide rule awarded by the club for excellence in first year mathematics. At the regular monthly meetings the topics discussed included: The slide rule; Vector analysis; Calculus of variations; Tensor algebra; Finite series; Complex functions.

Pi Mu Epsilon, University of Alabama

Director, Dr. B. G. Clarke; Vice-Director, W. M. Scott; Secretary, W. F. Adams; Librarian, Dr. F. A. Lewis. The club held its annual bridge party in December and a picnic in the spring but chiefly centered its attention on regular meetings with formal programs which included as topics for discussion: Types of equality; Quadratic equations in matrices and quaternions; An inverse problem in the calculus of variation; The non-isomorphism of two collineation groups of order 5184; Does nature follow exact laws?; The envelope of a system of circles; Linear differential equations; Conjugate sets of matrices; Conics generated by the polar with respect to a fixed circle of two ranges of points connected by a particular projectivity.

A. and M. Math. Club, Oklahoma Agricultural and Mechanical College

President, Roberta Adams; Vice-President, C. E. Abraham; Secretary, Wilma E. Meachem; Sponsor, Dr. E. F. Allen. Meetings were held on alternate Thursdays. This young club has made a determined effort to establish itself as a center of students interested in mathematics. Talks were given alternately by instructors and by students. Subjects ranged from mathematical phases of navigation and theory of approximations to games of chance and a contest in a definition of number. The year brought an increased membership and growing enthusiasm.

Kappa Mu Epsilon, Athens College

President, Elsie Medford; Vice-President, Martha Spence; Secretary, Lynda Christopher; Treasurer, Florence Tilman; Corresponding Secretary, Dr. Kathryn Wyant. The club has been chiefly interested in the organization of the Tennessee Valley Mathematics Association, drawing together those students who are preparing to be teachers of mathematics. The Chapter's own programs have been held regularly in addition to a spring alumni home-coming day and a banquet for new members.

Pi Mu Epsilon, University of California

Director, E. Lingafelter; Vice-Director, L. Aller; Secretary, E. Gorham; Treasurer, J. Foytik; Librarian, Dr. C. B. Morrey; Permanent Secretary, Sophia H. Levy. During the year twenty-eight new members were initiated, six in the fall, twenty-two in the spring. Besides initiations and annual picnic, ten regular meetings were held. Topics discussed included: The fundamental integral equation of stellar statistics; Spherical trigonometry by right triangles; Fixed and movable critical points of differential equations; A plane representation of rational ruled surfaces; Graphical representation of the solutions of Schroedinger's differential equation; Fundamentals of the theory of orbit determination; The transcendence of e and π; Mathematics in economics; Riemann and Cremona transformations; The additive number theory for arbitrary sets of numbers.

Euclid's Circle, Mount St. Scholastica College

Chairman, Mary A. Shirmer; Secretary, Ann Oberle. At the monthly meetings, topics discussed included: Non-euclidean geometry; Euclid, the pioneer geometer; Pythagoras and his philosophy of numbers; Aristotle's contributions to mathematics; Galileo, the mathematician; The human side of mathematics; Present position of mathematics in secondary schools; Classic problems solved by straight edge, compasses, and the graduated ruler. The club also presented a number of plays and original sings. The club attended in a body a lecture given at the University of Kansas by Professor W. D. Reeve, editor of *The Mathematics Teacher*. A picnic closed the first year of this energetic club.

The Mathematics Club of Northwestern University

Faculty Adviser, Professor H. L. Garabedian. First Semester: President, O. Hilton; Vice-President, J. Carr; Secretary; Alice Curtiss; Treasurer, Kay Rubens. Second Semester: President, C. Erikson; Vice-President, H. Wilson; Secretary, Esther Krughoff; Treasurer, Eleanor Cuffey. An informal tea preceded each regular program. Topics discussed included: Non-differentiable functions; Football ratings for the Big Ten Universities; Theorems on the use of correlation coefficients; Blaschke's asymptotes; Groups; Problem in Fourier's series; Present day trends in mathematics. Guest speakers of the year were Professor W. D. MacMillan, University of Chicago, who discussed "Permanent configurations of n-bodies in space," and Professor H. E. Buchanan, Tulane University, who spoke on "The three-body problem."

Pi Mu Epsilon, University of Georgia

Director, F. Cumming; President, J. M. Smythe; Vice-President, J. C. Ramsey; Secretary-Treasurer, F. Hair; Corresponding Secretary, Iris Callaway. The year was devoted largely to applications of mathematics to the fields of chemistry, physics, zoology, and research in education. Other topics included: Probability, based on returns of polls of *The Literary Digest* and of *Public Opinion;* Unique solution of certain bi-quartic equations; History of the Southeastern Section of the Mathematical Association of America. An initiation banquet closed a successful year.

Mathematics Club, Women's College of Delaware

A social hour precedes the regular program at each meeting. The topics for the year were chosen from the field of geometry and included: What is space?; The fourth dimension; Cross ratio; Geometry in modern education; Geometers, past and present. A banquet with special address closed the year's work.

Junior Mathematics Club, University of Wisconsin Extension Division

President, H. Hibscher; Secretary-Treasurer, first semester, Adrienne Schmidt, second semester, Viola Barbian; Faculty Adviser, Dr. Louise A. Wolf. An illustrated lecture on astronomy and a discussion of mathematics and mathematicians opened the year. Six of the regular meetings were devoted to the presentations of papers given in competition for the Euler prize which is offered each year by the Mathematics faculty. "Mechanical Integration," accompanied by a carefully constructed polar planimeter, by Edward Radtke won first prize; and honourable mention was given to Harold Hibscher's paper, "Life Probabilities." The remaining papers were: The Rhind Papyrus; Numerology; The life and works of Isaac Newton; and The trisection of the angle, squaring the circle and duplication of the cube. In addition to regular meetings, the Club sponsors the Mathematical and Astronomical Exhibit. Approximately 600 mathematics students from over thirty junior and senior high schools of Milwaukee and its suburbs were guests this year. A motion picture on geometry and one on the solar system were shown. Two student speakers gave talks on "Some geometrical fallacies" and "The distances and sizes of celestial bodies." Exhibits included: historic devices for noting the passage of time; calculating machines; astronomical models and instruments; slide rules; stringed, plaster, and wood models; charts; drawings; and problems. Several new models were made by students for this year's exhibit.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 304. *Proposed by Michael Goldberg, Washington, D. C.*

Superimpose a given circle upon a given polygon so that their common area is a maximum. Show that the segments which the circle intercepts on the sides of the polygon can form a closed polygon in which the directions of the segments are also preserved. Locate the center of the circle. When the polygon is a triangle, determine the locus of the center of the circle as its size is increased from that of the inscribed to that of the circumscribed circle.

E 305. *Proposed by D. L. MacKay, Evander Childs High School, N. Y.*

If the external angle bisectors at A and B are equal, must the triangle ABC be isosceles?

E 306. *Proposed by W. B. Campbell, Ithaca, New York.*

Each license plate in a certain state bears from one to five characters, of which not more than two are chosen from the 26 letters of the English alphabet, the remainder being chosen from the nine digits, 1, 2, 3, 4, 5, 6, 7, 8, and 9. Any letters used can be in any position. Find the number of different plates possible.

E 307. *Proposed by V. Thébault, Le Mans, France.*

Locate the point P in the plane of the given triangle ABC such that the triangles PAB, PBC, and PCA may have equal perimeters.

E 308. *Proposed by E. H. Clarke, Hiram College, Ohio.*

Find the triangle which contains an angle most nearly equal to one radian, from among all possible triangles whose sides are integers of one or two digits.

SOLUTIONS

E 266. *Proposed by N. A. Court, University of Oklahoma.*

On a given line, s, determine two points, P and Q, such that PQ shall have a given length, m, and that $AP:AQ=BP:BQ$, where A and B are two given points on a line skew to s.

Solution by Simon Vatriquant, Brussels, Belgium.

If $AP/AQ=BP/BQ$, the bisectors of angles PAQ and PBQ are concurrent, internally at the point X and externally at the point Y. XY is the diameter of a

sphere passing through A and B. Its center O lies in the mediating plane (that is, perpendicularly bisecting AB at M). We know the center and radius of this sphere, and wish to find P and Q such that $OP\cdot OQ=OX^2=OA^2$ and $OQ-OP=m$. The solution is classical.

Note. The relation $AP/AQ=BP/BQ$ may also be written $AP/BP=AQ/BQ$, so that the bisectors of angles APB and AQB are also concurrent on AB. If the bisectors of two face-angles of a tetrahedron intersect on the edge common to those two faces, the same is true for the other pair of faces.

Also solved by C. E. Springer and the proposer.

E 267. *Proposed by V. Thébault, Le Mans, France.*

If n is a positive integer such that $2n+1$ and $3n+1$ are each perfect squares, show that $n+1$ is the sum of two successive perfect squares, and that it is also the sum of a perfect square and twice the succeeding perfect square. The converse of this theorem is also true. Show further that the last digit of n is 0.

Solution by E. P. Starke, Rutgers University.

Set $2n+1=s^2$ and $3n+1=t^2$. Since s is odd, we may put it equal to $2u+1$. Then $n+1=(s^2+1)/2=u^2+(u+1)^2$, the sum of two successive perfect squares. Since t is not divisible by 3, we may put $t=3v\pm 1$. Then $n+1=(t^2+2)/3=(v\pm 1)^2+2v^2$, the sum of one of two successive perfect squares and twice the other. (To make this result fit the wording of the problem exactly in case $t=3v+1$, we may put $n+1=(-v-1)^2+2(-v)^2$.)

Conversely, from $n+1=a^2+(a+1)^2$ and $n+1=b^2+2(b+1)^2$ there follow at once respectively $2n+1=(2a+1)^2$ and $3n+1=(3b+2)^2$. If we now eliminate n from the equations defining s and t, we obtain

$$2t^2+1=3s^2. \tag{1}$$

Since s^2 is odd, its final digit must be 1, 5, or 9. But 5 and 9 require that $2t^2$ end in 4 or 6, and this is impossible. Hence s^2 must end in the digit 1. Therefore t^2 ends in 1 or 6. From (1) we may write $t^2-1=3(s-1)(s+1)/2$, so that t is obviously odd. Consequently t^2 also ends in the digit 1, and $n=t^2-s^2$ must terminate in 0.

The proposed conclusions are proved without demonstrating the existence of numbers n which satisfy the hypotheses. There exists however an infinite set of values for n. Suppose s and t, with $2<s<t$, integer solutions of (1). If we put $s=4t_1+5s_1$ and $t=5t_1+6s_1$, we obtain $2t_1^2+1=3s_1^2$. This implies also that $s_1=5s-4t$ and $t_1=5t-6s$. From $25s^2>24s^2>16t^2$ we have $5s>4t$, or $0<s_1$. Similarly, from $25t^2>24t^2+12=36s^2$, we have $0<t_1$. Thus from any solution (s, t) of (1) we derive another solution, (s_1, t_1), with $0<s_1<s$ and $0<t_1<t$. The process may be repeated as many times as necessary until the set (1, 1) is reached. Reversal of the process will produce all possible solutions of (1) from (1, 1) by repeated use of the relations, $s_{i+1}=4t_i+5s_i$, $t_{i+1}=5t_i+6s_i$. Thus the first few solutions are (1, 1), (9, 11), (89, 109), (881, 1079), with the corresponding values of n: 0; 40; 3960; 388,080.

It is worth noting that both s_i and t_i satisfy the difference equation,

$x_{i-1}+x_{i+1}=10x_i$. Note also that t_i/s_i are the alternate convergents of the continued fraction $1+1/4+1/2+1/4+1/2+1/4+\cdots=\sqrt{3/2}$, namely those convergents in which $\sqrt{3/2}<t_i/s_i$.

Also solved by Daniel Finkel, M. A. Heaslet, Yetta V. Maizlish, Simon Vatriquant, and the proposer.

E 268. *Proposed by J. E. Trevor, Cornell University.*

A quadrilateral inscribed in a semicircle consists of three chords, and the bounding diameter. Find the radius of the semicircle when the successive chords are of lengths a, b, and c. Then particularize when a, b, and c are one, two, and three feet respectively.

Solution by Robert Gaskell, Brooklyn, Michigan.

Let AC and BD be the diagonals of the quadrilateral, with d the diameter of the semicircle. Then by Ptolemy's Theorem, $AC\cdot BD=ac+bd$. But since ACD and ABD are right triangles, $(d^2-a^2)(d^2-c^2)=(ac+bd)^2$, so that, discarding the extraneous root, $d=0$, we find the cubic equation, $d^3-(a^2+b^2+c^2)d-2abc=0$, which may be used to find the diameter.

When a, b, and c are 1, 2, and 3 respectively, we use the equation, $d^3-14d-12=0$. The only positive root of this equation is $d=4.1131+$, so that the radius of the semicircle is $2.0565+$ feet.

Editorial Note. In his solution, W. E. Buker points out that this problem, in its numerical form, is similar to 1467, *School Science and Mathematics*, and to problem 9, p. 75 (solution p. 261), *Mathematical Nuts*, by S. I. Jones.

Also solved by Stanley Bolks, Wm. Douglas, H. R. Mutch, D. L. MacKay, C. E. Springer, E. P. Starke, and the proposer.

E 269. *Proposed by C. W. Trigg, Cummock College, Los Angeles.*

If a Cevian be drawn to a side of a triangle and circles inscribed in the two triangles thus formed, then (a) the sum of the Cevian and the side to which it is drawn is equal to the semiperimeter and the segment between the point of contact of the circles with that side; (b) the product of the radii is equal to the product of the parts into which said segment is divided by the Cevian; (c) if the circles are equal, then the area of the original triangle equals the product of a radius by the sum of the Cevian and the semiperimeter.

Solution by W. E. Buker, Pittsburgh, Pa.

Referring to the Figure on the next page, we prove the proposition as follows:

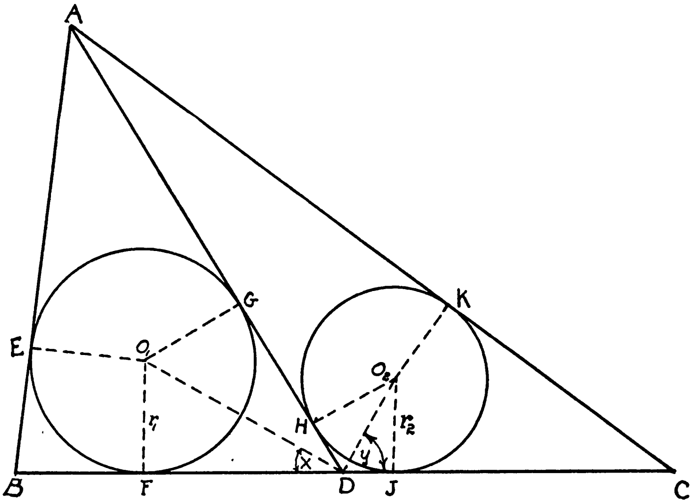
(a). $AG=AE$, $DG=FD$, $AH=AK$, and $DH=HJ$. These relations yield $AD=\frac{1}{2}(AE+AK)+\frac{1}{2}FJ$. Since $BF=BE$ and $JC=CK$, the semiperimeter s is equal to $\frac{1}{2}(AE+AK)+BF+JC+\frac{1}{2}FJ$. Therefore $AD+BC=s+FJ$ becomes the identity,

$$\tfrac{1}{2}(AE+AK)+\tfrac{1}{2}FJ+BF+FJ+JC=\tfrac{1}{2}(AE+AK)+BF+JC+\tfrac{1}{2}FJ+FJ.$$

(b). Angle X is the complement of angle Y, making right triangles FO_1D and JDO_2 similar. So $r_1/DJ=FD/r_2$. Then $r_1r_2=FD\cdot DJ$.

(c). Area $ABD=\frac{1}{2}r_1(AB+BD+AD)$; Area $ADC=\frac{1}{2}r_2(AD+DC+AC)$; and if $r_1=r_2$, addition gives the area of $ABC=r(AD+s)$.

Also solved by W. B. Clarke, K. W. Crain, D. L. MacKay, C. E. Springer, and the proposer.



E 270 [1937, 175]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

Stones of uniform size and weight are scattered evenly over a rectangular field 60 yards wide and 120 yards long. The owner hires Pat to carry the stones to the boundary of the field while he builds them into a wall of uniform height and thickness all around the field. Pat carries each stone separately straight from where he finds it to where it must fit into the wall, the middle stones for each of the four sides being brought perpendicularly to that side; but all the other stones are carried by routes at such angles as enable Pat to do as little work as possible. What is the average distance Pat carries a stone?

Solution by D. K. Pease, Student, Connecticut State College.

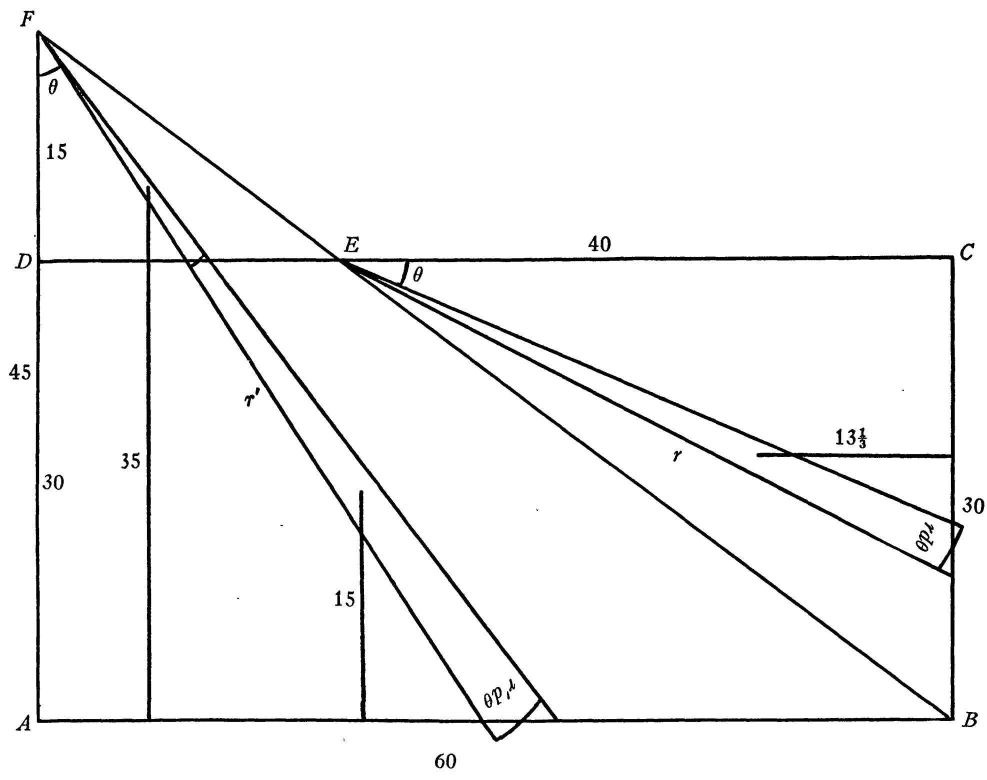
It is necessary to consider only one quarter of the field. All sections of wall of given length will require stones to be gathered from equal areas of the field. Since the stones are carried in straight lines the boundaries of these areas will be straight lines. Since BC is one-third of the wall length considered, the stones for BC will be gathered from a triangle with base BC and area equal to one-third of the area considered. Therefore $EC=40$ yds. The stones for AB will be gathered from the trapezoid $ABED$.

In the triangle BCE the stones for an element of the wall will be gathered from a triangular element with the area $\frac{1}{2}r^2d\theta$ and carried an average distance of $r/3$ where $r=40\sec\theta$. So for BCE we have

$$A_1X_1=\frac{(40)^3}{6}\int_0^{\arctan 3/4}\sec^3\theta\, d\theta,$$

where A_1 is the area, and X_1 is the average distance stones are carried for BCE. The same method applies to the trapezoid $DABE$ if we consider the triangle FAB and subtract the triangle FDE, remembering that the average distance the stones would be carried from triangle FDE would be $35 \sec\theta$. We have, using a similar notation,

$$A_2X_2 = \left[\frac{(45)^3}{6} - \frac{(35)(15)^2}{2}\right]\int_0^{\arctan 4/3} \sec^3\theta\, d\theta.$$



Adding and integrating we get

$$AX = \frac{64000}{12}\left[\sec\theta\tan\theta + ln\,(\sec\theta + \tan\theta)\right]_0^{\arctan 3/4}$$

$$+ \frac{67500}{12}\left[\sec\theta\tan\theta + ln\,(\sec\theta + \tan\theta)\right]_0^{\arctan 4/3}$$

$$= 27377.$$

Since $A = 1800$ sq. yds., we have $X = 15.21$ yds., the average distance the stones are carried.

Also solved by the proposer.

E 271 [1937, 245]. *Proposed by V. Thébault, Le Mans, France.*

Find the base B of a system of notation in which there exists a number of the form, *abba*, which is the square of the number, *bb*. Show that $b^2+B=aa$.

Solution by E. P. Starke, Rutgers University.

From the given conditions, we have at once $a(B^3+1)+bB(B+1)=b^2(B+1)^2$ from which the factor $(B+1)$ may be removed, giving $a(B^2-B+1)+bB=b^2(B+1)$, or $a(B+1)(B-2)+b(B+1)+(3a-b)=b^2(B+1)$. Thus $3a-b$ is divisible by $(B+1)$. The nature of the problem requires that $a\leqq b<B$. Hence we must have either (1) $b=3a$, or (2) $b=3a-B-1$.

If (1) holds, we have at once $a(B^2-B+1)+3aB=9a^2(B+1)$, or $B=9a-1$. Thus for any choice of a, $b=3a$ and $B=9a-1$, gives a solution. Then $b^2=9a^2=aB+a$, (when B is the base) $b^2=aa$. [Not $b^2+B=aa$.] For example, when $a=1$, $b=3$ and $B=8$, so that in the scale of 8, $33^2=1331$ and $3^2=11$. Again, when $a=2$, $b=6$, and $B=17$, so that in the scale of 17 we have $66^2=2662$ and $6^2=22$.

If (2) holds, we have the relation $9a^2-7a(B+1)+(B^2+3B+1)=0$, [after dividing out $(B+1)$]. Note that a and B must both be odd. Since this expression must factor rationally, its B-discriminant must be a perfect square, so that $13a^2-14a+5$ is a perfect square. Further, B^2+3B+1 must be divisible by a. Letting p be any prime factor of a, we may put $4B^2+12B+9\equiv 5 \pmod{p}$. Thus p is 5, or such a prime that 5 is a quadratic residue modulo p. Hence a is composed exclusively of prime factors chosen from 5, 11, 19, 29, 31, 41, 59, 61, 71, etc. At present, no solution coming under this case (2) has been determined, and it is not known whether or not such a solution exists.

Also solved by W. E. Buker, Mary L. Constable, Daniel Finkel, Walter Penney, W. R. Talbot, C. W. Trigg, and the proposer.

E 272 [1937, 245]. *Proposed by W. B. Campbell, Ithaca, New York.*

A certain railroad has n stations along the main line. How many different printed forms must be provided to take care of all possible one-way journeys, including provision by name for any or all stopovers which might be desired?

Solution by Daniel Finkel, New York, N. Y.

The simplest ticket a passenger can buy is from one station to another, without any stopover. The number of tickets necessary for this type of ride is the number of ways one may select two stations from n, or ${}_nC_2$. If one stopover is provided, we must select three stations from n, which requires ${}_nC_3$ different tickets. If stopovers are provided at all but one station, the number of forms equals the number of ways one can select $n-1$ stations from n, or ${}_nC_{n-1}$. And if stopovers at all stations are wanted, this takes ${}_nC_n$ or 1 more type of ticket. Now ${}_nC_0+{}_nC_1+{}_nC_2+\cdots+{}_nC_n=2^n$, and the first two terms, which must be excluded, are respectively 1 and n. Hence the number of forms, if we only go from east to west, is $2^n-(n+1)$. But the eastbound and westbound tickets will

ordinarily be different, so that the total number of different forms will then be $2^{n+1}-2(n+1)$.

Also solved by W. E. Buker, E. P. Starke, C. W. Trigg, and the proposer.

E 273 [1937, 245]. *Proposed by W. B. Clarke, San Jose, California.*

Last year three brothers discovered that if the age of each were deducted from the sum of the ages of the other two, and the three numbers so obtained were multiplied together, the result would equal sixteen times their combined ages. They went to tell their three sisters about this, but could only find Ida, who was the youngest of the six. She made a rapid calculation and announced that the same thing was true of the ages of herself and two sisters. In what years were the different boys and girls born?

Solution by C. W. Trigg, Cumnock College, Los Angeles.

Let the ages of the three members of the same sex be a, b, and c. Then $(a+b-c)(a-b+c)(b+c-a)=16(a+b+c)$. Restrict the solutions of this equation to positive integers, whence it is evident that the sum of any two is greater than the third, and all three may be even, or two may be odd and the other even. Set $a+b+c=2s$ and get $\sqrt{(s-a)(s-b)(s-c)/s}=2$. The problem is thus identical with that of finding the integral sides of a triangle with an inradius of 2. So no one of the ages is less than 4. In order that x be a member of a set of solutions of this problem, $4x/[(x-k)k-4]$ must be an integer for some integer k. Under the restrictions thus established we find the following sets of values for a, b and c: $A(5, 12, 13)$, $B(6, 8, 10)$, $C(6, 25, 29)$, $D(7, 15, 20)$ and $E(9, 10, 17)$. Now since a sister is the youngest of the six, there are nine solutions, AB, AC, AD, AE, BD, BE, CD, CE, and DE, all meeting the biological necessities. As Ida is capable of the calculation, DE is the most likely solution, so the girls were born in 1929, 1921 and 1916, while the boys were born in 1927, 1926 and 1919, provided only that they had passed their 1936 birthdays at the time of the momentous discovery.

Also solved by F. E. Allen, W. E. Buker, Daniel Finkel, E. P. Starke, J. E. Trevor, and the proposer.

E 274 [1937, 245]. *Proposed by R. P. Agnew, Cornell University.*

Let r be any positive rational number and n any positive integer. From r subtract the first term of the harmonic series, $1/n$, $1/(n+1)$, $1/(n+2)$, $\cdots$ which is $\leqq r$. From that difference deduct the next following term which is $\leqq$ that difference. Prove that a continuation of this process must produce a difference of zero in a finite number of subtractions.

Solution by C. H. W. Sedgewick, Connecticut State College.

Let the sequence, $1/n$, $1/(n+1)$, $1/(n+2)$, $\cdots$ be denoted by (A). Let r be written in lowest terms as the quotient of two integers; that is, $r=p/q$. Let $1/N$ be the first term of (A) which is $\leqq r$.

Subtracting from r all the successive terms of (A) possible under the given conditions, we have

$$\frac{p}{q} - \frac{1}{N} - \frac{1}{N+1} - \cdots - \frac{1}{N+k} = \frac{p_1}{q_1} = r_1.$$

Let the first term of (A) which can be subtracted from r_1 under the given conditions be denoted by $1/N_1$. We then have

$$\frac{1}{N_1} \leqq \frac{p_1}{q_1} < \frac{1}{N_1 - 1}, \quad \text{or} \quad \frac{p_1}{p_1N_1} \leqq \frac{p_1}{q_1} < \frac{p_1}{p_1(N_1-1)}.$$

Thus $q_1 = p_1N_1 - k_1$, where k_1 is one of the integers $0, 1, 2, \cdots, p_1 - 1$. Then

$$\frac{p_1}{p_1N_1 - k_1} - \frac{1}{N_1} = \frac{k_1}{(p_1N_1 - k_1)N_1} = r_2.$$

Similarly, if $1/N_2$ is the first term of (A) which can be subtracted from r_2 under the given conditions, we have

$$\frac{1}{N_2} \leqq \frac{k_1}{(p_1N_1 - k_1)N_1} < \frac{1}{N_2 - 1}, \quad \text{or} \quad \frac{k_1}{k_1N_2} \leqq \frac{k_1}{(p_1N_1 - k_1)N_1} < \frac{k_1}{k_1(N_2-1)}$$

and $(p_1N_1 - k_1)N_1 = k_1N_2 - k_2$, where k_2 is one of the integers, $0, 1, 2, 3, \cdots, k_1 - 1$. Then

$$\frac{k_1}{k_1N_2 - k_2} - \frac{1}{N_2} = \frac{k_2}{(k_1N_2 - k_2)N_2} = r_3.$$

Continuing in this way, we get a sequence of remainders, $r_1, r_2, r_3, \cdots$. Each remainder is a proper fraction whose numerator is less than the numerator of the preceeding remainder. Hence, we will obtain a remainder whose numerator is equal to unity, after a finite number of subtractions.

Editorial Note. D. T. Sigley calls attention to the fact that G. A. Miller published in this MONTHLY (1931, 194) his proof of the theorem, "Every positive rational number may be expressed as the sum of a finite number of distinct terms from the harmonic series, in an infinite number of ways." A. J. Kempner notes that the Rhind Papyrus contains related results.

Also solved by Daniel Finkel, J. Rosenbaum, and the proposer.

E 275 [1937, 245]. *Proposed by J. A. Benner, Lafayette College, Easton, Pa.*

In a certain town it began snowing before noon and continued at a constant rate until dark. At noon a crew of men set out along the highway, clearing the snow from it as they went. They cleared two miles in the first two hours, but only one mile in the next two hours. If the crew clears equal volumes of snow in equal times, at what time did it begin to snow?

Solution by W. B. Campbell, Ithaca, New York.

Let b represent the snowfall, in inches per hour; c the rate of removal in inch-miles per hour, with snowfall starting at time h hours before noon. At time t, the depth of snow attacked by the crew is $b(h+t)$, and v_t, the rate of

advance of the crew, is $c/[b(h+t)]$ miles per hour. Then x_t, the distance from town at time t, is given by

$$x_t = \int_0^t v_t dt = \frac{c}{b}[\log(h+t) - \log h].$$

From the given data, $x_2=2$, $x_4=3$, we find that $h=(\sqrt{5}-1)$ hours, or 1 hour, 14 minutes and 9.8 seconds. Hence it started snowing at 10:45:50.2 A.M.

The values of b and c are indeterminate, but c/b, the number of miles the crew could keep clear with snowfall and removal starting from scratch, is $2/[\log(\sqrt{5}+1)-\log(\sqrt{5}-1)]=2.078$. In the given problem, the crew, or its equivalent relief, would make some progress so long as snowfall and removal continue at the given rates. As $t\to\infty$, $v_t\to 0$, but $x_t\to\infty$.

Also solved by William Douglas, C. E. Springer, E. P. Starke, W. J. Taylor, and the proposer.

Advanced Problems

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3847. *Proposed by N. A. Court, University of Oklahoma.*

Three vertices and the circumradius of a variable tetrahedron are fixed, and the Monge point of the tetrahedron lies on the circumsphere. Find the locus of the fourth vertex.

3848. *Proposed by P. Erdös, Budapest, Hungary.*

Let O be an arbitrary point in the plane of the triangle ABC, and let A', B', C' be the points in which AO cuts BC, etc. If $AA'\geqq BB'$ and $AA'\geqq CC'$, then $AA'\geqq OA'+OB'+OC'$, where equality holds only if $AA'=BB'=CC'$.

3849. *Proposed by V. Thébault, Le Mans, France.*

Let O be the circumcenter of the tetrahedron $ABCD$, and P an arbitrary point in space. The segments of straight lines PA, PB, PC, PD are divided in the same ratio u, and the points of division are taken as centers of four spheres with radii $v\cdot PA$, $v\cdot PB$, $v\cdot PC$, $v\cdot PD$, respectively. The radical center R of these four spheres is on the straight line OP so that

$$OR:OP = (v^2 - u^2 + u):u.$$

This is a generalization of a proposition by N. A. Court in which $P\equiv G$. See this MONTHLY, 1932, p. 198.

3850. *Proposed by V. Thébault, Le Mans, France.*

Let BCA_1A_2, CAB_1B_2, ABC_1C_2 be squares constructed interiorly on the sides of a triangle ABC for which V is the angle of Brocard. If $\cot V=2$, the lines which join A, B, C, respectively, to the symmetrics of A_1, B_1, C_1 with respect to A_2, B_2, C_2, meet in a point.

Correction. In the first line of problem 3820 [1937, 179] replace $<2n$ by $\leqq 2n$.

SOLUTIONS

3758 [1935, 626]. *Proposed by V. Thébault, Le Mans, France.*

Let P be a point on the nine-point circle of ABC, and A_1, B_1, C_1, the midpoints of BC, CA, AB: through the orthocenter of ABC are drawn perpendiculars (or parallels) to PA_1, PB_1, PC_1 which cut BC, CA, AB in α_1, β_1, γ_1 (or α_2, β_2, γ_2). (a) Prove that the points α_1, β_1, γ_1 (or α_2, β_2, γ_2) lie on a straight line Δ_1 (or Δ_2) and that Δ_1 passes through P. Find the envelope of Δ_1 (or Δ_2) when P describes the nine-point circle. (b) Determine the position of P so that Δ_1 and Δ_2 are parallel.

I. *Solution by J. R. Musselman, Western Reserve University.*

Let the vertices of the triangle ABC be represented by the complex numbers t_i, where $|t_i|=1$. Then the orthocenter H will be given by σ_1; the point P by $(\sigma_1-t)/2$, and the point Q, which lies on the nine-point circle diametrically opposite to P, by $(\sigma_1+t)/2$. One will find without difficulty that the complex number representing α_1 is

$$x=\frac{\sigma_3(\sigma_1-t)}{\sigma_3+t_1^2t},$$

where the σ's are the elementary symmetric functions of the t_i. Expressions for the coördinates of β_1 and γ_1 are obtained from x_1 by replacing t_1^2 by t_2^2 and t_3^2 respectively. Hence the parametric equation of the line Δ_1 (parameter T) is

$$x=\frac{\sigma_3(\sigma_1-t)}{\sigma_3+tT}\cdot \tag{1}$$

The self-conjugate equation of Δ_1 can be written as

$$x(t\sigma_2-\sigma_3)+\bar{x}t\sigma_3(\sigma_1-t)-(t\sigma_2-\sigma_3)(\sigma_1-t)=0. \tag{2}$$

The parametric equation of the line Δ_2 (parameter T) is

$$x=\frac{\sigma_3(\sigma_1+t)}{\sigma_3+tT}, \tag{3}$$

which in self-conjugate form becomes

$$x(t\sigma_2+\sigma_3)+\bar{x}t\sigma_3(\sigma_1+t)-(t\sigma_2+\sigma_3)(\sigma_1+t)=0. \tag{4}$$

Now it can easily be verified in equations (2) and (4) that Δ_1 is nothing else but the perpendicular through P to HQ, and that Δ_2 is the perpendicular through Q to HP. This fact gives us a simple method for constructing the lines Δ_1 and Δ_2.

If R be the join of Δ_1 and Δ_2, then H is the orthocenter of the triangle PQR and since in any triangle the angles PHQ and PRQ are either supplementary or equal, it is evident that for Δ_1 and Δ_2 to be parallel H must lie on the line PQ or in other words, if P and Q are the intersections of the nine-point circle and the Euler line of ABC then Δ_1 and Δ_2 will be parallel, and both perpendicular to the Euler line.

Since Δ_2 with reference to P would be a Δ_1 as to Q, the lines will have the same envelope. To obtain its equation we find $\partial x/\partial t$ from equation (2) and eliminating $\bar{x}$ between this result and its conjugate we obtain

$$x = \frac{(\sigma_1 - t)^2}{\sigma_1 - 2t + t^2\sigma_2/\sigma_3}\,. \tag{5}$$

This is the equation of a conic inscribed in ABC, with foci at the orthocenter and the circumcenter of the triangle ABC. The type of conic depends upon the triangle ABC. For the equilateral triangle, the envelope of Δ_1 or Δ_2 is the inscribed circle, for a right triangle the envelope is just a point, the vertex of the right angle.

II. *Solution by R. Goormaghtigh, Bruges, Belgium.*

This question is connected with Droz-Farny's theorem: Two perpendicular straight lines δ_1, δ_2, drawn through the orthocenter H of a triangle ABC mark on the sides three segments having their mid-points α_1, β_1, γ_1 on a straight line Δ_1.

This theorem has been completed by another one due to Neuberg: the envelope of Δ_1 is Euler's conic, *i.e.*, the inscribed conic having its foci at the orthocenter and the circumcenter.*

We have given the following proof of these two theorems in *Gazeta Matematica* (Bucarest, vol. 35, 1930, p. 202):

Let α, β, γ be the mid-points of AH, BH, CH, A_1', B_1', C_1' the projections of A, B, C on δ_1; then $\alpha A_1'$, $\beta B_1'$, $\gamma C_1'$ are concurrent at a point N on the nine-point circle $A_1B_1C_1$, for the angles $\beta B_1'H$ and $\gamma C_1'H$ are respectively equal to $B_1'H\beta$ and $C_1'H\gamma$ and therefore $B_1'NC_1'$ is equal to BHC.

By an inversion with respect to the conjugate circle, the straight line $\alpha A_1'$, passing through the projections of A on δ_1 and δ_2 is transformed into a circle C_a passing through H and through the points where δ_1 and δ_2 meet BC; therefore

* About these theorems see: Mantel, Mathesis, 1889, p. 219; Cesàro, Ibid., 1890, p. 184; Noyer, Journal de Mathématiques spéciales, 1893, p. 39; Droz-Farny, Educational Times, 1899; Neuberg, Mathesis, 1899, p. 39; Visschers, Ibid., 1913, p. 256; Neuberg, Ibid., 1913, p. 262; Barisien, Jeřabek, Goormaghtigh, Ibid., 1915, p. 20; Servais, Ibid., 1915, p. 91; Sebban, Journal de Vuibert, 1916–1917, p. 56; Barisien, Intermédiaire des Mathématiciens, 1916, p. 124; Goormaghtigh, Ibid., 1917, p. 19; Mathesis, 1922, p. 52; Mineur, Mathesis, 1925, p. 17; Servais, Ibid., 1925, p. 98; Goormaghtigh, Mathesis, 1926, p. 196; Gazeta Matematica, XXXV, 1930, p. 202; Thébault, Ibid., XXXV, 1930, p. 441.

the circle C_a and the two other analogous circles C_b and C_c meet at one of the intersections T of HN with the circumcircle. Hence their centers α_1, β_1, γ_1 are on the perpendicular Δ_1 to HT at its mid-point J, and, J being on the nine-point circle, this proves both Droz-Farny's and Neuberg's theorems.

Further, $N\alpha$ being perpendicular to $H\alpha_1$, the perpendicular to $H\alpha_1$ drawn through A passes through the point Q where HT meets again the circumcircle, Q being the image of H through N; if S is the other end of the circumdiameter passing through J, QTS is a right angle and the mid-point P of HS lies on Δ_1; hence P is on the parallel to $N\alpha$ drawn through A_1, and also the second end of the diameter of the nine-point circle passing through N.

Therefore, P being a point on the nine-point circle, the perpendiculars $H\alpha_1$, $H\beta_1$, $H\gamma_1$ to PA_1, PB_1, PC_1 cut the sides at α_1, β_1, γ_1 on a straight line Δ_1, and the same property applied to the point N shows that the points α_2, β_2, γ_2 are on a straight line Δ_2 perpendicular to HP. The envelope of Δ_1 and Δ_2 is the antipedal of the nine-point circle with respect to H, that is, Euler's conic.

It follows also that Δ_1 and Δ_2 are parallel when H, N, P are collinear, *i.e.*, when P is one of the points where the Euler line meets the nine-point circle.

Solved also by J. W. Clawson and the proposer.

Editorial Note. The proposer considered an equilateral hyperbola circumscribed about ABC with its center at Q, the point diametrically opposite to P on the nine-point circle. The figure is projected upon a plane so that the hyperbola projects into a circle whose center is the projection of Q. Goormaghtigh stated that his solution is not perhaps the most direct, but it is intended to show the relation of the problem to other known theorems. The polar properties of the conjugate circle, which is the basis of his solution, were also used in the solution of 3258 [1928, 210] in the proof of the collinearity of three points. See also the solution of 3342 [1929, 449] for similar theorems in regard to the tetrahedron. Clawson used a system of rectangular coördinates with the origin at the center of the nine-point circle. This solution showed great care and skill in the laborious reductions. Musselman's solution has somewhat the character of a vector solution, and it will be seen below that a purely vector solution is quite simple.

The solution of the problem and the simultaneous proof of the related theorems in solution II may be obtained without the use of the conjugate circle by projective methods applied to the characteristic property of the three lines l, m, n defined below. This method is simple, but it is still simpler and briefer to use vectors. Before proceeding to the proof, we state some preliminary facts. If P is any point in the plane of the triangle ABC, there is associated with P a second point P', the inverse, or isogonal conjugate, of P. The mid-point M of the segment PP' is the center of the circle which passes through the six projections of P and P' on the sides of ABC; the point M is the center of a conic with foci at P and P' tangent to the sides of the triangle, and the pedal circle (M) is the major auxiliary circle of the conic. If P is at the circumcenter O, P' is at the orthocenter H, and (M) is the nine-point circle which we designate as (N).

If any straight line is drawn through O (or H) cutting (N) in R, the perpendicular to OR (or HR) at R is tangent to the conic (see the solution of 3658 [1935, 257]).

Through H, any point in the plane of AB and AC, draw two perpendicular lines δ_1 and δ_2; they determine by their intersections with AB and AC two segments on these lines whose middle points are γ_1 and β_1, respectively. Draw through H two rays HB_2 and HC_2 parallel to and in the directions of AB and AC. The reflections of HB_2 and HC_2, or their extensions, in δ_1, pass, respectively, through γ_1 and β_1; and the angle between these reflections is obviously constant as δ_1 varies. These reflections are along two rays n and m proceeding from H such that the angle from m to n is $180°-A$ in the sense opposite to that of AB to AC through the angle A. Now let H be the orthocenter of triangle ABC, and through C', the mid-point of AB, draw a perpendicular to n; similarly from B' draw a perpendicular to m. These two perpendiculars meet in a point P on (N). When n passes through B, m passes through C, and P is at A'. If from C' and B' we draw parallels to n and m, respectively, these parallels meet on (N) in Q, which is diametrically opposite to P. Let O, the circumcenter, be the origin of vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ to the vertices. Then, if R is the circumradius and $\mathbf{h}$ is the vector to H, we have

$$\mathbf{a}^2 = \mathbf{b}^2 = \mathbf{c}^2 = R^2, \qquad \mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{h}. \tag{1}$$

The second equation is true since $(\mathbf{h}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{c}) = \mathbf{b}^2-\mathbf{c}^2=0$; or $(\mathbf{h}-\mathbf{a}-\mathbf{b}-\mathbf{c})\cdot(\mathbf{b}-\mathbf{c})=0$. Since this is also true for a circular permutation of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, the second equation of (1) must be true. The equations of n, AB, and (N) are, if $\mathbf{y}$ is the vector of P,

$$\begin{gathered}(\mathbf{x} - \mathbf{h})\cdot\left(\mathbf{y} - \frac{\mathbf{h} - \mathbf{c}}{2}\right) = 0, \qquad \left(\mathbf{x} - \frac{\mathbf{h} - \mathbf{c}}{2}\right)\cdot\left(\frac{\mathbf{h} - \mathbf{c}}{2}\right) = 0, \\ \left(\mathbf{y} - \frac{\mathbf{h}}{2}\right)^2 - \frac{\mathbf{c}^2}{4} = 0.\end{gathered} \tag{2}$$

The sum of the first two equations minus the last easily reduces to

$$(\mathbf{x} - \mathbf{y})\cdot\mathbf{y} = 0; \tag{3}$$

and the vector of γ_1 must satisfy this equation. There are three rays l, m, n which in this order have a sense of rotation opposite to that of A, B, C, and such that the angle from l to m is $180°-C$; from m to n, $180°-A$; from n to l, $180°-B$; and they pass, respectively, through α_1, β_1, γ_1. For any one, for a given set δ_1, δ_2, we get (3). Hence (3) is the equation of a straight line Δ_1 passing through α_1, β_1, γ_1; and Δ_1 is perpendicular to OP at P. Moreover, Δ_1 is tangent to the inscribed conic with foci at O and H. For the parallels to l, m, n through the respective mid-points, we have merely to replace $\mathbf{y}$ in (3) by $\mathbf{z}$, the vector to Q, where

$$\mathbf{y} + \mathbf{z} = \mathbf{h}, \tag{4}$$

and we have the equation of Δ_2 which is also tangent to the conic. Obviously Δ_1 and Δ_2 are parallel, if and only if P is at an extremity of the diameter of (N) through O and H. If H is on (N), the last equation of (2) gives $\mathbf{h}^2=\mathbf{c}^2$ and $(\mathbf{h}-\mathbf{y})\cdot\mathbf{y}=0$. Hence both Δ_1 and Δ_2 pass through H; H must be a vertex and the angle of the triangle at this vertex is a right angle. This finishes the solution of the problem and the proofs of the theorems in solution II.

There is one result remaining. The locus of the intersection of Δ_1 and Δ_2 is the directrix of the conic for the focus H. The reader will easily find from (3), a similar equation with $\mathbf{z}$ in place of $\mathbf{y}$, equation (4), and the last one in (2) for $\mathbf{y}$ and $\mathbf{z}$, that this locus has the equation

$$(2\mathbf{x}-\mathbf{h})\cdot\mathbf{h}-R^2=0; \tag{5}$$

and this is the equation of the polar of H with respect to (N) or to the conic. It will be observed that this shows that, if from a point on the directrix of a central conic corresponding to a given focus, two tangents are drawn to the conic, the feet of the perpendiculars to the two tangents from the other focus lie at the extremities of a diameter of the major auxiliary circle.

3759 [1935, 626]. *Proposed by V. Thébault, Le Mans, France.*

In the tetrahedron $ABCD$ the trihedral angle $G(ABC)$ is trirectangular at the centroid G. Prove that (a) the parallelopiped constructed with the medians of $ABCD$ is rectangular; (b) the center of the circumsphere of $GABC$ lies upon the circumsphere (O) of $ABCD$; (c) the sum $AB^2+BC^2+CA^2$ is equal to the product of the diameter of (O) and the altitude of the tetrahedron from the vertex D.

Solution by J. W. Clawson, Ursinus College.

Take G for the origin of a system of rectangular coördinates. Let A, B, C be respectively $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$. Then D', the centroid of face ABC is $(a/3, b/3, c/3)$; so D, the fourth vertex, is $(-a, -b, -c)$, since D, G, D' are collinear, with $DG=3\cdot GD'$.

(a) I am not sure exactly what the proposer means here. Naturally three of the four medians are along GA, GB, GC, which are given to be mutually at right angles. If we should extend GA to A'', making it equal to $A'A$, and so on, the diagonal of this rectangular parallelopiped through G would be equal in length to the fourth median, DD'. (The three bimedians are not rectangular.)

(b) The center of the sphere $GABC$ is $(a/2, b/2, c/2)$. The equation of sphere $ABCD$ is

$$x^2+y^2+z^2+(b^2+c^2-a^2)x/2a+(a^2+c^2-b^2)y/2b$$
$$+(a^2+b^2-c^2)z/2c=(a^2+b^2+c^2)/2.$$

Direct substitution proves that the theorem is true.

(c) $AB^2=a^2+b^2$, etc. Hence $AB^2+BC^2+CA^2=2\sum a^2$.

The diameter of sphere $ABCD$ is

$$\frac{\sqrt{\sum a^6b^2 + 2\sum a^4b^4 + 5\sum a^4b^2c^2}}{2abc}.$$

The perpendicular distance from D to the face ABC is

$$\frac{4abc}{\sqrt{\sum a^2b^2}}.$$

Now

$$\sum a^2b^2\cdot\left(\sum a^4 + 2\sum a^2b^2\right) = \sum a^6b^2 + 2\sum a^4b^4 + 5\sum a^4b^2c^2.$$

The last part of the theorem follows at once.

Solved also by the proposer.

Editorial Note. The proposer indicated briefly his method of proof, which for the last two parts was as follows: Let G_d be the centroid of ABC and P_d the point where DG_d cuts (O), the circumsphere. Then we have $G_dP_d = GG_d/2 = DG_d/8$. The center ω of the sphere $GABC$ is upon GG_d and $GG_d = 2G_d\omega$. Hence $\omega \equiv P_d$. Then (c) follows easily. Since the argument is not quite clear, we give another proof.

Let $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{d}$ be the vectors of A, B, C, D with their origin at the centroid G. Then

$$(1) \qquad -\mathbf{d} = \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_1, \qquad \mathbf{d}^2 = \mathbf{a}_1^2 + \mathbf{a}_2^2 + \mathbf{a}_3^2 = (a^2 + b^2 + c^2)/2.$$

Let M_a, M_b, M_c, M_d be the lengths of the medians from A, B, C, D; then $M_a^2 = (-\mathbf{a}_1/3)^2$, $M_b^2 = (-\mathbf{a}_2/3)^2$, $M_c^2 = (-\mathbf{a}_3/3)^2$, $M_d^2 = (-\mathbf{d}/3)^2$, and

$$(2) \qquad M_a^2 + M_b^2 + M_c^2 = M_d^2.$$

This is part (a). Let S and E be the circumcenters of $ABCD$ and $GABC$. E is also the circumcenter of the rectangular parallelopiped with edges GA, GB, GC. Since S and E are each equally distant from A, B, C, SE is perpendicular to the plane of ABC. Moreover E lies on DG produced so that $GE = GD/2$. To prove that E lies on (S), let $\mathbf{x}$ be the vector of S; then we have

$$(\mathbf{x} - \mathbf{a}_1)^2 = R^2, \qquad (\mathbf{x} - \mathbf{a}_2)^2 = R^2, \qquad (\mathbf{x} - \mathbf{a}_3)^2 = R^2, \qquad (\mathbf{x} - \mathbf{d})^2 = R^2,$$

where R is the radius of (S). Three times the sum of the first three equations minus the last gives after division by 8

$$\left(\mathbf{x} + \frac{\mathbf{d}}{2}\right)^2 = R^2.$$

Since $-\mathbf{d}/2$ is the vector of E, the latter point lies on (S).

The altitude DH_d from D is parallel to SE, and the plane of these two parallels contains the similar triangles DH_dG_d and $ES'S$, where S' is the mid-point of DE. Thus $SE/DG_d = S'E/DH_d$. Hence

$$2R(DH_d) = 2(\tfrac{3}{4}\mathbf{d})\cdot(\tfrac{4}{3}\mathbf{d}) = 2\mathbf{d}^2 = a^2 + b^2 + c^2.$$

3760 [1935, 626]. *Proposed by R. E. Gaines, University of Richmond, Va.*

If a variable chord of a conic subtends at a focus a constant angle 2α, its envelope and the locus of its pole are conics. These two with the given conic belong to an infinite series of conics which have one focus and directrix in common, and whose eccentricities are in geometric progression with $\cos\alpha$ as the common ratio. If α is a divisor of $360°$ then between any two consecutive conics an infinite number of polygons may be drawn each of which is inscribed in one conic and circumscribed about the other.

Solution by C. E. Springer, University of Oklahoma.

Let the given conic be

$$\frac{\ }{r} = 1 + e\cos\theta. \tag{1}$$

The variable chord subtending a constant angle 2α at a focus is

$$\frac{l}{r} = e\cos\theta + \sec\alpha\cos(\theta - \beta). \tag{2}$$

This is seen to be tangent to the conic

$$\frac{l\cos\alpha}{r} = 1 + e\cos\alpha\cdot\cos\theta. \tag{3}$$

Now both the lines

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \beta + \alpha),$$

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \beta - \alpha)$$

meet the conic

$$\frac{\ }{r} = e\cos\theta + \cos\alpha$$

at the same point, viz., where $\theta = \beta$ and $lr^{-1} = e\cos\beta + \cos\alpha$. Therefore, the locus of the pole of the variable chord (2) is

$$\frac{l\sec\alpha}{r} = 1 + e\sec\alpha\cdot\cos\theta. \tag{4}$$

This much of the solution can be found in C. Smith, *Conic Sections*, page 220. Now let (3) be considered the initial conic, and treat it as (1) was treated above. The variable chord of (3) subtending the constant angle 2α at the focus envelops the conic

$$\frac{l \cos^2 \alpha}{r} = 1 + e \cos^2 \alpha \cos \theta. \tag{5}$$

The locus of the pole of the variable chord is the given conic (1).

Proceeding in this manner, we arrive at an infinite series of conics.

The conics (4), (1), (3), (5) $\cdots$ have eccentricities given by the geometric progression $e \sec \alpha$, e, $e \cos \alpha$, $e \cos^2 \alpha$, $\cdots$ respectively, and all the conics of the series have the common directrix $lr^{-1} = e \cos \theta$. For every α which is a divisor of 360°, a closed polygon can be inscribed in any one of the conics and the above analysis shows that these sides of the polygon are tangents to a consecutive conic of the series. As the chord varies, we have an infinite number of such polygons for any given conic of the series.

Solved also by W. B. Campbell and the proposer.

3761 [1935, 626]. *Proposed by H. D. Grossman, New York.*

If $\sum_{i=1}^{s} n_i = n$, and if the greatest common divisor of the positive integers n_i is unity, then $(n-1)!/(n_1!n_2! \cdots n_s!)$ is an integer.

I. *Solution by S. B. Townes, Norman, Okla.*

Let p^a be the highest power of any prime factor p of the denominator. And let p^b be the highest power of p which is a factor of the numerator. We must prove that $b \geqq a$.

Now

$$a = \sum_{r=1}^{\infty} \sum_{i=1}^{s} \left[\frac{n_i}{p^r}\right], \qquad b = \sum_{r=1}^{\infty} \left[\frac{n-1}{p^r}\right],$$

where $[c/d]$ represents the largest integer less than or equal to c/d. By hypothesis, at least one of the n_i, say n_k, is prime to p; therefore

$$\left[\frac{n_k}{p^r}\right] = \left[\frac{n_k - 1}{p^r}\right], \qquad (r = 1, 2, \cdots).$$

It follows that

$$\left[\frac{n-1}{p^r}\right] \geqq \sum_{i=1}^{s} \left[\frac{n_i}{p^r}\right], \qquad (r = 1, 2, \cdots),$$

and therefore $b \geqq a$. The condition stated in the hypothesis is sufficient but not necessary since

$$\frac{9!}{6!4!} = 21.$$

Determination of the necessary conditions might be a good problem.

II. *Solution by E. P. Starke, Rutgers University.*

Set $J = n!/(n_1!n_2! \cdots n_s!)$, $K_r = n_r(n-1)!/(n_1!n_2! \cdots n_s!)$, $(r = 1, 2, \cdots, s)$. Then J and K_r are integers; for example, K_1 is the number of permutations of $n-1$ objects taken all at a time of which $n_1 - 1$ are alike, n_2 are alike, $\cdots$, n_s are alike. Now $n_r J = nK_r$. Hence n is a divisor of all the integers $n_r J$, and therefore of

their highest common divisor J. Thus J/n, which is the proposed number, is an integer.

Solved also by J. Barinaga, Hansraj Gupta, J. Milkman, E. G. Olds, and H. D. Ruderman.

Editorial Note. Olds's solution was similar to II. Two of the remaining solutions used the method in I. Ruderman showed that the theorem is true if two of the n_i are relatively prime by writing the given expression as a product of integers. Baringa and Gupta pointed out that, if d is the greatest common divisor of the set of n_i, then the given expression multiplied by d is an integer. It will be observed that this is obvious from the proof in II.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

Dr. O. B. Ader and Dr. K. L. Palmquist have accepted positions at the Southern Methodist University.

Dean G. D. Birkhoff of Harvard University was awarded a doctorate *honoris causa* at the one hundredth anniversary of the University of Athens.

In the November issue of this Monthly, it was incorrectly stated that Assistant Professors H. J. Miles and G. E. Moore of the University of Illinois had been promoted to associate professorships. The promotions were from the rank of an instructor to the rank of an associate.

Miss A. Marguerite Risley of Randolph-Macon Woman's College is on leave of absence for the current year.

Assistant Professor Wladimir Seidel of the University of Rochester is on leave of absence for the current academic year, and is at Brown University.

Associate Professors M. H. Stone and D. V. Widder of Harvard University have been promoted to full professorships.

Professor L. A. H. Warren, professor of mathematics in the University of Manitoba, has been appointed Professor and Head of the Department of Actuarial Science recently established in that University.

C. W. Williams of the University of Kentucky has accepted a position at Washington and Lee University.

Professor J. D. Bond, for the past fifteen years a member of the faculty at the University of Tennessee, died November 10, 1937. He was a charter member of the Mathematical Association.

Mr. C. B. Haldeman of Ross, Ohio, a charter member and a life member of the Mathematical Association, died November 3, 1937.

Frank Morley

In the passing of Professor Morley on October 17, 1937, American mathematics suffered the loss of an outstanding personality. Born in Woodbridge, Suffolk, England, on September 9, 1860, Frank Morley received the A.B. degree at King's College, Cambridge, in 1883, the A.M. degree in 1887 and the degree of Sc.D. in 1897. After spending the years 1884 to 1887 as master at Bath College, England, he accepted in 1887 a call from America to an instructorship at Haverford College. In 1888 he became professor of mathematics at Haverford and continued there in this capacity until called to head the department of mathematics as professor at the Johns Hopkins University in 1900. He held this latter position until his retirement, as professor emeritus, in 1928. During these years he acted as an editor of the *American Journal of Mathematics*, and after his retirement he continued his connection with the *Journal* as a cooperating editor, being a constant contributor to its volumes.

Professor Morley was keenly interested in watching the development of the MONTHLY and more especially in its Questions Section. He rarely missed a meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association, when he was not abroad, and frequently presented papers and entered into the discussion of the papers of others. He was a past President of the American Mathematical Society and was also a member of other mathematical and learned societies in this country and abroad, and was an active contributor to their proceedings and their publications.

He possessed a notable power of geometric intuition, and most of his work was in the field of geometry. However, he also possessed great algebraic facility, and made a valuable contribution in algebra to the theory of elimination. During his residence at Haverford, he collaborated with Professor James Harkness, then of Bryn Mawr, in the publication of two outstanding treatises entitled *Elementary treatise on the theory of functions* and *Introduction to the theory of analytic functions*. In them, Professor Morley's artistic and skilful geometric treatment of many topics is plainly evident.

While at Johns Hopkins he developed and refined a powerful method for the study of problems in inversive geometry. Many of his own articles and those of his students on this subject during this time are devoted to the development of the subject. After his retirement from active professional duties, he finally found time to carry out a project he had long contemplated, and with the cooperation of his son, Frank V., he published in 1933 his treatise on *Inversive geometry*. In addition, during the nine years after his retirement in 1928, he continued his active interest in research, a joint paper by him and a former student having appeared in the most recent October number of the *American Journal of Mathematics*.

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BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of addresss to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-first Summer Meeting, Pennsylvania State College, Sept. 6–7, 1937.
Twenty-second Annual Meeting, Indianapolis, Ind., December 30–31, 1937.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1937 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Waynesburg, Pa., May 1; Pittsburgh, October 23.
ILLINOIS, DeKalb, May 14–15.
INDIANA, Greencastle, April 30–May 1.
IOWA, Dubuque, April 16–17.
KANSAS, Wichita, April 3.
KENTUCKY, Louisville, May 1.
LOUISIANA-MISSISSIPPI, Hammond, La., March 5–6.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Lynchburg, Va., May 8; Baltimore, Md., December 4.
MICHIGAN, Ann Arbor, March 20.
MINNESOTA, St. Paul, May 15.
MISSOURI.
NEBRASKA, Lincoln, May 7.
OHIO, Columbus, April 1.
OKLAHOMA, Tulsa, February 5.
PHILADELPHIA, Haverford, Nov. 27.
ROCKY MOUNTAIN, Greeley, Colo., April 16–17.
SOUTHEASTERN, Nashville, Tenn., April 16–17.
SOUTHERN CALIFORNIA, Los Angeles, March 6.
SOUTHWESTERN, State College, N.M., April 2–3.
TEXAS, Houston, April 23–24.
WISCONSIN, Milwaukee, May 8.

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